ON CODIMENSION ONE ANOSOV DIFFEOMORPHISMS.

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1. Throughout this paper, we let $M$ be a compact connected $C^\infty$ manifold without boundary. A $C^r$ diffeomorphism $f: M \to M$, $1 \leq r \leq \infty$, is called an Anosov diffeomorphism if there is a continuous splitting of the tangent bundle $TM = E^s \oplus E^u$, a riemannian metric $\| \cdot \|$ on $TM$, and constants $c, c' > 0$, $0 < \lambda < 1$, such that

(i) $T_xf(E^s) = E_{f(x)}^s$, $T_xf(E^u) = E_{f(x)}^u$

(ii) For $v \in E^s$, $\| T^f_n(v) \| \leq c \lambda^n \| v \|$, and for $v \in E^u$, $\| T^f_n(v) \| \leq c' \lambda^n \| v \|$

where $T_xf$ refers to the derivative of $f$ at the point $x$. It can be shown that condition (ii) is independent of the riemannian metric on $TM$.

In the past few years, Anosov diffeomorphisms have been studied to a great extent. We refer the reader to [2] and [8] for background information, general references, and terms which are not defined here.

An Anosov diffeomorphism is said to be of codimension 1 if either $\dim E^u = 1$ or $\dim E^s = 1$ where $E^u$ and $E^s$ are as in the above definition.

Let $A$ be an $n \times n$ matrix with integer entries such that $\det A = \pm 1$ and the eigenvalues of $A$ are off the unit circle. Then $A$ induces a diffeomorphism $\hat{A}$ of the $n$-dimensional torus $T^n$. The map $\hat{A}$ is called a toral automorphism. Two maps $f: M \to M$, $g: N \to N$ are called $\pi_1$-conjugate if there is an isomorphism $\phi: \pi_1(M) \to \pi_1(N)$ such that $\phi f_* = g_* \phi$ where $\pi_1(M)$, $\pi_1(N)$ are the fundamental groups and $f_*$, $g_*$ are the induced maps. The maps $f$, $g$ are called topologically conjugate if there is a homeomorphism $h: M \to N$ such that $hf = gh$.

For a diffeomorphism $f: M \to M$, we let $NW(f)$ denote the set of non-wandering points of $f$ which is defined by $NW(f) = \{ x \in M: \text{for any neighborhood } U \text{ of } x, \text{there is a positive integer } n(U) \text{ such that } f^n(U) \cap U \neq \emptyset \}$. In [2], Franks proves the following theorem.

(1.1) Theorem. Let $f: M \to M$ be a codimension 1 Anosov diffeomorphism such that $NW(f) = M$. Then $f$ is topologically conjugate to a toral
automorphism. Any two codimension 1 Anosov diffeomorphisms \( f: M \to M, \quad g: N \to N \), such that \( NW(f) = M, \ NW(g) = N \) are topologically conjugate if and only if they are \( \pi_1 \)-conjugate.

Under the assumption that the stable and unstable foliations were of class \( C^2 \), partial results in the direction of Theorem (1.1) were obtained independently by H. Rosenberg.

In this paper, we prove

(1.2) Theorem. Let \( f: M \to M \) be any codimension 1 Anosov diffeomorphism. Then \( NW(f) = M \).

Theorem (1.2) was proved earlier by Smale in the case where \( \dim M = 2 \) (see [2, (7.2)]).

In view of (1.1) and (1.2), we obtain

(1.3) Corollary. Any codimension 1 Anosov diffeomorphism is topologically conjugate to a toral automorphism. Any two codimension one Anosov diffeomorphisms are topologically conjugate if and only if they are \( \pi_1 \)-conjugate.

Applying well-known facts we obtain

(1.4) Corollary. If \( f: M \to M \) is any codimension 1 Anosov diffeomorphism, then

1. the periodic points of \( f \) are dense in \( M \) [8], and
2. \( f \) has an invariant Lebesgue measure and \( f \) is ergodic ([7] or [1]).

I wish to thank J. Franks, M. Hirsch, Z. Nitecki, and C. Pugh for helpful comments.

2. In this section, we prove Theorem (1.2). It is well-known that an Anosov diffeomorphism satisfies Smale's Axioms A and B [8; (6.1) and (6.4)], and we wish to make use of this fact and several of its consequences. Thus we assume familiarity with §§I.3, I.6, and I.7 of [8].

