Bifurcations of Morse–Smale Dynamical Systems†

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1 Introduction

Our primary purpose here is to describe a phenomenon which occurs in the bifurcation theory of one-parameter families of diffeomorphisms. This is the appearance of infinitely many different topological conjugacy classes of structurally stable diffeomorphisms, each class containing a diffeomorphism with an infinite nonwandering set, in every neighborhood of certain diffeomorphisms in the boundary of the Morse–Smale diffeomorphisms. This phenomenon occurs quite frequently in the following sense: Any Morse–Smale diffeomorphism (on a manifold of dimension greater than one) can be moved through a one-parameter family which first exhibits some simple phase portrait changes to a new Morse–Smale diffeomorphism, and then has the above behavior near the first subsequent unstable diffeomorphism it reaches. The natural place where the phenomenon appears is in the construction of a cycle, and we conjecture that it occurs in that situation generically. However, the methods given here require some additional hypotheses.

On the other hand, under a reasonably general condition, which we also feel is generic, if one approaches the boundary of the Morse–Smale diffeomorphisms without creating a cycle, then the only new structurally

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stable diffeomorphisms one can find nearby will also be Morse–Smale. In this case as well, one can encounter an infinite number of topologically different Morse–Smale diffeomorphisms.

It will be shown that all of the conditions assumed for these results are true for open subsets of the space of one-parameter families of diffeomorphisms.

The contents of the chapter are as follows: Section 2 contains definitions, some notation, and detailed statements of the main results. In Section 3 we derive conditions for a closed invariant subset of a diffeomorphism to be hyperbolic which generalize the conditions in [5]. We replace the uniformity assumptions in [5] with pointwise assumptions which turn out to be more convenient for our applications. In Sections 4–6 the main results of the paper are proved. Section 7 contains some examples related to the results in the previous sections and a description of the fact that the phenomena described earlier may be obtained with very simple isotopies beginning with any Morse–Smale diffeomorphism.

While we are concerned here with diffeomorphisms, one can get similar results for flows by analogous methods.†

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We recall some definitions and fix some terminology and notation. Let M be a compact connected C^∞ manifold without boundary. The space $\mathscr{D}^r$ of C^r diffeomorphisms of M with the C^r topology, is an infinite dimensional metrizable C^∞ Banach manifold for $1 \leq r < \infty$, and a Frechet manifold for $r = \infty$. Let $\Phi^{k,r} = C^k(I, \mathscr{D}^r)$ be the space of C^k mappings of the unit interval I into $\mathscr{D}^r$ with the uniform C^k topology, $1 \leq k < \infty$, $1 \leq r$. This topology is defined as in the finite dimensional case with respect to a riemannian (or Finsler) structure on $\mathscr{D}^r$ canonically induced by one on M (see [1]). An element $\xi \in \Phi^{k,r}$ is called a one-parameter family of diffeomorphisms of M.‡ A subset $\mathscr{F}$ of $\Phi^{k,r}$ is called residual if it contains a countable intersection of dense open sets. We will sometimes use the words "for most ξ" to mean for ξ in some residual set.

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‡ It will be evident that, if desired, one could suppress the Banach manifold structure of $\mathscr{D}^r$ and state most of the results in this chapter for C^r mappings $\xi : I \times M \to M$ such that $\xi_t : \{t\} \times M \to M$ is a diffeomorphism for all t.

For $f \in \mathscr{D}^r$, let $P(f)$ denote the set of periodic points of f; i.e., $x \in P(f)$ if and only if there is an integer $n > 0$ such that $f^n(x) = x$.

Suppose $f \in \mathscr{D}^r$, $r \geq 5$.

A fixed point x of f is said to be *quasi hyperbolic* if it has the following properties:

(2.1) (1) All of the eigenvalues of the derivative $T_x f$ have multiplicity one, and, if λ_1 and λ_2 are distinct eigenvalues such that $|\lambda_1| = |\lambda_2|$, then λ_1 is the complex conjugate of λ_2.

(2) $T_x f$ has an eigenvalue λ such that $|\lambda| = 1$.

(3) If $\lambda = 1$ and v is a nonzero eigenvector of $T_x f$ corresponding to λ, then the second derivative of f at x on v, $T_x^2 f(v, v)$, is not in the sum of the eigenspaces of the other eigenvalues of $T_x f$.

