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DIFFEOMORPHISMS WITH INFINITELY MANY SINKS

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(Dedicated to my father, Bernard C. Newhouse 1916–1971)

LET M be a compact C^∞ manifold without boundary, and let $\text{Diff}^r(M)$ denote the space of C^r diffeomorphisms of M with the uniform C^r topology, $1 \le r < \infty$. Suppose $f \in \text{Diff}^r(M)$ and p is a periodic point of f of period ν; that is, $f^\nu(p) = p$ but $f^k(p) \neq p$ for $0 < k < \nu$. We say that p is a *sink* if each eigenvalue of the derivative f^ν at p has absolute value less than one. S. Smale suggested in [12] that most diffeomorphisms on S^2 with the C^r topology ought to have only finitely many sinks. We show here that this is not the case for $r \ge 2$ by proving the following

THEOREM. *On any manifold M of dimension greater than one, there is a residual subset* $\mathscr{B}$ *of an open set in* $\text{Diff}^r(M)$, $r \ge 2$, *such that every element of* $\mathscr{B}$ *has infinitely many sinks.*

A residual subset of an open set $\mathscr{N}$, of course, means a set which contains a countable intersection of dense open subsets of $\mathscr{N}$.

Our theorem will be proved as follows. First, sufficient conditions for the appearance of infinitely many sinks will be given. Second, it will be shown that these conditions can be obtained in a two dimensional disk D^2 using the examples in [5]. Then, using well-known techniques, the conditions can be constructed in a disk of the same dimension as M, after which they may be conjugated into M via a local coordinate system about a contracting fixed point of a given diffeomorphism.

The theorem was obtained in response to a question raised by M. Shub at the beginning of the Symposium on Dynamical Systems in Salvador, Brazil, in July 1971. The question was whether most diffeomorphisms have an Ω-spectral decomposition with no cycles (see [10]). One can see from the examples described here that this question also has a negative answer.

We take this opportunity to point out that while the statement of Lemma 5.3 in [5] is correct, the proof given there is not. A correct proof similar to that of Lemma 5.1 in [5] can be given. Alternatively, the persistence of the tangency condition required can be obtained directly from Lemmas 3.7 and 5.1 as follows. With the notation of [5], one can show that for f C^2 near L, the sets $\overline{W^s(f)}$ and $\overline{W^u(f)}$ extend to C^1 foliations $\mathscr{F}^s(f)$, $\mathscr{F}^u(f)$ on a neighborhood V of ρ_0. This is done by using methods in [3; §6] and [2] to show that the tangents to

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$\overline{W^s(f)}$ and $\overline{W^u(f)}$ extend to C^1 vector fields on a neighborhood V of ρ_0 and then integrating these vector fields. The vector fields obtained will have tangencies on a C^1 arc γ which is C^1 near ρ_0. Then one may apply Lemmas 3.7 and 5.1 to get the tangency condition.

We proceed to give conditions for the appearance of infinitely many sinks.

Fix $r \geq 1$. Given a hyperbolic periodic point p of $f \in \mathrm{Diff}^r(M)$, $W^s(p)$ and $W^s(o(p))$ will denote, respectively, the stable manifold of the point p and the orbit $o(p)$, and $W^u(p)$, $W^u(o(p))$ will denote the corresponding unstable manifolds. When we wish to emphasize the dependence of these manifolds on f, we will write $W^s(p) = W^s(p, f)$, etc. Let p be a hyperbolic periodic point of f of period v such that $\dim W^s(p) = \dim M - 1$. Let $\mu_1, \ldots, \mu_s, \lambda$ be the eigenvalues of the derivative $T_p f^v$ with $|\mu_1| \leq |\mu_2| \leq \cdots \leq |\mu_s| < 1 < |\lambda|$, and set $\mu(p) = |\mu_s|$ $\lambda(p) = |\lambda|$. A point of tangency of $W^s(o(p))$ and $W^u(o(p))$ is a point $x \in W^s(o(p) \cap W^u(o(p))$ such that $\dim(T_x W^s(o(p)) \cap T_x W^u(o(p))) > 0$. Since $\dim W^s(o(p)) + \dim W^u(o(p)) = \dim M$, a point of tangency is a non-transversal intersection.

PROPOSITION 1. *With the notation above, assume x is a point of tangency of $W^s(o(p))$ and $W^u(o(p))$ and $\mu(p) \cdot \lambda(p) < 1$. Then given any neighborhoods U of x in M and $\mathcal{N}$ of f in* $\mathrm{Diff}^r(M)$, *there is a g in $\mathcal{N}$ which has a sink in U.*

Before proceeding with the proof of Proposition 1, it is perhaps worthwhile to say a few words of motivation. Let $p_1, \ldots, p_n$ be hyperbolic periodic points of a diffeomorphism g. The orbits form a *cycle* if $o(p_1) = o(p_n)$ and, for $1 \leq j < n$, $W^u(o(p_j)) - o(p_j) \cap W^s(o(p_{j+1})) - o(p_{j+1}) \neq \varnothing$. A basic question in bifurcation theory is to describe the formation of cycles. For a general discussion of this problem we refer the reader to [7] and a sequel to that paper which is now being written. Here we wish to consider what is at first glance the simplest possible way of forming a cycle. It was, in fact, in this context that Proposition 1 was discovered.