To begin the proof of (1.2), we observe that since there are no Anosov diffeomorphisms of \( S^1 \), we may assume \( \dim M \geq 2 \). Since \( NW(f) = NW(f^{-1}) \), we may assume \( \dim E^u = 1 \) and \( \dim E^s = n - 1 \) where \( \dim M = n \geq 2 \). Further, by taking two two to one coverings if necessary, we may assume \( M \) is orientable, \( TM \) is oriented, and the line bundle \( E^u \) is oriented. Thus the unstable manifolds of \( f \) are oriented arcs. By replacing \( f \) by \( f^2 \) or \( f^4 \), assume \( Tf \) preserves the orientations of \( E^u \) and \( TM \).
Let $NW(f) = \Omega_1 \cup \cdots \cup \Omega_n$ be the spectral decomposition of $NW(f)$. We let $W^s(x)(W^u(x))$ denote the stable (unstable) manifold of $f$ at the point $x \in M$. For a subset $\Lambda$ of $M$ let

$$W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x), \text{ and let } W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x).$$

A source is a basic set $\Omega_i$ such that $W^s(\Omega_i) = \Omega_i$. A sink is a basic set $\Omega_i$ such that $W^u(\Omega_i) = \Omega_i$. It is easy to see that if $\Omega_i$ is a source, then $W^u(\Omega_i)$ is an open subset of $M$, and, if $\Omega_i$ is a sink, then $W^s(\Omega_i)$ is an open subset of $M$. If we show some source is a sink, then it follows that $NW(f) = M$. For, if $\Omega_i$ is a source and a sink, then $W^s(\Omega_i) = \Omega_i = W^u(\Omega_i)$ is an open and closed subset of $M$. Thus $\Omega_i = M$. We proceed to show, in fact, that any source must also be a sink.

Let $\Omega_i$ be a source. We will show

(2.1) \hspace{1cm} W^u(\Omega_i) = \Omega_i.

Given $y_1, y_2 \in W^u(x)$, we say $y_1 < y_2$ if $y_1 \neq y_2$ and the subarc of $W^u(x)$ from $y_1$ to $y_2$ has the same orientation as $W^u(x)$. For $y_1 < y_2$, let $[y_1, y_2]$ denote the compact subarc of $W^u(x)$ from $y_1$ to $y_2$. Let $l[y_1, y_2]$ denote the length of $[y_1, y_2]$. Given $0 < \alpha < \infty$, and $y \in W^u(x)$, define

$$B_+(y_1) = \{y \in W^u(x): y_1 < y \text{ and } l[y_1, y] < \alpha\},$$

and

$$B^+(y_1) = \{y \in W^u(x): y_1 < y\}.$$

Similarly, define

$$B_-(y_1) = \{y \in W^u(x): y < y_1 \text{ and } l[y, y_1] < \alpha\},$$

and

$$B^-(y_1) = \{y \in W^u(x): y < y_1\}.$$

(2.2) \hspace{1cm} \textbf{Lemma.} (1) For each $x \in \Omega_i$, $B^+(x) \cap \Omega_i \neq \emptyset$

(2) For each $x \in \Omega_i$, $B^-(x) \cap \Omega_i \neq \emptyset$.

Before proving (2.2), we show how (2.1) follows from (2.2). Comments by M. Hirsch and J. Franks were useful in simplifying the original proof.