(4) If $\lambda = -1$ and v is a nonzero eigenvector corresponding to λ, the third derivative of f^2 at x on v, $T_x^3 f^2(v, v, v)$, is not in the sum of the eigenspaces of the other eigenvalues of $T_x f^2$.

(5) If $\lambda = e^{i\alpha}$, $\alpha \neq 0, \pi$, then $\lambda^k \neq 1$ for $0 < k \leq 5$, and there are coordinates $\varphi: U \to R^m$ on a neighborhood U of x such that $\varphi(x) = 0$ and

$$\varphi f \varphi^{-1}(y_1, \ldots, y_m) = (f_1(y_1, y_2), f_2(y_3, \ldots, y_m)) \in R^2 \times R^{m-2},$$

where $(y_1, \ldots, y_m)$ are coordinates on $R^m = R^2 \times R^{m-2}$ and

(a) f_1 is a diffeomorphism of R^2 with a fixed point at 0 such that λ is an eigenvalue of $T_0 f_1$, and if $\varphi_1: R^2 \to R^2$ is the polar coordinate map $\varphi_1(r, \theta) = (r \cos\theta, r \sin\theta)$, then

$$\varphi_1^{-1} f_1 \varphi_1(r, \theta) = (r + ar^3, \theta + \alpha + br^2) + o(r^5),$$

where a, α, b are constants and $a \neq 0$;

(b) f_2 is a diffeomorphism of R^{m-2} with a hyperbolic fixed point at 0.

A periodic point x such that $f^n(x) = x$ but $f^k(x) \neq x$, for $0 < k < n$, is said to be *quasi hyperbolic* if it is a quasi-hyperbolic fixed point of f^n (n is called the period of x).

Definitions (2.1.1)–(2.1.4) above are due to Sotomayor [19]. Definition (2.1.5) is motivated by Section 7 of the paper of Ruelle and Takens [14].

As is well known [5], each hyperbolic periodic point x has a smooth stable manifold $W^s(x)$ and a smooth unstable manifold $W^u(x)$. It also follows from the center manifold theory of [6] that each quasi-hyperbolic periodic point x has smooth stable and unstable manifolds defined by

$$W^s(x) = \{y \in M : f^{mn}(y) \to x \text{ as } m \to \infty\},$$
$$W^u(x) = \{y \in M : f^{-mn}(y) \to x \text{ as } m \to \infty\},$$

where n is the period of x. If x is as in (2.1.3), $W^s(x)$ will be diffeomorphic to a closed half-space of a euclidean space. Its boundary is the set of those points y such that $f^{mn}(y)$ tends to x exponentially, i.e.,

$$\partial W^s(x) = \{y \in M : d(f^{mn}(y), x) \leq c\lambda^m, m \geq 0\},$$

where $c > 0$, $0 < \lambda < 1$ are constants, and d is some metric on M. If x is as in (2.1.4) or (2.1.5), $W^s(x)$ is diffeomorphic to a euclidean space, and $f^n \mid W^s(x)$ is topologically conjugate to a hyperbolic contraction. Similar statements are true for $W^u(x)$.

Let R^{n_1}, R^{m-n_1} be the euclidean spaces of dimensions n_1, $m - n_1$, and let $\pi: R^{n_1} \times R^{m-n_1} \to R^{m-n_1}$ be the natural projection. Let $\varphi_i: N_i \to M$, $i = 1, 2$, be two smooth embeddings into M with $\dim N_i = n_i$, $\dim M = m$, and let $y \in \varphi_1(N_1) \cap \varphi_2(N_2)$. Say that y is a quasi-transversal intersection of $\varphi_1(N_1)$ and $\varphi_2(N_2)$ if the following statement is true: There is a diffeomorphism ψ mapping $R^{n_1} \times R^{m-n_1}$ onto a neighborhood U of y in M such that:

(1) $\psi(R^{n_1} \times 0) \subset \varphi_1(N_1)$, $\psi((0, 0)) = y$;

(2) The linear map

$$A_y = T_{\varphi_2^{-1}(y)}\pi\psi^{-1}\varphi_2 \to R^{m-n_1}$$

has rank $m - n_1 - 1$, and letting

$$\pi_1: R^{m-n_1} \to R^{m-n_1}/\text{Image}(A_y) \approx R$$

be the natural projection onto the quotient space;

(3) (a) The map

$$\pi_1\pi\psi^{-1}\varphi_2 \mid_{\varphi_2^{-1}(U)}$$

has $\varphi_2^{-1}(y)$ as a nondegenerate critical point,

(b) If $T_y\varphi_1(N_1) \cap T_y\varphi_2(N_2) \neq (0)$, there is a curve $\gamma\colon [-1, 1] \to N_2$ such that $\varphi_2\gamma(0) = y$,

$$T_{\varphi_2^{-1}(y)}\varphi_2(\gamma'(0))$$

is a nonzero vector in $T_y\varphi_1(N_1) \cap T_y\varphi_2(N_2)$, and the map $\pi_1\pi\psi^{-1}\varphi_2\gamma$ has a nonzero second derivative at zero.