Consider a diffeomorphism g of a two dimensional manifold having a hyperbolic fixed point p_1 such that in local coordinates near p_1, g is expressed as

$$g(u, v) = (\mu u, \lambda v) \quad \text{with} \quad 0 < \mu < 1 < \lambda.$$

The local stable manifold of p_1 may be identified with the line segment $v = 0$, and the local unstable manifold of p_1 may be identified with the line segment $u = 0$ as in Fig. 1. Let $(o, v_o) \in W^u(p_1)$ and $(u_o, o) \in W^s(p_1)$

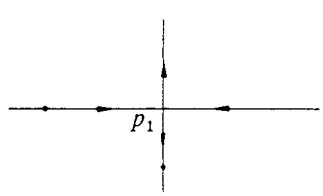

FIG. 1.

With a smooth one parameter family η_t, $o \leq t \leq 1$, of diffeomorphisms leaving a neighborhood of p_1 fixed, let us move a small line segment $l \subset W^u(p_1)$ containing (o, v_o) through the first quadrant so that it becomes tangent to $W^s(p_1)$ at (u_o, o) as in Fig. 2.

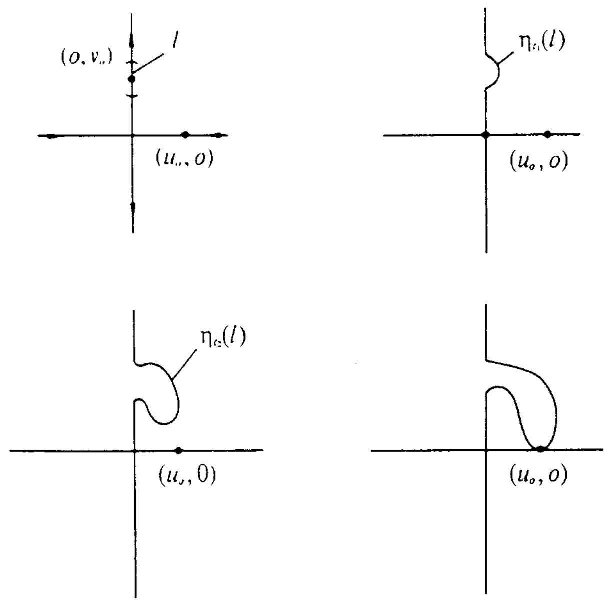

FIG. 2.

This may be done so that the curves in Fig. 2 represent the local stable and unstable manifolds of $g_t = \eta_t \circ g$ at p_1 for the values $t = o, t_1, t_2, 1$.

Now one can see that near (u_o, o) there is a rectangle D whose images under high powers of various g_t's are as in Fig. 3. Provided D is chosen properly, the diffeomorphism $g_1{}^n | D$ is a "horseshoe" diffeomorphism as considered in [13] and $g_1{}^n | D$ will have infinitely many hyperbolic periodic points. Thus in moving from g_o to g_1, we have introduced infinitely many new periodic points in D. If $\mu\lambda < 1$, and g_{t_o} is the first diffeomorphism among the g_t for which $g_{t_o}{}^n$ has a fixed point in D, then for $t > t_o$ and near t_o, $g_{t_o}{}^n$ will have a fixed sink in D. The proof of this is essentially a two dimensional version of proof Proposition 1, and the reader may find it helpful to keep the above figures in mind.

Proof of Proposition 1. Let dim $M = m$, and let (u, v) denote coordinates on $R^{m-1} \times R$ where R^{m-1} is the $(m-1)$-dimensional Euclidean space and R is the real line.

Making a preliminary approximation and using Sternberg's linearization theorem [1], we may assume that f is C^∞ and there are a neighborhood V of p and a C^∞ diffeomorphism $\psi : V \to R^{m-1} \times R$ such that $\psi(p) = (0, 0)$ and $\psi f^\nu \psi^{-1}(u, v) = (Au, \lambda v)$ for (u, v) in $\psi(V \cap f^{-\nu} V)$ where $A : R^{m-1} \to R^{m-1}$ is a linear contraction. Replacing f^ν by $f^{2\nu}$, if necessary, and making a further linear change of coordinates on $R^{m-1} \times R$, we may assume that $\lambda > 1$ and $|A|\lambda < 1$. The preliminary approximation may be done so that $W^s(o(p))$ is still tangent to $W^u(o(p))$ at x.

