4 (b) - Derivatives of Trigonometric Functions

Theorem 4.3.

(a) $\dfrac{d}{dx}(\sin(x)) = \cos(x)$

(b) $\dfrac{d}{dx}(\cos(x)) = -\sin(x)$

(c) $\dfrac{d}{dx}(\tan(x)) = \sec^2(x) = [\sec(x)]^2$

(d) $\dfrac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$

(e) $\dfrac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$

(f) $\dfrac{d}{dx}(\cot(x)) = -\csc^2(x) = -[\csc(x)]^2$

Proof of (a) and (b)

Use the sum and difference identities for $\sin(x)$ and $\cos(x)$ to write $\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$ and $\lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$ in terms of $\frac{\sin(h)}{h}$ and $\frac{\cos(h)-1}{h}$ and apply **Theorem 4.1**.

5 The Chain Rule

Theorem 5.1.
If g is differentiable at x and f is differentiable at $g(x)$, then the composition $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, the same thing can be written as

$$\frac{dF}{dx} = \frac{dF}{dg} \cdot \frac{dg}{dx}$$

Remark 5.2.

Although the Leibniz notation should not literally be thought of as a fraction, it still gives a good way to remember the chain rule: it looks like the dg's just cancel.

Example 5.3.

If $y = f(x)$ is differentiable, then by combining the chain rule and the power rule, we can differentiate powers of f:

$$\frac{d}{dx}(y^n) = n \cdot y^{n-1} \cdot \frac{dy}{dx}$$

Or, written in terms of $f(x)$, this would be

$$\frac{d}{dx}(f(x)^n) = n \cdot [f(x)]^{n-1} \cdot f'(x)$$

Example 5.4 (Instructor)**.**

Prove part **(c)** of **Theorem 4.3**.

Example 5.5 (Instructor)**.**

Calculate the derivatives of the following functions by first writing them as a composition of simpler functions, and then applying the chain rule:

(a) $f(x) = \left(7x^3 + 2x^2 - x + 3\right)^5$

(b) $g(x) = \sin\left(3x^2 + 5x + 8\right)$

(c) $h(x) = x\tan\left(2\sqrt{x}\right) + 7$

Example 5.6 (Instructor)**.**

Prove the quotient rule by writing $\dfrac{f(x)}{g(x)}$ as $f(x) \cdot \dfrac{1}{g(x)}$ and using the product rule and chain rule.

Example 5.7 (Instructor)**.**

Recall that f is an even function if $f(-x) = f(x)$ for all x, and that f is an odd function if $f(-x) = -f(x)$ for all x. Use the chain rule to prove that the derivative of an even function is an odd function.

Example 5.8 (Instructor)**.**

A table of values for f, g, f', and g' is given:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $h(x) = f(g(x))$, find $h'(1)$.

(b) If $H(x) = g(f(x))$, find $H'(1)$.

Example 5.9 (Student)**.**

Calculate the derivatives of the following functions:

(a) $y = \sin(x\cos(x))$

(b) $y = \sqrt{x + \sqrt{x}}$

(c) $F(x) = \left(4x - x^2\right)^{100}$

Example 5.10 (Student)**.**

Prove part **(d)** of **Theorem 4.3**.

Example 5.11 (Student)**.**

Prove that the derivative of an odd function is an even function.

Example 5.12 (Student)**.**

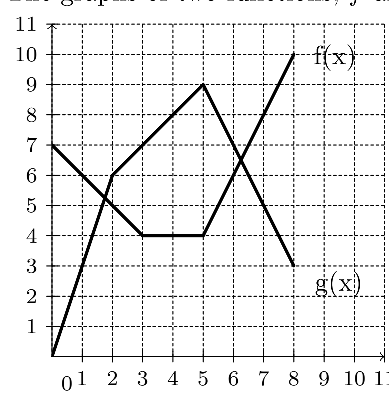
A ball is expanding over time, and the radius (as a function of time in minutes) is given by

$$r(t) = \sqrt{t} + 1$$

How fast is the volume of the ball increasing at $t = 4$ minutes?

Example 5.13 (Student)**.**

The graphs of two functions, f and g, are pictured below.



Let $h(x) = f(g(x))$, and $p(x) = g(f(x))$. Compute the following derivatives, or explain why they do not exist:

(a) $h'(2)$

(b) $h'(7)$

(c) $p'(1)$