Let $x \in \Omega_i$. We prove that $W^u(x) \subset \Omega_i$. Choose an increasing sequence of integers $n_1 < n_2 < n_3 \cdots$ and a point $y \in M$ such that $f^{n_i}(x) \to y$ as $i \to \infty$. 
Since $\Omega_1$ is closed and invariant, $y \in \Omega_1$. By (2.2), we may choose $\alpha > 0$ such that $B_{\alpha}^+(y) \cap \Omega_1 \neq \emptyset$. But then there is an integer $N > 0$ such that for $i \geq N$, $B_{\alpha}^+(f^{ni}(x)) \cap \Omega_1 \neq \emptyset$. Since $f^{-ni}$ contracts the unstable manifolds, we see that $x$ is an accumulation point of $B^+(x) \cap \Omega_1$. That is, $x$ is an accumulation point of $W^u(x) \cap \Omega_1$ from both sides in $W^u(x)$. The same argument shows that any point $v \in W^u(x) \cap \Omega_1$ is an accumulation point of $W^u(x) \cap \Omega_1$ from both sides in $W^u(x)$. Now, since $W^u(x) \cap \Omega_1$ is closed in $W^u(x)$, if $y$ were a point of $W^u(x) - \Omega_1$, then one could find an arc $[x_1, x_2]$ in $W^u(x)$ such that $y \in [x_1, x_2]$, $(x_1, x_2) \cap \Omega_1 = \emptyset$, and $x_1, x_2 \in \Omega_1$. But $\Omega_1$ must accumulate on $x_1$ and $x_2$ from both sides in $W^u(x)$. This would contradict the fact that $(x_1, x_2) \cap \Omega_1 = \emptyset$. Thus no such $y$ exists, so $W^u(x) \subseteq \Omega_1$ and (2.1) is proved.

Now we recall some definitions and results we will need for the proof of (2.2).

For each $x$, we let $W^e_u(x)$ be the intrinsic closed $\epsilon$-ball about $x$ in $W^u(x)$. This is obtained as follows. The Riemannian metric on $TM$ induces one on $TW^u(x)$. This in turn induces a topological metric on $W^u(x)$ which makes $W^u(x)$ homeomorphic to the real line. Then $W^e_u(x)$ is to be the ball of radius $\epsilon$ about $x$ in this topological metric. Similarly, let $W^e_s(x)$ be the intrinsic closed $\epsilon$-ball about $x$ in $W^s(x)$. Thus $W^e_u(x)$ is diffeomorphic to a closed $(n-1)$-disk where $\dim M = n$. For any subset $\Lambda$ of $M$, and $y \in \Lambda$, let $W^s(y, \Lambda)$ be the connected component of $W^s(y) \cap \Lambda$ which contains $y$. Similarly, let $W^u(y, \Lambda)$ be the connected component of $W^u(y) \cap \Lambda$ which contains $y$. The following fact is an easy consequence of Theorem (7.3) of [8]. This theorem is proved in [4].

(2.3) There is an $\epsilon > 0$ such that for each $x \in M$, there is a neighborhood $V^e(x)$ of $x$ satisfying

(2.3.1) $V^e(x)$ is homeomorphic to $W^e_u(x) \times W^e_s(x)$.

(2.3.2) for $y_1, y_2 \in V^e(x)$, if $W^u(y_1, V^e(x)) \cap W^u(y_2, V^e(x)) \neq \emptyset$, then $W^u(y_1, V^e(x)) = W^u(y_2, V^e(x))$; similarly, if $W^s(y_1, V^e(x)) \cap W^s(y_2, V^e(x)) \neq \emptyset$, then $W^s(y_1, V^e(x)) = W^s(y_2, V^e(x))$.

(2.3.3) for $y_1, y_2 \in V^e(x)$, $W^u(y_1, V^e(x)) \cap W^s(y_2, V^e(x))$ is a single point.

The interior of $V^e(x)$ is usually referred to as a local product neighborhood, and (2.3) is referred to as the local product theorem.
For our purposes it is convenient to use the notion of a product set which is a sort of elongated closed local product neighborhood. Let \( x \in M \), and \( \epsilon > 0 \). A stable product set relative to \( W^s_\epsilon(x) \) is a set, denoted by \( N \) or \( N(W^s_\epsilon(x)) \), satisfying the following conditions.

\[
(2.4) \quad N = \bigcup \{ W^u(y, N) : y \in W^s_\epsilon(x) \}.
\]

(2.5) For \( y_1, y_2 \in N \), either \( W^u(y_1, N) \cap W^u(y_2, N) = \emptyset \) or \( W^u(y_1, N) = W^u(y_2, N) \); similarly, either \( W^s(y_1, N) \cap W^s(y_2, N) = \emptyset \) or \( W^s(y_1, N) = W^s(y_2, N) \).

(2.6) For \( y_1 \in N \), \( W^s(y_1, N) \) \( (W^u(y_1)) \) is homeomorphic to a closed ball in \( W^s(y_1)(W^u(y_1)) \).

(2.7) There exists \( \epsilon_1 > 0 \) such that \( W^u_\epsilon_1(y_1) \subseteq W^u(y_1, N) \) for all \( y_1 \in W^s(y_1, N) \).