Choose integers $n_1 < n_2$ such that $f^{n_1}(x), f^{n_2}(x) \in V$, $\psi f^{n_2}(x) = (u_o, o)$, and $\psi f^{n_1}(x) = (o, v_o)$ with $u_o \neq 0$ in R^{m-1} and $v_o \neq 0$ in R. Adjusting ψ, we may take $v_o > 0$. Let D_1 be a

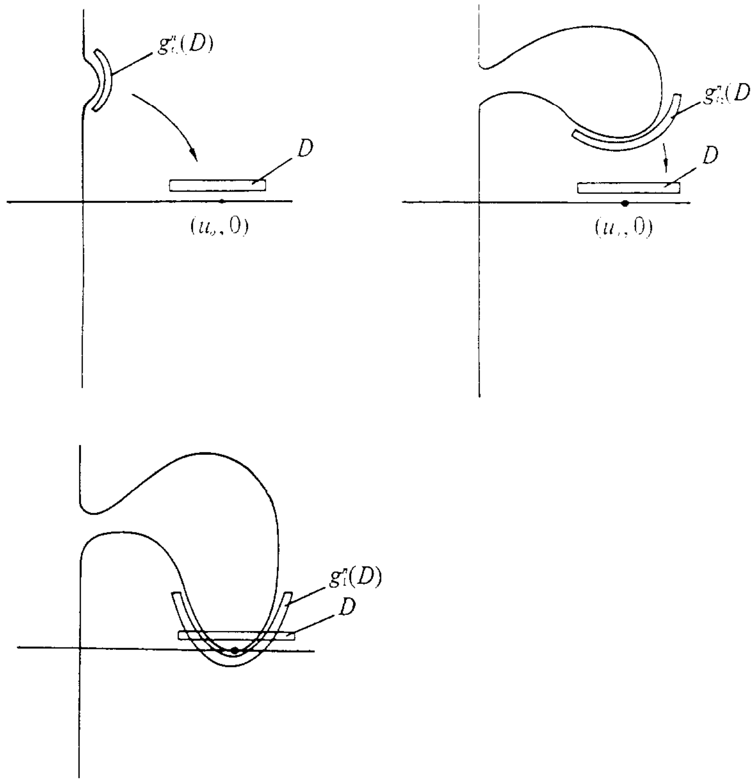

FIG. 3.

disk in R^{m-1} and D_2 be an interval in R such that $u_0 \in \text{int } D_1$, $v_o \in \text{int } D_2$. $A^n(D_1) \cap D_1 = \varnothing$ and $\lambda^{-n} D_2 \cap D_2 = \varnothing$ for $n > 0$, and $D_1 \times \{0\} \cup \{0\} \times D_2 \subset \psi V$.

Changing f slightly near $f^{n_2-1}(x)$ and decreasing D_2, we may assume there is a neighborhood V_o of $\{0\} \times D_2$ in ψV such that if $g : V_o \to R^{m-1} \times R$ is the map $(u, v) \to \psi f^{n_2-n_1} \psi^{-1}(u, v) = (g_1(u, v), g_2(u, v))$, then $g(o, v_o) = (u_o, o)$, $g_{2v}(o, v_o) = 0$, and $g_{2vv}(o, v_o) \neq 0$ where g_{2v} and g_{2vv} denote the first and second partial derivatives of g_2 with respect to v. We assume $g_{2vv}(o, v_o) < 0$, the other case being handled similarly. Shrinking V_o, we suppose there is a constant $K > 0$ such that $g_{2vv}(u, v) < -K < 0$ for $(u, v) \in V_o$.

Choose $n_o > 0$ such that $A^n(D_1) \times D_2 \subset V_o$ for $n \geq n_o$, and let $D_n = D_1 \times \lambda^{-n} D_2$. Taking n_1, n_2, D_1 and D_2 appropriately, we may assume that $(A^j \times \lambda^j)(D_n) \subset \psi V$ for $0 \leq j \leq n$. Let $f_{n,t} : D_n \to g V_o$ be defined by

$$f_{n,t}(u, v) = (g_1(A^n u, \lambda^n v), g_2(A^n u, \lambda^n v) + t) \qquad \text{for } t \text{ small.}$$

LEMMA 2. *For any neighborhood U of (u_o, o) there is an integer $N > 0$ such that if $n \geq N$, there is a number t_n such that $0 < t_n < \lambda^{-n+1} v_o$ and f_{n,t_n} has a fixed sink in U.*