Let KS denote the set of Kupka–Smale diffeomorphisms on M which is defined by $f \in KS$ if and only if all the periodic points of f are hyperbolic and their stable and unstable manifolds meet transversely.

As a preliminary, we state the following approximation theorem which is due to Sotomayor [19]. The paper of Brunovsky [2] also contains results related to part (2a) of the theorem, and Robinson proved a fact similar to part (1) for Hamiltonian systems.

(2.2) THEOREM For $k \geq 1$, $r \geq 5$, there is a residual subset $\mathscr{F}_1$ of $\Phi^{k,r}$ such that if $\xi \in \mathscr{F}_1$, then:

(1) $\xi_t \in KS$ for all but countably many $t \in I$;

(2) If $\xi_t \notin KS$, then exactly one of the following occurs:
(a) ξ_t has one orbit of quasi-hyperbolic periodic points, the other periodic orbits of ξ_t are hyperbolic, and the stable and unstable manifolds of all the periodic points meet transversely;
(b) All the periodic orbits of ξ_t are hyperbolic, there is one orbit of quasi-transversal intersections of stable and unstable manifolds of periodic points, and all the other intersections of these manifolds are transversal;

(3) The set Q of diffeomorphisms satisfying either (2a) or (2b), but not both, is a countable (increasing) union of residual subsets of embedded codimension-one submanifolds Q^n or $\mathscr{D}^r$ and ξ is transverse to each of these submanifolds.

We recall the definitions of the manifolds Q^n. For $f \in \mathscr{D}^r$, $n \geq 1$, let $P_n(f)$ be the set of periodic points of f of period less than $n + 1$. Let $Q_1{}^n = \{f \in \mathscr{D}^r : P_n(f)$ has at most one quasi-hyperbolic orbit and the other orbits in $P_n(f)$ are hyperbolic$\}$. Then $Q_1{}^n$ is a dense open subset of $\mathscr{D}^r$ and $P_n(f)$ is finite for f in $Q_1{}^n$. Given $x \in P_n(f)$, $f \in Q_1{}^n$, let $F_n{}^u(x, f) = \{y \in W^u(x, f) : d_u(y, x) < n\}$ where d_u is the distance in $W^u(x, f)$ defined by some riemannian metric on $W^u(x, f)$. Let $F_n{}^u(f) = \bigcup_{x \in P_n(f)} F_n{}^u(x, f)$. Define $F_n{}^s(f)$ similarly. Let $Q_2{}^n = \{f \in Q_1{}^n : P_n(f)$ has one quasi-hyperbolic orbit, the other orbits in $P_n(f)$ are hyperbolic,

and $F_n^s(f)$ meets $F_n^u(f)$ transversely}, and set $Q_3^n = \{f \in Q_1^n : P_n(f)$ is hyperbolic, and $F_n^s(f) \cap F_n^u(f)$ has one orbit of quasi-transversal intersections, the other intersections being transversal}. Then $Q^n = Q_2^n \cup Q_3^n$.

(2.3) REMARK Some parts of Theorem 2.2 hold under weaker differentiability assumptions: $r \geq 3$ is needed for Definition (2.1.4), and $r \geq 5$ is needed for Definition (2.1.5).

For $f \in \mathscr{D}^r$, let $\Omega(f)$, $L^-(f)$, $L^+(f)$, $L(f)$ denote, respectively, the nonwandering set, negative limit set, positive limit set, and limit set of f. These are defined as follows: $\Omega(f)$ is the set of x in M such that for any neighborhood U of x, there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$, $L^-(f)$ is the closure of the set of α-limit points of f, $L^+(f)$ is the closure of the set of ω-limit points of f, and $L(f) = L^-(f) \cup L^+(f)$.

It follows from [5] that if $L^-(f)$ is hyperbolic, then each point x in $L^-(f)$ has a smooth stable manifold $W^s(x)$ and a smooth unstable manifold $W^u(x)$. In this case, we say that f satisfies the strong transversality condition if $W^s(x)$ is transverse to $W^u(y)$ for all x and y in $L^-(f)$.