(2.8) If \( y_1, y_2 \in N \), then \( W^u(y_1, N) \cap W^s(y_2, N) \) is a single point.

For \( K \subseteq W^s_\epsilon(x) \), a stable product set \( N \) or \( N(K) \) relative to \( K \) is defined to be \( \bigcup_{y \in K} W^u(y, N(W^s_\epsilon(x))) \) where \( N(W^s_\epsilon(x)) \) is some stable product set relative to \( W^s_\epsilon(x) \).

Similarly, for \( x \in M \), \( \epsilon > 0 \), we may define an unstable product set relative to \( W^u_\epsilon(x) \) or \( K \) where \( K \subseteq W^u_\epsilon(x) \).

2.9) Remark. 1. Using (2.3), and the compactness of \( W^u_\epsilon(x) \) and \( W^s_\epsilon(x) \), it is easy to check that for any \( x \in M \), and any \( \epsilon > 0 \), product sets relative to \( W^u_\epsilon(x) \) and \( W^s_\epsilon(x) \) exist.

2. If \( N \) is a stable product set relative to \( W^s_\epsilon(x) \), \( K \subseteq W^s_\epsilon(x) \), and \( y \in N \), then \( N(K) \cap W^s(y, N) \) is homeomorphic to \( K \). A similar statement holds for unstable product sets.

Now we prove Lemma (2.2). We prove (2.2.1) since the same methods yield (2.2.2). Let \( A = \{ x \in \Omega : B^s(x) \cap \Omega \neq \emptyset \} \). We proceed to show \( A = \Omega \).

Step 1. \( A \) is an \( f \)-invariant subset of \( \Omega \), i.e. \( f(A) = A \).

Proof. This follows easily from the fact that \( f \) preserves the orientation of \( E^u \) and the facts that for any \( x \in M \), \( f(W^s(x)) = W^s(f(x)) \) and \( f(W^u(x)) = W^u(f(x)) \).

Step 2. If \( x \in \Omega \) is not periodic, then \( x \in A \).
Proof. Choose $\epsilon > 0$ as in (2.3). Since $x$ is not periodic, the orbit of $x$, $o(x)$, is infinite. We claim

$$(2.10) \text{there is an infinite sequence } \{x_i\}_{i \geq 1} \subset o(x) \text{ such that if } x_i \neq x_j \text{ then } W^s(\epsilon x_i) \cap W^s(\epsilon x_j) = \emptyset.$$  

If for each pair of integers $n_1 > n_2$, $W^s(f^{n_1}(x)) \cap W^s(f^{n_2}(x)) = \emptyset$, (2.10) is true. If there are integers $n_1 > n_2$ such that

$$W^s(f^{n_1}(x)) \cap W^s(f^{n_2}(x)) \neq \emptyset,$$

then $W^s(f^{n_1}(x)) = W^s(f^{n_2}(x))$, and so $f^{n_1-n_2}(W^s(x)) = W^s(x)$. Since $f^{-(n_1-n_2)}$ expands $W^s(x)$, there is an increasing sequence of integers $m_1, m_2, \cdots$ such that if $i \neq j$, the distance in $W^s(x)$ between $f^{-m_i(n_1-n_2)}(x)$ and $f^{-m_j(n_1-n_2)}(x)$ is larger than $2\epsilon$. Thus $W^s(f^{-m_i(n_1-n_2)}(x)) \cap W^s(f^{-m_j(n_1-n_2)}(x)) = \emptyset$, and we take $x_i = f^{-m_i(n_1-n_2)}(x)$ for (2.10).

Now let $y$ be a limit point of $\{x_i\}$, and choose a subsequence $\{x_{i_j}\}$ such that $x_{i_j} \to y$ as $j \to \infty$. Let $V_\epsilon(y)$ be a neighborhood of $y$, as described in (2.3). By (2.3) and the fact that the stable manifolds are $(n-1)$-dimensional, it is clear that if $x_{i_j} \neq x_{i_k}$ and $x_{i_j}, x_{i_k} \in V_\epsilon(y)$, then either $B^+(x_{i_j}) \cap W^s(x_{i_k}) \neq \emptyset$ or $B^+(x_{i_k}) \cap W^s(x_{i_j}) \neq \emptyset$, so either $x_{i_j} \in A$ or $x_{i_k} \in A$. In either case, since $\{x_{i_j}, x_{i_k}\} \subset o(x)$ and $A$ is $f$-invariant (step 1), we obtain that $x \in A$.