Before proving Lemma 2, we note that Proposition 1 follows from it. To see this, first observe that $f_{n,t}(u, v) = \psi f^{\nu n+n_2-n_1}\psi^{-1}(u, v) + (o, t)$. Now given a small C^r neighborhood $\mathcal{N}$ of f, there are neighborhoods $V_1 \subset \text{int } V_2$ of $f^{-1}\psi^{-1}(u_o, o)$, a real number $c > 0$, and an integer $N > 0$ such that for $n > N$ and $0 < t < c$, there is a diffeomorphism $f_1 \in \mathcal{N}$ such that $f_1|_{M-V_2} = f|_{M-V_2}$ and

$$\psi f_1^{\nu n+n_2-n_1}\psi^{-1}|_{\psi(fV_1)\cap D_n} = f_{n,t}|_{\psi(fV_1)\cap D_n}.$$

Choosing N big enough and c small enough, Lemma 2 implies that one can get $f_1^{\nu n+n_2-n_1}$ to have a sink near $f^{n_2}(x)$ which in turn yields Proposition 1.

We now prove Lemma 2. In what follows, g_{iu}, g_{iv} will denote the partial derivatives and $g_{iuu}, g_{iuv}, g_{ivv}$ will denote the second order partial derivatives of g_i, $i = 1, 2$. g_{2v} and g_{2vv} will be considered as linear maps $R \to R$ or as real numbers.

The fixed point set of $f_{n,t}$ is given by

$$u = g_1(A^n u, \lambda^n v)$$

$$v = g_2(A^n u, \lambda^n v) + t$$

where $(u, v) \in D_n$.

Setting $\psi_1(u, v) = u - g_1(A^n u, \lambda^n v)$ for $(u, v) \in D_n$, one has that $\frac{\partial \psi_1}{\partial u} = I - g_{1u}A^n$ with I the $(m-1) \times (m-1)$ identity matrix. Thus, for large n, $\frac{\partial \psi_1}{\partial u}$ is an isomorphism, and, hence, there is a C^∞ function $u(v)$ for $v \in \lambda^{-n}D_2$ such that $\psi_1(u, v) = 0$ if and only if $u = u(v)$. Also, $u'(v) = (I - g_{1u}A^n)^{-1}g_{1v}\lambda^n$ where $u'(v)$ is the derivative of $u(v)$ at v. Now the fixed point set of $f_{n,t}$ is the set of $(u(v), v)$ such that $\phi_{n,t}(v) = 0$ where

$$\phi_{n,t}(v) = g_2(A^n u(v), \lambda^n v) - v + t \quad \text{with} \quad v \in \lambda^{-n}D_2.$$

We consider the zeros of $\phi_{n,t}$ for n large as t varies.

We have

$$\phi'_{n,t}(v) = g_{2u}A^n u'(v) + g_{2v}\lambda^n - 1.$$

From the condition $|A|\lambda < 1$, we get that $|A^n|\,|u'(v)| \to 0$ as $n \to \infty$. This and the facts that $g_{2v}(o, v_o) = 0$ and $g_{2vv}(u, v) < -K$ for $(u, v) \in V_o$ imply that there is a constant $K_1 > 0$ such that, for large n, there are points $v_1, v_2 \in \lambda^{-n}D_2$ such that $g_{2v}(A^n u(v_1), \lambda^n v_1) < -K_1$ and $g_{2v}(A^n u(v_2), \lambda^n v_2) > K_1$. Thus, for large n, there is a point $v_n \in \lambda^{-n}D_2$ such that $\phi'_{n,t}(v_n) = 0$ and $g_{2v}(A^n u(v_n), \lambda^n v_n)\lambda^n$ is near 1.

Now we show that $\phi''_{n,t}(v)$ is strictly negative for $v \in \lambda^{-n}D_2$, n large. This implies that $\phi_{n,t}$ has a unique maximum at v_n. Then, since the arcs $(A^n u(v), \lambda^n v)$, $v \in \lambda^{-n}D_2$, converge in the C^1 sense to $\{0\} \times D_2$ as $n \to \infty$, it will follow that there are numbers $0 < s_n < t_n < \lambda^{-n+1}v_o$ such that $\phi_{n,t}$ has no zero for $t < s_n$, one zero for $t = s_n$, and two zeros for $s_n < t \le t_n$.

Also, since the point $(u(v_n), v_n)$ is a fixed point of f_{n,s_n}, $(u(v_n), v_n)$ approaches (u_o, o) as $n \to \infty$. Thus, one can choose the numbers t_n such that, letting $v_{1nt} < v_{2nt}$ be the zeros of $\phi_{n,t}$ for $s_n < t \le t_n$, the points $(u(v_{1nt}), v_{1nt})$, $(u(v_{2nt}, v_{2nt})$ approach (u_o, o) as $n \to \infty$. These points are fixed points of $f_{n,t}$. The lemma will then be proved by showing that $(u(v_{2nt}), v_{2nt})$ is a sink for n large, $s_n < t \leq t_n$, and t_n close to s_n.

