1 Derivatives and Rates of Change

Definition(s) 1.1.

The **derivative of a function f at a number** c, denoted by $f'(c)$, is the number

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

if the limit exists. An equivalent formulation is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Remark 1.2.

The tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ is the line through $(c, f(c))$ whose slope is equal to $f'(c)$.

Example 1.3.

If $f(t)$ measures distance of a moving object, and t is time, then the **velocity** (or **instantaneous velocity**) of the moving object, denoted $v(t)$, is the limit of the average velocities (as defined in **Section 1.4**).

$$v(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t}$$

2 The Derivative as a Function

Definition(s) 2.1.
The **derivative of a function** f, denoted by f', is the function $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. Another common notation is to write $\dfrac{df}{dx}$ or $\dfrac{d}{dx}f(x)$ instead of $f'(x)$. The derivative at $x = c$ in this notation is written $\left.\dfrac{df}{dx}\right|_{x=c}$.

Definition(s) 2.2.

A function f is **differentiable at** c if $f'(c)$ exists. It is **differentiable on the interval** (a, b) if it is differentiable at every number in (a, b).

Theorem 2.3.

If f is differentiable at c, then f is continuous at c.

How Can a Function Fail to be Differentiable (at a point c)?

- f is discontinuous at c
- $\displaystyle \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \neq \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$
- $\displaystyle \lim_{x \to c} |f'(x)| = \infty$

Definition(s) 2.4.

The **second derivative** of f is the derivative of $f'(x)$, denoted by $f''(x)$ or $\dfrac{d^2 f}{x^2}$. In general, the n^{th} **derivative**, denoted by $f^{(n)}(x)$ or $\dfrac{d^n f}{dx^n}$, is the derivative of $f^{(n-1)}(x)$.

Example 2.5.

If $f(t)$ measures distance of a moving object, then the **acceleration** of the object, $a(t)$, is the second derivative of f, and the first derivative of the velocity, $v(t)$.

Example 2.6 (Instructor).

Compute (using the limit definition) the derivative of the following functions:

(a) $f(x) = x^2 + 3x + 7$

(b) $f(x) = \dfrac{1}{x}$

(c) $f(x) = \sqrt{x}$

Example 2.7 (Instructor).

Explain why $f(x) = |x|$ is *not* differentiable at $x = 0$.

Example 2.8 (Instructor).

Are the following functions differentiable at $x = 2$? Why or why not?

(a) $f(x) = \begin{cases} x & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$

(b) $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4x - 4 & \text{if } x \geq 2 \end{cases}$

(c) $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$

Example 2.9 (Student).

Compute (using the limit definition) the derivative of the following functions:

(a) $f(x) = 3x^2 + x - 8$

(b) $f(x) = \dfrac{5}{x - 3}$

(c) $f(x) = \dfrac{1}{x^2}$

Example 2.10 (Student).

Is the function $f(x) = \begin{cases} 2x + 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$ differentiable at $x = 0$? Why or why not?

Example 2.11 (Student).

Assuming a function $f(x)$ is differentiable at $x = c$, come up with a general equation for the tangent line to f at c.

Example 2.12 (Student).

The graph of a function $f(x)$ is shown on the left. Use it to sketch a graph of $f'(x)$.