It remains to show if $p \in \Omega_1$ is periodic, then $p \in A$. The arguments for this fact for the case where $\dim M = 2$ are different from those for the case where $\dim M > 2$.

First assume $\dim M = 2$, and $p \in \Omega_1$ is periodic of period $m$, i.e., $f^m(p) = p$. Since $M$ has a nowhere vanishing line field and is orientable, we may assume $M$ is the two torus $T^2$. Suppose $p \notin A$. Then $B^+(p) \cap W^s(p) = \emptyset$. Since $\Omega_1$ is a source, $W^s(p) \subset \Omega_1 \subset NW(f) = NW(f^m)$. Thus there is a source $\Omega_1'$ for $f^m$ such that $p \in \Omega_1'$ and $p$ is a fixed point of $f^m$. Since $W^s(p)$ is dense in $\Omega_1'$, it recovers on itself.

Let $V_\epsilon(p)$ be as in (2.3). Since the unstable bundle $E^u$ on $M$ is oriented, so is the stable bundle $E^s$. Let $q$ be a point of $W^s(p) \cap B^-(\epsilon p)$, and consider the loop $\gamma$ consisting of the arc in $W^s(p)$ from $p$ to $q$ followed by the arc in $B^-(\epsilon p)$ from $q$ to $p$. If $\gamma$ is null-homotopic, then, since it is a topological circle embedded in a topological torus, it bounds a topological 2-disk. But then if we follow $W^s(p)$ in the direction from $p$ to $q$ and go beyond $q$ to the first point $q_1$ where $W^s(p)$ meets $B^-(\epsilon p)$ again, it is easy to see that the arc from $p$ to $q$ must cross $B^-(\epsilon p)$ in the direction opposite to that of the arc from $q$ to $q_1$. This contradicts the fact that the stable
bundle \( E^s \) is oriented. Thus \( \gamma \) is not null-homotopic. Again since \( M \) is a torus, \( M^2 - \gamma \) is a topological cylinder which has an unstable foliation induced by \( \{ W^u(x) \}_{x \in M} \). Now it is easy to see how to define \( \omega \)-limit sets and set up a Poincare-Bendixon theory (see [3] or [6]) for this continuous foliation on \( M^2 - \gamma \). (Alternatively, using the structurally stability of \( f \), one could, by approximating, assume \( f \) is \( C^1 \) and use theorem (6.5) of [4] to get \( E^u \) is \( C^1 \) and apply the standard Poincare-Bendixon theory.) Since closure \( (B^+(p) - B^+(p)) \subset M^2 - \gamma \), the \( \omega \)-limit set of the leaf \( B^+(p) \) is a non-empty subset of \( M^2 - \gamma \). Since the foliation \( \{ W^u(x) \} \) has no singularities, Poincare-Bendixon theory says the \( \omega \)-limit set of \( B^+(p) \) must be a circle in \( M^2 - \gamma \). But this is obviously impossible since all the unstable manifolds of \( f \) are injectively immersed cells. Thus the assumption that \( p \notin A \) leads to a contradiction. This completes the proof of (2.2.1) for \( \dim M = 2 \).

For the remainder of the proof of (2.2.1), we assume \( \dim M \geq 3 \), and, as before, \( p \in \Omega_1 \) is periodic.

**Step 3.** By step 2, if \( x \in W^s(p) - \{ p \} \), then \( x \in A \). For \( x \in W^s(p) - \{ p \} \), define \( \tilde{\phi}(x) = \inf \{ l[x,y] : y \in B^+(x) \cap \Omega_1 \} \). If there is an \( x \) such that \( \tilde{\phi}(x) = 0 \), then \( p \in A \). Hence we may assume \( \tilde{\phi}(x) > 0 \) for \( x \in W^s(p) - \{ p \} \).

**Proof.** This follows immediately from the continuous dependence of the stable manifolds on compact sets and the fact that if \( x, y \in \Omega_1 \), then \( W^u(x) \cap W^s(y) \subset \Omega_1 \).