With suitable identifications, we have

$$\phi''_{n,t}(v) = g_{2uu}A^n u'(v)A^n u'(v) + g_{2uv}\lambda^n A^n u'(v) + g_{2u}A^n u''(v) + g_{2vu}A^n u'(v)\lambda^n + g_{2vv}\lambda^{2n}$$

and

$$u''(v) = -(I - g_{1u}A^n)^{-1}(-g_{1uu}A^n u'(v)A^n - g_{1uv}\lambda^n A^n)(I - g_{1u}A^n)^{-1}g_{1v}\lambda^n + (I - g_{1u}A^n)^{-1}(g_{1vu}A^n u'(v)\lambda^n + g_{1vv}\lambda^{2n}).$$

Since $|A^n|\lambda^n \to 0$ and $|A^n||u'(v)| \to 0$, there are constants $e_1, e_2 > 0$ such that $|u''(v)| \leq e_1\lambda^n + e_2\lambda^{2n}$.

Then one sees easily that there is a constant $c_2 > 0$ such that $\phi''_{n,t}(v) < c_2\lambda^n + g_{2vv}(A^n u(v), \lambda^n v)\lambda^{2n}$ which implies that, for large n, $\phi''_{n,t}(v)$ is negative on $\lambda^{-n}D_2$ as desired.

It will now be shown that $(u(v_{2nt}), v_{2nt})$ is a sink for $f_{n,t}$ with $t > s_n$ and near s_n. Fix $(u, v) \in D_n$. Then, the derivative $f'_{n,t}(u, v)$ as a linear map from $R^{m-1} \times R$ to $R^{m-1} \times R$ has the form

$$f'_{n,t}(u, v) = \begin{pmatrix} g_{1u}A^n & g_{1v}\lambda^n \\ g_{2u}A^n & g_{2v}\lambda^n \end{pmatrix}.$$

We compute the eigenvalues of $f'_{n,t}(u(v), v)$ for v near v_n and t near s_n.

Recall that $\phi'_{n,t}(v) = g_{2u}A^n u'(v) + g_{2v}\lambda^n - 1$, and $u'(v) = (I - g_{1u}A^n)^{-1}g_{1v}\lambda^n$. Looking at the function $\psi_2(v, x) = \phi'_{n,t}(v) - g_{2u}A^n(I - g_{1u}A^n)^{-1}g_{1v}\lambda^n + 1 + g_{2u}A^n(xI - g_{1u}A^n)^{-1}g_{1v}\lambda^n - x$, one sees that $\psi_2(v_n, 1) = 0$ and $\dfrac{\partial\psi_2}{\partial x}(v_n, 1) < 0$. Then the implicit function theorem gives a smooth function $x(v)$ for v near v_n such that $x(v_n) = 1$ and $\psi_2(v, x(v)) = 0$. Also, one calculates easily that $x'(v_n) < 0$ since $\phi''_{n,t}(v_n) < 0$. Hence, $x(v) < 1$ for $v > v_n$ and near v_n.

From the equation $\psi_2(v, x(v)) = 0$ we get $g_{2v}\lambda^n = \phi'_{n,t}(v) - g_{2u}A^n(I - g_{1u}A^n)^{-1}g_{1v}\lambda^n + 1 = -g_{2u}A^n(x(v)I - g_{1u}A^n)^{-1}g_{1v}\lambda^n + x(v)$. Thus, for v near v_n,

$$f'_{n,t}(u(v), v) = \begin{pmatrix} g_{1u}A^n & g_{1v}\lambda^n \\ g_{2u}A^n & -g_{2u}A^n(x(v)I - g_{1u}A^n)^{-1}g_{1v}\lambda^n + x(v) \end{pmatrix}.$$

Letting I_1 be the $m \times m$ identity matrix, we have

$$f'_{n,t}(u(v), v) - x(v)I_1 = \begin{pmatrix} g_{1u}A^n - x(v)I & g_{1v}\lambda^n \\ g_{2u}A^n & -g_{2u}A^n(x(v)I - g_{1u}A^n)^{-1}g_{1v}\lambda^n \end{pmatrix}$$
$$= \begin{pmatrix} g_{1u}A^n - x(v)I \\ g_{2u}A^n \end{pmatrix}(I\,(g_{1u}A^n - x(v)I)^{-1}g_{1v}\lambda^n).$$

Thus, $f'_{n,t}(u(v), v) - x(v)I_1$ may be written as a product of an $m \times (m-1)$ matrix and an $(m-1) \times m$ matrix, and, hence, has rank less than m. (Actually, it is clear that rank $f'_{n,t}(u(v), v) = m - 1$.) This implies that $x(v)$ is an eigenvalue of $f'_{n,t}(u(v), v)$.

