3 The Fundamental Theorem of Calculus

Theorem 3.1 (FTC, Part 1)**.** If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\ dt, \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

Remark 3.2. Here is an idea of the proof:

Theorem 3.3 (FTC, Part 2)**.** If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Remark 3.4. Here is an idea of the proof:

Remark 3.5. The two parts of the FTC together state that differentiation and integration are inverse processes.

Remark 3.6. From our perspective **FTC, Part 2** is the most important because it allows us to calculate definite integrals without having to take limits of Riemann sums!

Example 3.7 (Instructor). Evaluate the following integrals

(a) $\displaystyle\int_1^9 \sqrt{x}\ dx$

(b) $\displaystyle\int_{-1}^5 (u+1)(u-2)\ dt$

(c) $\displaystyle\int_1^2 \frac{x^4+1}{x^2}\ dx$

(d) $\displaystyle\int_{-1}^2 |x|\ dx$

Example 3.8 (Instructor). Find the derivatives of the following functions:

(a) $f(x) = \displaystyle\int_1^x t^2\ dt$

(b) $g(x) = \displaystyle\int_x^2 \sqrt{t^2+3}\ dt$

(c) $h(x) = \displaystyle\int_{3x}^{2x} \frac{u^2-1}{u^2+1}\ du$

Example 3.9 (Student). Evaluate the following integrals

(a) $\displaystyle\int_1^8 \frac{1}{\sqrt[3]{x^2}}\ dx$

(b) $\displaystyle\int_0^{\pi/3} \sin\theta\ d\theta$

(c) $\displaystyle\int_{-2}^2 f(x)\ dx \quad \text{where } f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4-x^2 & \text{if } 0 < x \le 2 \end{cases}$

(d) $\displaystyle\int_{-1}^5 |3x-6|\ dx$

Example 3.10 (Student). Identify what is wrong with the evaluation:

$$\begin{aligned} \int_{-1}^1 \frac{1}{x^2}\ dx &= \left[-x^{-1}\right]_{-1}^1 \\ &= \left[-(1)^{-1} + (-1)^{-1}\right] = [-1-1] = -2 \end{aligned}$$

Example 3.11 (Student). Find the derivatives of the following functions:

(a) $F(x) = \displaystyle\int_2^{1/x} \sin^4 t\ dt$

(b) $G(x) = \displaystyle\int_{\sin x}^1 \cos^2\theta\ d\theta$

(c) $H(x) = \displaystyle\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\ dt$