**Step 4.** For \( x \in W^s(p) - \{ p \} \), let \( \phi(x) \) be the point in \( B^+(x) \) such that \( l[x,\phi(x)] = \tilde{\phi}(x) \). Then there is a \( z_0 \in M \) such that \( \phi(x) \in W^s(z_0) \) for all \( x \in W^s(p) - \{ p \} \); i.e., all the \( \phi(x) \) lie in the same stable manifold as \( x \) varies in \( W^s(p) - \{ p \} \).

**Proof.** For any \( z \in M \), let \( U_z = \{ x \in W^s(p) - \{ p \} : \phi(x) \in W^s(z) \} \). It is clear that \( U_z \) is open in \( W^s(p) - \{ p \} \) for all \( z \). Clearly, if \( W^s(z_1) \neq W^s(z_2) \), then \( U_{z_1} \cap U_{z_2} = \emptyset \). Now since \( \dim W^s(p) \geq 2 \), \( W^s(p) - \{ p \} \) is connected. Hence, if \( z_0 \) is such that \( U_{z_0} \neq \emptyset \), then \( U_{z_0} = W^s(p) - \{ p \} \).

**Step 5.** The map \( \phi : W^s(p) - \{ p \} \to W^s(z_0) \) is continuous and injective.

**Proof.** Let \( x \in W^s(p) - \{ p \} \), and let \( \gamma \) be the arc \([x,\phi(x)]\). Let \( \epsilon > 0 \). There is an unstable product set \( N \) relative to \( \gamma \) such that \( W^s(\phi(x),N) \subset W^s(\phi(x)) \) and \( p \notin W^s(x,N) \). Then if \( y \in W^s(z_0,N), \phi(y) \in W^s(\phi(x),N) \subset W^s(\phi(x)) \). So \( \phi \) is continuous. Injectivity follows immediately from the definition of \( \phi \).
Thus if \( K \subseteq W^s(p) - \{p\} \) is compact, \( \phi \mid_K \) is a homeomorphism. Let \( E_0 \) be a closed \((n-1)\)-ball in \( W^s(p) \) containing \( p \) in its interior relative to \( W^s(p) \). Let \( \Sigma_0 \) be the boundary of \( E_0 \) in \( W^s(p) \) so that \( \Sigma_0 \) is an \((n-2)\)-sphere in \( W^s(p) - \{p\} \). Let \( H = \bigcup_{a \in \Sigma_0} [x, \phi(x)] \). Then, clearly, \( H \) is homeomorphic to \( \Sigma_0 \times I \) where \( I \) is the unit interval.

Step 6. For \( x \in \Sigma_0 \), let \( \gamma_x = [x, \phi(x)] \). For \( y \in \gamma_x \), let \( \Sigma_y \) be the path component of \( W^s(y) \cap H \) which contains \( y \). Then

\[(2.11) \quad \Sigma_y \text{ is homeomorphic to an } (n-2)\text{-sphere} \text{; hence, by the Jordan-Brouwer separation theorem, \cite[9, p. 198]{9}, } \Sigma_y \text{ separates } W^s(y).\]

Proof. We first assert

\[(2.12) \quad \text{For each } x, x_1 \in \Sigma_0, \text{ and each } y \in \gamma_x, \Sigma_y \cap \gamma_{x_1} \text{ is exactly one point}.\]

If \( (2.12) \) is true, fix \( x \) and define \( \phi_y : \Sigma_0 \to \Sigma_y \) by \( \{\phi_y(x)\} = \Sigma_y \cap \gamma_{x_1} \) for \( x_1 \in \Sigma_0 \). Then it is easy to check that \( \phi_y \) is injective, surjective, and continuous, so it is a homeomorphism. Thus we need to prove \( (2.12) \).

We first prove

\[(2.13) \quad \text{if } y \in \gamma_x, \text{ then } \Sigma_y \cap \gamma_{x_1} \neq \emptyset \text{ for all } x_1 \in \Sigma_0.\]

Fix \( y \in \gamma_x \). Let \( \Sigma_0' = \{z \in \Sigma_0 : \Sigma_y \cap \gamma_z \neq \emptyset\} \). By the local product theorem \( (2.3) \), \( \Sigma_0' \) is open in \( \Sigma_0 \). We show \( \Sigma_0 - \Sigma_0' \) is open in \( \Sigma_0 \).