The proof of Lemma 3 will be completed by showing that for $\varepsilon > 0$, v near v_n, and n large, $f'_{n,t}(u(v), v)$ has an invariant $(m-1)$-dimensional subspace such that all the eigenvalues of $f'_{n,t}(u(v), v)$ restricted to this subspace have absolute value less than or equal to ε.

From this and the fact that $x'(v_n) < 0$, it follows easily that $(u(v_{2nt}), v_{2nt})$ is a sink for $f'_{n,t}$ if $t > s_n$ is near s_n.

For v near v_n, $\phi'_{n,t}(v)$ is near 0, and, for n large, $g_{2v}(A^n u(v), \lambda^n v)\lambda^n$ is near 1. Also, $g_{1v} \neq 0$ since g_{2v} is near 0 and g is a diffeomorphism. Let

$$B_n = \begin{pmatrix} I & \dfrac{g_{1v}\lambda^n}{|g_{1v}\lambda^n|} \\ 0 & \dfrac{g_{2v}\lambda^n}{|g_{1v}\lambda^n|} \end{pmatrix}$$

so that

$$B_n^{-1} = \begin{pmatrix} I & \dfrac{-g_{1v}\lambda^n}{g_{2v}\lambda^n} \\ 0 & \dfrac{|g_{1v}\lambda^n|}{g_{2v}\lambda^n} \end{pmatrix}.$$

Setting $J_n = B_n^{-1} f'_{n,t} B_n$, one has

$$J_n = \begin{pmatrix} E_n & F_n \\ G_n & H_n \end{pmatrix}$$

where

$$E_n = g_{1u}A^n - \frac{g_{1v}\lambda^n}{g_{2v}\lambda^n} g_{2u}A^n, \qquad G_n = \frac{|g_{1v}\lambda^n|}{g_{2v}\lambda^n} g_{2u}A^n,$$

$$F_n = \left(g_{1u}A^n - \frac{g_{1v}\lambda^n g_{2u}A^n}{g_{2v}\lambda^n}\right)\frac{g_{1v}\lambda^n}{|g_{1v}\lambda^n|} \quad \text{and} \quad H_n = \frac{g_{2u}A^n g_{1v}\lambda^n}{g_{2v}\lambda^n} + g_{2v}\lambda^n.$$

Thus $|E_n|$, $|F_n|$, $|G_n| \to 0$, and $|H_n| \to 1$ as $n \to \infty$.

The matrices B_n, B_n^{-1}, and $f'_{n,t}$ represent linear mappings from $R^{m-1} \times R \to R^{m-1} \times R$. For the following, give $R^{m-1} \times R$ the norm $|(u, v)| = \max\{|u|, |v|\}$.

Using known techniques [3], we show that given $\varepsilon > 0$, there is a subspace $W \subset R^{m-1} \times R$ such that $J_n(W) = W$ and $|J_n| W| \leq \varepsilon$. From this it follows that each eigenvalue of $J_n| W$ has absolute value less than or equal to ε, and, hence, the same is true of each eigenvalue of $f'_{n,t}| B_n(W)$.

Assume $\frac{\varepsilon}{2} < 1$. Choose n large enough so that

(1) $|H_n^{-1}||E_n| + |H_n^{-1}||F_n| + |H_n^{-1}G_n| \leq \frac{\varepsilon}{2}$,

(2) $|H_n^{-1}||E_n| + 2|H_n^{-1}||F_n| < 1$, and

(3) $|E_n| + |F_n|\frac{\varepsilon}{2} \leq \varepsilon$.

We will find W as the graph of a linear map $P : R^{m-1} \to R$ such that $|P| \leq \frac{\varepsilon}{2}$. The condition that J_n (graph P) = graph P is equivalent to saying that P is a fixed point of the map $\Phi_n : P \mapsto H_n^{-1}PE_n + H_n^{-1}PF_nP - H_n^{-1}G_n$. Let $\mathscr{H}$ be the space of linear maps from R^{m-1} to R with norm less than or equal to $\frac{\varepsilon}{2}$.

By (1), Φ_n maps $\mathscr{H}$ into itself, and, by (2), it is a contraction. Thus the desired map P exists.

Now, if $v = (v_1, v_2) \in W$, then $|v| = \max\{|v_1|, |v_2|\} = |v_1|$ since $\frac{|v_2|}{|v_1|} \leqq \frac{\varepsilon}{2} < 1$. Also, $J_n v = (E_n v_1 + F_n v_2,\ G_n v_1 + H_n v_2) \in W$. Thus, $|J_n v| = |E_n v_1 + F_n v_2| \leq \left(|E_n| + |F_n| \frac{\varepsilon}{2}\right) \times |v_1| \leq \varepsilon |v_1| = \varepsilon |v|$ by (3). This completes the proof of Lemma 2.

