6. Linear Differential Equations of the Second Order

A differential equation of the form

$$L(y) = g$$

is called *linear* if L is a linear operator and $g = g(t)$ is continuous.

The most general second order linear differential equations has the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

where P, Q, R, G are continuous functions defined on an interval I. Assuming that $P(t) \neq 0$ for $t \in I$, we can divide through by $P(t)$ and rewrite this d.e. as

$$y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

where p, q, g are all continuous on the interval I.

Analogously, we write the IVP

$$y'' + p(t)y' + q(t)y = g(t),\ y(t_0) = y_0,\ y'(t_0) = y_0' \tag{2}$$

where p, q, g are all continuous on the interval I, $t_0 \in I$, and y_0, y_0' are given constants.

The following is an important theorem, usually proved in a more advanced course.

Theorem(Existence-Uniqueness Theorem for Second Order Linear Differential Equations). *Let* $p(t), q(t), g(t)$ *be continuous functions on the interval* I*, let* $t_0 \in I$*, and let* y_0, y_0' *be given constants. Then, there is a unique solution* $y(t)$ *to the IVP (2) which is defined on the whole interval* I*.*

We are concerned with finding the general solution to (1) , and solving initial value problems.

Given equation (1), the *associated homogeneous* equation is the d.e.

$$y'' + p(t)y' + q(t)y = 0 \tag{3}$$

A consequence of the next result is that, in order to find the general solution to (1), it suffices to

1. find the general solution y_h to (3), (4)

and

2. find a particular solution y_p to (1). (5)

The general solution to (1) is then obtained as

$$y = y_h + y_p.$$

Theorem. *Let* $y_p(t)$ *be a particular solution to (1). Then, every solution* $y(t)$ *to (1) can be expressed as*

$y(t) = y_1(t) + y_p(t)$ where $y_1(t)$ is a solution to (3). Conversely, for any solution $y_1(t)$ of (3), the function $y(t) = y_1(t) + y_p(t)$ is a solution to (1).

Proof.

Let $y_p(t)$ be a particular solution to (1), and let $y(t)$ be any other solution to (1). Consider the function

$$y_1(t) = y(t) - y_p(t).$$

We clearly have $y(t) = y_1(t) + y_p(t)$. Let us verify that

$$y_1 \text{ is a solution to (3).} \tag{6}$$

By linearity,

$$L(y_1) = L(y(t) - y_p(t)) = L(y(t)) - L(y_p(t)) = g(t) - g(t) = 0,$$

which verifies (6).

Converse:

Let $y_1(t)$ be solution to (3), and let $y(t) = y_1(t) + y_p(t)$.

Then,

$$L(y) = L(y_1 + y_p) = L(y_1) + L(y_p) = 0 + g(t) = g(t),$$

so, y is a solution to (1). QED.

In view of the preceding theorem, we need to study methods to handle the problems (4) and (5).

We begin with (4).

It turns out that to solve this problem, it suffices to find *two* solutions which satisfy a condition called *linear independence.*

Defintion. A pair of functions $y_1(t),\ y_2(t)$ defined on an interval I is called a *linearly independent* pair of functions (on I) if whenever there are constants c_1, c_2 such that

$$c_1 y_1(t) + c_2 y_2(t) = 0, \quad \forall t \in I,$$

we have $c_1 = c_2 = 0$.

This means that if $c_1 y_1 + c_2 y_2$ is the zero function on I, it follows that $c_1 = c_2 = 0$.

We state some theorems which allow us to find the general solution to second order homogeneous linear differential equations.

We will justify the theorems later.

Theorem. *Let $y_1(t),\ y_2(t)$ be a linearly independent pair of solutions to (3) on the interval I. Then, the general solution to (3) has the form*

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

Definition. A 2×2 matrix is an array of the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where the a_{ij} are real or complex numbers. When they are real, we say that A is a *real* matrix.

Definition. The determinant $det(A)$ of the 2×2 matrix A is the number

$$det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Definition. The *Wronskian* at t_0 of the two functions $y_1,\ y_2$ is the determinant

$$W(y_1, y_2)(t_0) = det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}$$

We also call the function $W(y_1, y_2)(t)$ the *Wronksian* or *Wronskian function* of y_1 and y_2.

Theorem(Abel). *Two solutions $y_1,\ y_2$ of the the equation (3) are linearly independent on I if and only if $W(y_1, y_2)(t) \neq 0$ for some (or any) $t \in I$.*

Proof. We will show that $W = W(t)$ satisfies the differential equation

$$W(t)' = -p(t)W(t) \tag{7}$$

for $t \in I$.

Once this is done, let t_0 and t_1 be two points in I.

Then,

$$\frac{W'}{W} = -p(t)$$

$$\frac{d(log(W(t))}{dt} = -p(t)$$
$$\log(W(t_1)) - \log(W(t_0)) = \int_{t_0}^{t_1} -p(s)ds$$
$$\frac{W(t_1)}{W(t_0)} = exp(\int_{t_0}^{t_1} -p(s)ds)$$
$$W(t_1) = W(t_0)exp(\int_{t_0}^{t_1} -p(s)ds)$$

Since the exponential term is never zero, we see that $W(t_1) = 0$ if and only if $W(t_0) = 0$.

