6 (b) - Calculating Limits Using the Limit Laws

Definition(s) 6.9.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Theorem 6.10.

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Theorem 6.11 (Squeeze Theorem)**.**

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) \text{ exists, and } \lim_{x \to a} g(x) = L.$$

7 The Precise Definition of a Limit

Definition(s) 7.1.
Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Remark 7.2 (Strategy for solving problems)**.**

Find an δ based on ε solving the absolute value inequality. Check your work (NOT OPTIONAL).

Remark 7.3 (Check your work for linear functions)**.**

If $f(x) = mx + b$ then $\delta = \varepsilon/m$ is a good choice.

Remark 7.4 (δ for quadratic functions)**.**

First check if you have a perfect square, otherwise take $\delta < 1$.

Example 7.5 (Instructor).

Show that $\lim_{x \to 0} 2x^2 \sin \frac{3}{x} = 0$

Example 7.6 (Instructor).

Prove that $\lim_{x \to 1} 3x - 1 = 2$ using the formal definition of a limit.

Example 7.7 (Instructor).

Given $\varepsilon = 0.1$, find the largest value of δ in the proof of $\lim_{x \to 5}(x^2 - 10x + 41) = 16$.

Example 7.8 (Instructor).

Prove that $\lim_{x \to 2} x^2 = 4$ using the formal definition of a limit.

Example 7.9 (Instructor).

Setup the absolute value inequalities in the proof that $\lim_{x \to \pi/2} \sin x = 1$.

Example 7.10 (Students).

If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find $\lim_{x \to 4} f(x)$.

Example 7.11 (Students).

Find $\lim_{x \to 0} x^4 \cos \frac{-1}{x}$

Example 7.12 (Students).

Prove that $\lim_{x \to 2}(5 - x) = 3$ using the formal definition of a limit.

Example 7.13 (Students (If time: stop at 6:45pm)).

In 1.6 we learned that $\lim_{x \to a} c = c$ and $\lim_{x \to a} x = a$. Prove these using the formal definition of the limit.