Recall that a compact f-invariant set Λ is a basic set for f if it is hyperbolic, topologically transitive, the periodic points of f in Λ are dense in Λ, and Λ has a local product structure [2]. The basic set is *non-trivial* if it contains more than one orbit. In this case it must be infinite. Given a basic set Λ_f for f, there is a neighborhood $\mathscr{N}_1$ of f in $\mathrm{Diff}^r(M)$ such that any $g \in \mathscr{N}_1$ has a unique basic set Λ_g near Λ_f, and there is a homeomorphism $h : \Lambda_f \to \Lambda_g$ such that $gh = hf$. For $x \in \Lambda_f$ and $g \in \mathscr{N}_1$, we denote $h(x)$ by x_g. $W^u(\Lambda_g)$ and $W^s(\Lambda_g)$ will denote the stable and unstable manifolds of Λ_g which are defined by $W^u(\Lambda_g) = \{y \in M : \alpha(y, g) \subset \Lambda_g\}$, $W^s(\Lambda_g) = \{y \in M : \omega(y, g) \subset \Lambda_g\}$. Here $\alpha(y, g)$ and $\omega(y, g)$ are, respectively, the α-limit set and ω-limit sets of y by g. Also, for $x \in \Lambda_g$, $W^u(x, g) = \{y \in M : d(g^{-n}y, g^{-n}x) \to 0$ as $n \to \infty\}$, $W^s(x, g) = \{y \in M : d(g^n y, g^n x) \to 0$ as $n \to \infty\}$ where d is some topological metric on M. It is known that each $W^u(x, g)$, $W^s(x, g)$ is an immersed copy of a Euclidean space and $W^u(\Lambda_g) = \bigcup_{x \in \Lambda_g} W^u(x, g)$, $W^s(\Lambda_g) = \bigcup_{x \in \Lambda_g} W^s(x, f)$ [2], [3]. A point of tangency of $W^s(\Lambda_g)$ and $W^u(\Lambda_g)$ is defined as above. If $D \subset M$, Cl D will denote its closure in M.

PROPOSITION 3. *Suppose Λ_f is a non-trivial basic set for $f \in \mathrm{Diff}^r(M)$ which contains a periodic point p such that* $\dim W^s(p) = \dim M - 1$. *Let $\mathscr{N}_1$ be the small neighborhood of f described above such that each $g \in \mathscr{N}_1$ has a unique basic set Λ_g near Λ_f. Assume there is a neighborhood $\mathscr{N} \subset \mathscr{N}_1$ of f in* $\mathrm{Diff}^r(M)$ *such that if $g \in \mathscr{N}$, then $W^u(\Lambda_g)$ and $W^s(\Lambda_g)$ have a point of tangency and $\mu(p_g) \cdot \lambda(p_g) < 1$. Then there is a residual subset $\mathscr{B}$ of $\mathscr{N}$ such that for $g \in \mathscr{B}$, each point in* Cl $(W^u(\Lambda_g) \cap W^s(\Lambda_g))$ *is a limit of a sequence of sinks.*

Proof. For $g \in \mathscr{N}$, let H_{p_g} denote the homoclinic class of p_g; that is, H_{p_g} consists of all periodic points q of g such that $W^u(o(q))$ has a point of transversal intersection with $W^s(o(p))$ and $W^u(o(p))$ has a point of transversal intersection with $W^s(o(q))$. Then H_{p_g} contains the periodic points of Λ_g and Cl H_{p_g} is a closed invariant topologically transitive set which equals the closure of the transversal homoclinic points of $o(p_g)$ [6]. Let $KS(\mathscr{N})$ denote the residual set of Kupka–Smale diffeomorphisms in $\mathscr{N}$. Using methods similar to those in [8], one sees that the mapping $g \mapsto$ Cl H_{p_g} from $KS(\mathscr{N})$ into the compact subsets of M is lower semi-continuous. Thus it is continuous on a residual subset $\mathscr{B}_1 \subset KS(\mathscr{N})$. Also, if $g \in \mathscr{B}_1$, Cl $H_{p_g} = $ Cl$(W^u(\Lambda_g) \cap W^s(\Lambda_g))$. Let $S(g)$ denote the set of sinks of g for $g \in \mathscr{N}$. Similar reasoning shows that Cl $S(g)$ is a continuous function of g for g in some residual subset