Now, we turn to the proof of (7).

We have

$$W = y_1y_2' - y_2y_1'.$$

So,

$$\begin{aligned} W' &= y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1'' \\ &= y_1y_2'' - y_2y_1'' \\ &= y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1) \\ &= -p(y_1y_2' - y_2y_1') \\ &= -pW \end{aligned}$$

as required. QED.

Second Order Linear Homogeneous Differential Equations with Constant Coefficients:

These have the form

$$ay'' + by' + cy = 0 \tag{8}$$

where a, b, c are constants and $a \neq 0$.
Let us first try to find a solution of the form

$$y = e^{rt} \tag{9}$$

where r is a constant.
Differentiating, we get

$$\begin{aligned} ay'' + by' + cy &= ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \\ &= (ar^2 + br + c)e^{rt} = 0 \end{aligned}$$

Since e^{rt} is never zero, the only way we could possibly get a solution of the form (9) is for r to be a root of the polynomial

$$\rho(r) = ar^2 + br + c.$$

This last polynomial is called the *characteristic polynomial* of the d.e. (8), and the equation

$$\rho(r) = ar^2 + br + c = 0 \tag{10}$$

is called the *characteristic* equation of (8).

Proceding in the reverse order, we also see that if r_1 is a root of the characteristic equation, then, indeed, $y(t) = e^{r_1 t}$ is a solution of (8).

Also, if the characteristic equation (10) has two distinct real roots, r_1, r_2, then we get two solutions of the form

$$y_1(t) = e^{r_1 t},\ y_2(t) = e^{r_2 t}.$$

Note that this occurs if and only if $b^2 - 4ac > 0$.

We label the roots r_1, r_2 of $\rho(r)$ so that $r_1 > r_2$.

Thus, from the quadratic formula, we have

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We will say that $y_1(t) = e^{r_1 t}$ is the *first fundamental solution* to (8), and $y_2(t) = e^{r_2 t}$ is the *second fundamental solution* to (8).

Let us see that these turn out to be linearly independent solutions.

We compute the Wronskian at $t = 0$.

$$\begin{aligned} W(y_1, y_2)(t) &= det \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} \\ &= det \begin{pmatrix} 1 & 1 \\ r_1 & r_2 \end{pmatrix} \\ &= r_2 - r_1 \\ &\neq 0. \end{aligned}$$

Since this non-zero at $t = 0$ it is non-zero everywhere, so we do have linearly independent solutions.

Hence, the general solution in the case of real distinct roots r_1, r_2 of (10) is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Examples:

1. $y'' - 3y' - 4y = 0$.

   Find the roots of $r^2 - 3r - 4$.

   Factoring the polynomial, we get

   $$r^2 - 3r - 4 = (r - 4)(r + 1).$$

   So, the roots are $r_1 = 4, r_2 = -1$.

   General Solution: $y = c_1 e^{4t} + c_2 e^{-t}$.

   Writing $y_1(t), y_2(t)$ for the first and second fundamental solutions, we have

   $$y_1(t) = e^{4t},$$

   and

   $$y_2(t) = e^{-t}.$$

2. $y'' + 3y' + y = 0$

   Characteristic equation: $r^2 + 3r + 1 = 0$.

   Use the quadratic formula:

$$r = \frac{-3 \pm \sqrt{5}}{2}.$$

   So, general solution:

$$y(t) = c_1 e^{(\frac{-3+\sqrt{5}}{2})t} + c_2 e^{(\frac{-3-\sqrt{5}}{2})t},$$

   and the first and second fundamental solutions $y_1(t), y_2(t)$, respectively, are

$$y_1(t) = e^{(\frac{-3+\sqrt{5}}{2})t}$$

$$y_2(t) = e^{(\frac{-3-\sqrt{5}}{2})t}$$

Now, we know that, given a second degree polynomial $\rho(r)$, we have three possibilities for its roots r_1, r_2.

Case 1. $r_1 \neq r_2$ and both are real

Case 2. $r_1 = r_2$,

Case 3. $r_1 = \alpha + \beta i,\ r_2 = \alpha - \beta i$ where $i = \sqrt{-1}$.

So, finding the general solution to a homogeneous second order linear d.e. with constant coefficients, also involves those three cases.

We have already dealt with Case 1.

Case 2: $r_1 = r_2$. That is, $\rho(r) = a(r - r_1)^2$.

Here we already have one non-zero solution $y_1(t) = e^{r_1 t}$.

We claim that the function $y_2(t) = te^{r_1 t}$ is a second linearly independent solution.

In proceding to verifty this, it will be useful to recall the formula for the second derivative of a product.

$$(fg)^{''} = f''g + 2f'g' + fg''.$$

Now, let us verify that y_2 is a solution.