Let \( z \in \Sigma_0 - \Sigma_0' \). Then \( \Sigma_y \cap \gamma_z = \emptyset \), so there is an unstable product set \( N \) relative to \( \gamma_z \) such that \( N \cap \Sigma_y = \emptyset \). Now, by the version of \( (2.7) \) for the unstable product set \( N \), \( N \cap \Sigma_0 \) is a neighborhood of \( z \) in \( \Sigma_0 \). But \( N \cap \Sigma_0 \subseteq \Sigma_0 - \Sigma_0' \), so \( \Sigma_0 - \Sigma_0' \) is open in \( \Sigma_0 \). Since \( \dim \Sigma_0 \geq 1 \), and \( \Sigma_0' \neq \emptyset \), \( \Sigma_0' = \Sigma_0 \). Thus \( (2.13) \) is proved.

Now for fixed \( x \in \Sigma_0 \), let \( D_x = \{y \in \gamma_x : \Sigma_y \cap \gamma_{x_1} \text{ is one point for all } x_1 \in \Sigma_0\} \).
We show $D_x$ and $\gamma_x = D_x$ are open in $\gamma_x$. By taking a stable product set relative to $E_0$, we see that $D_x \neq \emptyset$, so $D_x = \gamma_x$ which proves (2.13).

$D_x$ open: Let $y \in D_x$. Then the map $\phi_y : \Sigma_0 \to \Sigma_y$ defined above is injective, surjective, and continuous, so it is a homeomorphism. Let $N_y$ be a stable product set relative to $\Sigma_y$. Then $W^u(y, N_y) \cap \gamma_x$ is a neighborhood of $y$ in $\gamma_x$ which is contained in $D_x$.

$\gamma_x = D_x$ open: Let $y \in \gamma_x = D_x$. By (2.13), there exists an $x_1 \in \Sigma_0$ such that $\Sigma_y \cap \gamma_{x_1}$ has at least two points. Then using a stable product set relative to an arc in $\Sigma_y$ which connects $y$ and two points of $\Sigma_y \cap \gamma_{x_1}$, we see that $y$ is an interior point of $\gamma_x = D_x$.

Thus (2.12), and hence (2.11) is proved.

The next step uses an argument similar to one frequently used by Haefliger.

**Step 7.** Fix $x \in \Sigma_0$. It is clear that the sets $\Sigma_y$ vary continuously with $y \in \gamma_x$. That is, if $y_1$ is close to $y_2$, and $\{y_1, y_2\} \subset \gamma_x$, then there is a homeomorphism $\Psi : \Sigma_{y_1} \to \Sigma_{y_2}$ such that $\Psi$ is $C^0$ close to the inclusion $i_{y_2} : \Sigma_{y_2} \to M$. Now each $\Sigma_y$, being a homeomorph of an $(n-2)$-sphere in $W^s(y)$, is the boundary of a bounded open path connected set $V_y$ in $W^s(y)$. One can show the $V_y$ vary continuously with $y$, but we do not need this. Let $\gamma = \gamma_x$. We claim

\begin{equation}
(2.14) \quad B^+(p) \cap V_y \neq \emptyset \quad \text{for all} \quad y \in \gamma.
\end{equation}

Notice that (2.14) implies that $B^+(p) \cap V_{\phi(y)} \neq \emptyset$. But since $\phi(x) \in \Omega_1$, $V_{\phi(y)} \subset \Omega_1$, so $B^+(p) \cap \Omega_1 \neq \emptyset$ which means that $p \in A$. Thus to complete the proof of (2.2.1) we need only prove (2.14).

**Proof of (2.14).** Let $D = \{y \in \gamma : B^+(p) \cap V_y \neq \emptyset\}$. Clearly, $D$ is open in $\gamma$. Since $D$ contains a neighborhood of $x$ in $\gamma$, we are done if we prove $\gamma - D$ is open in $\gamma$, for then $D = \gamma$.

Let $y \in \gamma - D$. Take a stable product set $N_y$ relative to $V_y$ such that $B^+(p) \cap N_y = \emptyset$. By (2.7), continuous dependence of $\Sigma_y$ on $y$, and the definition of $V_y$, we see that if $y_1$ is close to $y$, then $V_{y_1} \subset N_y$. Thus $y$ is an interior point of $\gamma - D$.
REFERENCES.