$\mathscr{B}_2 \subset KS(\mathscr{N})$. The proposition will be proved with $\mathscr{B} = \mathscr{B}_1 \cap \mathscr{B}_2$ once we show that for any $g \in \mathscr{N}$ and any $q \in H_{p_g}$, there is an arbitrarily small approximation g' of g which has a sink arbitrarily close to q. This is accomplished as follows. Fix a tangency x_1 of $W^u(\Lambda_g)$ and $W^s(\Lambda_g)$. We make a series of perturbations of g in the complement of a neighborhood of the orbit of q. First, we perturb g to g_1 so that there is a periodic point $q_1 \in \Lambda_{g_1}$ such that $W^u(o(q_1), g_1)$ has a tangency x_2 with $W^s(o(q_1), g_1)$ near x_1. Next, since q is homoclinically related to q_1, we may perturb g_1 to g_2 so that $W^u(o(q), g_2)$ has a tangency x_3 with $W^s(o(q), g_2)$ near x_2. Then, we change g_2 to g_3 so that $W^u(o(p_{g_3}), g_3)$ has a tangency x_4 with $W^s(o(p_{g_3}), g_3)$ near x_3. Finally, we move g_3 to g' to introduce a sink near x_4 using Proposition 1. Since the sink may be gotten arbitrarily close to a certain disk in $W^s(o(q), g')$ of fixed diameter (depending only on the position of x_3), its orbit under g' will get close to $o(q)$.

To prove the theorem, it remains to construct open sets in $\mathrm{Diff}^r(M)$, $r \geq 2$, satisfying the hypotheses of Proposition 3.

Let D^m be a closed m-disk about 0 in R^m with $m = \dim M$. If $m \geq 2$, we write $D^m = D^2 \times D^{m-2}$. Using the examples in [5], for $r \geq 2$, one may construct a C^r diffeomorphism f_1 mapping D^2 into its interior with the following properties.

(1) There is a basic set $\Lambda_{f_1} \subset \operatorname{int} D^2$ such that $f_1|_{\Lambda_{f_1}}$ is topologically conjugate to a shift automorphism on two symbols.

(2) There is a C^2 neighborhood $\mathscr{N}(f_1)$ such that for each g in $\mathscr{N}(f_1)$, $W^u(\Lambda_g)$ has a tangency with $W^s(\Lambda_g)$ and there is a fixed point p_g such that $\det(T_{p_g} g) < 1$.

For $m > 2$, let f_2 be a contracting diffeomorphism of D^{m-2} into its interior with fixed point at 0. Define $f\colon D^m \to D^m$ by $f(x) = f_1(x)$ if $m = 2$, $f(x, y) = (f_1(x), f_2(y))$ if $m > 2$. If the Lipschitz constant of f_2 is taken small enough, there is a C^r neighborhood $\mathscr{N}(f)$ such that any g in $\mathscr{N}(f)$ will have an invariant 2-disk C^2 near $D^2 \times \{0\}$ [4], [9]. Thus f has a neighborhood in which hypothesis similar to those of Proposition 3 are satisfied. Now let F be a Morse–Smale diffeomorphism on M with a contracting fixed point p_1. For instance, we may choose F to be the time-one map of a Morse–Smale gradient flow on M. Fixing a small disk $D_1 \subset M$ such that $p_1 \in \operatorname{int} D_1$ and $F(D_1) \subset \operatorname{int} D_1$, we may find a diffeomorphism $\phi : D_1 \to D^m$ and a diffeomorphism $f\colon D^m \to D^m$ as above so that the mapping

$$F_1(x) = \begin{cases} F(x) & x \notin D_1 \\ \phi^{-1} f \phi(x) & x \in D_1 \end{cases}$$

defines a diffeomorphism of M. Then there are small C^r neighborhoods of F_1 satisfying the hypotheses of Proposition 3.

Remarks. (1) If T^3 is the 3-torus, it is easily seen that examples of residual subsets of open sets in $\mathrm{Diff}^1(T^3)$ with infinitely many sinks may be gotten using Propositions 1 and 3 and the examples of C. Simon [11].

(2) In [14], R. Thom suggested that for most dynamical systems on a manifold, the orbits of almost all points would tend to a finite set of structurally stable attractors. This was shown to be false in 1968 when M. Shub exhibited an open subset of non-structurally stable diffeomorphisms on the 4-torus T^4 each having all of T^4 as an attractor. The

examples here go in the other direction by showing that one can have infinitely many structurally stable attractors. But the question of whether almost all orbits tend to attractors remains unanswered. More precisely, for $f \in \mathrm{Diff}^r(M)$, say that a closed f-invariant set Λ is an attractor for f if $f|_\Lambda$ is topologically transitive and there is a compact neighborhood U of Λ in M such that $f(U) \subset U$ and $\bigcap_{n \geq 0} f^n(U) = \Lambda$. The basin $B(\Lambda)$ of Λ is $\bigcup_{n \geq 0} f^{-n}(U) = \{y \in M : w(y,f) \subset \Lambda\}$. Then is it true for most f that the union of the basins of the attractors of f is dense in M?

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