Note that, since, r_1 is a root of multiplicity two, we have

$$ar_1^2 + br_1 + c = 0, \quad \text{and } 2ar_1 + b = 0.$$

We have

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2r_1e^{r_1 t} + tr_1^2 e^{r_1 t}) + b(e^{r_1 t} + tr_1 e^{r_1 t}) + cte^{r_1 t} \\ &= (a2r_1 + b)e^{r_1 t} + (ar_1^2 + br_1 + c)te^{r_1 t} \\ &= 0e^{r_1 t} + 0te^{r_1 t} = 0. \end{aligned}$$

Hence, y_2 is a solution.

Now, let us verify that the pair $y_1(t), y_2(t)$ is a linearly independent pair.

We compute the Wronskian:

$$\begin{aligned} W(y_1, y_2) &= det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \\ &= det \begin{pmatrix} e^{r_1 t} & te^{r_1 t} \\ r_1 e^{r_1 t} & e^{r_1 t} + tr_1 e^{r_1 t} \end{pmatrix} \\ &= e^{2r_1 t} + tr_1 e^{2r_1 t} - tr_1 e^{2r_1 t} \\ &= e^{2r_1 t} \neq 0. \end{aligned}$$

Hence, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 te^{r_1 t}.$$

In Case 2, we define the first and second fundamental solutions $y_1(t), y_2(t)$, respectively, by

first fundamental solution: $y_1(t) = e^{r_1 t}$
second fundamental solution: $y_2(t) = te^{r_1 t}$

Case 3: $r_1 = \alpha + \beta i$ with $\beta \neq 0$.
Here we will make use of complex variables.
Recall the formula

$$e^{\alpha + \beta i} = e^{\alpha}(\cos(\beta) + i \sin(\beta)).$$

We first verify that the complex valued function

$$y(t) = e^{(\alpha+\beta i)t}$$

is a solution to our d.e. It turns out that the real and imaginary parts of this complex solution give linearly independent solutions to the d.e.

What are these real and imaginary parts:

$$e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \tag{11}$$

So, the real part is

$$e^{\alpha t}\cos(\beta t)$$

and the imaginary part is

$$e^{\alpha t}\sin(\beta t).$$

Hence, the general solution is

$$y(t) = e^{\alpha t}(c_1 cos(\beta t) + c_2 sin(\beta t)).$$

In the case of complex roots, we define the first and second fundamental solutions to (8).

First fundamental solution: $y_1(t) = e^{\alpha t} cos(\beta t)$

Second fundamental solution: $y_2(t) = e^{\alpha t} sin(\beta t)$

Examples:

1. Find the general solution and the first and second fundamental solutions $y_1(t), y_2(t)$ to

   $$y'' + 6y' + 9y = 0.$$

   Solution: $r^2 + 6r + 9 = (r+3)^2$, so the general solution is:

   $$y(t) = c_1 e^{-3t} + c_2 t e^{-3t}.$$

   The first and second fundamental solutions are:

   $$y_1(t) = e^{-3t},\ y_2(t) = te^{-3t}$$

2. Find the general solution and first and second fundamental solutions to

   $$y'' + y' + 3y = 0.$$

   Solution:

   Step 1: Roots of characteristic equation.

   $$r^2 + r + 3 = 0.$$

   $$\begin{aligned} r &= \frac{-1 \pm \sqrt{1-12}}{2} \\ &= \frac{-1}{2} \pm \frac{\sqrt{11}i}{2} \end{aligned}$$

General Solution:

$$y(t) = e^{-\frac{t}{2}}(c_1 \cos(\frac{\sqrt{11}t}{2}) + c_2 \sin(\frac{\sqrt{11}t}{2})).$$

First and second fundamental solutions:

$$y_1(t) = e^{-\frac{t}{2}} \cos(\frac{\sqrt{11}t}{2}),\ y_2(t) = e^{-\frac{t}{2}} \sin(\frac{\sqrt{11}t}{2})$$

1 Summary

Consider the differential equation

$$ay'' + by' + cy = 0$$

where a, b, c are real constants and $a \neq 0$.

Let $q(r) = ar^2 + br + c = 0$ be the characteristic equation.

Let $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ be the roots of $q(r) = 0$.

Case 1: $r_1 \neq r_2$, both real, $r_1 > r_2$,

General Solution:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

First and second fundamental solutions:

$$y_1(t) = e^{r_1 t},\ y_2(t) = e^{r_2 t}$$

Case 2: $r_1 = r_2 \neq 0$

General Solution:

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

First and second fundamental solutions:

$$y_1(t) = e^{r_1 t},\ y_2(t) = t e^{r_2 t}$$

Case 3: $r_1 = \alpha + \beta i$, $r_2 = \alpha - \beta i$, complex with $\beta \neq 0$

General Solution:

$$y(t) = e^{\alpha t}(c_1 cos(\beta t) + c_2 sin(\beta t))$$

First and second fundamental solutions:

$$y_1(t) = e^{\alpha t} cos(\beta t),\ y_2(t) = e^{\alpha t} sin(\beta t)$$