9 (b) - Antiderivatives

Definition(s) 9.1. A **differential equation** is an equation involving the derivatives of an unknown function.

Definition(s) 9.2. An **initial value problem** is a differential equation for $y = f(x)$ along with an **initial condition**, such as $f(c) = a$ for some constants c and a. The solution to the initial value problem is a solution to the differential equation that also satisfies the initial condition.

Remark 9.3. In an initial value problem, if the unknown function $f(t)$ represents position as a function of time, then the initial condition $f(t_0) = a$ means that at time $t = t_0$, the object is at position a. If the unknown function represents velocity, then the same initial condition means the velocity is a at time $t = t_0$.

1 Areas and Distances

The goal of this section is to understand how to approximate areas with rectangles. Suppose that we want to approximate the area between the graph of a continuous function $f(x)$ and the x-axis between $x = a$ and $x = b$ (suppose for now that f is positive). Let's sub-divide the interval $[a, b]$ into n sub-intervals $[x_1, x_2]$ through $[x_{n-1}, x_n]$ of equal width Δx. If we pick a point x_i^* in each interval $[x_i, x_{i+1}]$, then we can estimate the area under the graph by the sum of the areas of the rectangles with width Δx and height $f(x_i^*)$:

$$\text{Area} \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

Definition(s) 1.1.

- An **upper sum** is when the x_i^* are all chosen to be the global maximum on $[x_i, x_{i+1}]$ for each i.
- A **lower sum** is when the x_i^* are all chosen to be the global minimum on $[x_i, x_{i+1}]$ for each i.
- A **left-hand sum** is when x_i^* is the left endpoint of $[x_i, x_{i+1}]$ for each i.
- A **right-hand sum** is when x_i^* is the right endpoint of $[x_i, x_{i+1}]$ for each i.

Remark 1.2. To get better approximations of area, use more rectangles.

Remark 1.3. If the velocity of an object is constant, then we have

$$\text{distance} = \text{velocity} \times \text{time}$$

We can think of this product as being the area of a rectangle of width "time" and height "velocity". Thought of in this way, the distance traveled by an object from time $t = a$ to $t = b$ is given by the area under the graph of the velocity from $t = a$ to $t = b$.

Example 1.4 (Instructor)**.** A ball is thrown upward from an initial height of 1 meter, with an initial velocity of 2 meters per second. Given that the acceleration due to gravity on earth is −9.8 meters per second per second, come up with the function $h(t)$ giving the height of the ball at time t.

Example 1.5 (Instructor)**.** Solve the initial value problem

$$f'(x) = 5x^4 - 3x^2 + 4, \quad f(-1) = 2$$

Example 1.6 (Instructor)**.** Solve the initial value problem

$$f''(x) = 8x^3 + 5, \quad f(1) = 0, \quad f'(1) = 8$$

Example 1.7 (Instructor)**.** Approximate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using 4 rectangles of equal width, using a:

(a) left-hand sum

(b) right-hand sum

Example 1.8 (Instructor)**.** The speed of a runner (in ft/s) is given in the table below at different times (in seconds).

t	0	0.5	1.0	1.5	2.0	2.5	3.0
$v(t)$	0	6.2	10.8	14.9	18.1	19.4	20.2

(a) Give an upper estimate of the distance traveled by the runner.

(b) Give a lower estimate of the distance traveled by the runner.

Example 1.9 (Student)**.** Solve the initial value problem

$$f'(x) = 1 + 3\sqrt{x}, \quad f(4) = 25$$

Example 1.10 (Student)**.** Solve the initial value problem

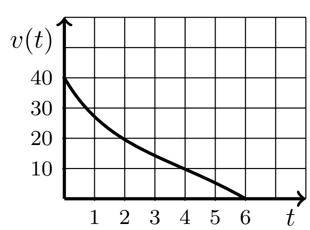
$$f''(x) = 2 + \cos(x), \quad f(0) = -1, \quad f(\pi/2) = 0$$

Example 1.11 (Student)**.** Estimate the area under the graph of $f(x) = 28 + 12x - x^2$ from $x = -2$ to $x = 14$ using:

(a) An upper sum

(b) A lower sum

Example 1.12 (Student)**.** Below is the graph of the velocity of a car (in ft/s) as it is coming to a stop.



Estimate the distance the car travels as it comes to a stop using the various types of sums.