Let AS be the set of diffeomorphisms f satisfying Axiom A and the strong transversality condition (see [17]). It follows from [8] that $f \in AS$ if and only if (1) $L^-(f)$ is hyperbolic, and (2) f satisfies the strong transversality condition.

The Morse–Smale diffeomorphisms, denoted MS, are those $f \in AS$ such that $P(f)$ is finite, or, equivalently, those f for which $L^-(f)$ is finite hyperbolic and the strong transversality condition holds.

For $\xi \in \Phi^{k,r}$, $\xi_0 \in AS$, let $B(\xi) = \{t \in I : \xi_t \notin AS\}$, and let $b_0 = b_0(\xi) = \inf B(\xi)$. We call $B(\xi)$ the bifurcation set of ξ and $b_0(\xi)$ the first bifurcation point of ξ. Note that since AS is an open subset of $\mathscr{D}^r$, $B(\xi)$ is closed in I. It will always be assumed that $b_0(\xi) < 1$.

We are interested in the structure of $B(\xi)$ near $b_0 = b_0(\xi)$ for most ξ. Our main considerations will be under the assumption that $L^-(\xi_{b_0})$ is finite. The results also apply when $L^+(\xi_{b_0})$ is finite by replacing ξ_t by ξ_t^{-1}. It is an important unsolved problem whether for most ξ with $\xi_0 \in MS$, either $L^-(\xi_{b_0})$ or $L^+(\xi_{b_0})$ is finite. The next proposition, which is an immediate corollary of results of Brunovsky [2], shows that this problem is essentially a type of C^r closing lemma for the boundary of MS. We remark that even in the C^1 case, the technique will have to be different from that in Pugh's *closing lemma* [12] because, as simple examples show, there are open sets in the boundary of some elements of MS all of whose

diffeomorphisms have infinitely many nonwandering points, but only finitely many α and ω limit points (see Example 1 in Section 7).

(2.4) PROPOSITION For $k \geq 1$, $r \geq 2$, there is a residual set $\mathscr{T} \subset \Phi^{k,r}$ such that if $\xi \in \mathscr{T}$ and $\xi_0 \in MS$, then ξ_{b_0} has finitely many periodic points. On the other hand, if $\xi \in \mathscr{T}$, $\xi_0 \in AS$, and ξ_{b_0} has finitely many periodic points, then $\xi_0 \in MS$.

We note that, for $\mathscr{T}_1$, Q as in (2.2), if $\xi \in \mathscr{T}_1$, $\xi_0 \in AS$, and $L^-(\xi_{b_0})$ is finite, then $\xi_{b_0} \in Q$. To see this, observe that since $L^-(\xi_{b_0})$ is finite, $\xi_{b_0} \in KS$ if and only if $\xi_{b_0} \in AS$, and $\xi_{b_0} \notin AS$, since $b_0 \in B(\xi)$.

For $f \in Q$, choose Q^n containing f. A coordinate neighborhood Q_1 of f in Q^n will be called an intrinsic neighborhood of f in Q^n.

(2.5) THEOREM For $r \geq 5$, $k \geq 1$, there is a residual set $\mathscr{T}_2 \subset \Phi^{k,r}$ such that if $\xi \in \mathscr{T}_2$, $\xi_0 \in MS$, and $L^-(\xi_{b_0})$ is finite, then there are a neighborhood $\mathscr{N}$ of ξ in $\Phi^{k,r}$, an integer n, and an intrinsic neighborhood Q_1 of ξ_{b_0} in Q^n such that if $\eta \in \mathscr{N}$, then $\eta_0 \in MS$, $L^-(\eta_{b_0(\eta)})$ is finite, $b_0(\eta) = \inf\{t \in I : \eta_t \in Q_1\}$, and $\eta_{b_0(\eta)} \in Q$.

(2.6) REMARK We emphasize that Theorem (2.5) says that the set $\mathscr{A}^{k,r}$ of ξ in $\mathscr{T}_2$ such that $\xi_0 \in MS$ and $L^-(\xi_{b_0})$ is finite is an open subset of $\Phi^{k,r}$ for $r \geq 5$, $k \geq 1$.

Given an invariant subset $F \subset M$, set

$$W^s(F) = \{y \in M : \omega(y) \subset F\}, \qquad W^u(F) = \{y \in M : \alpha(y) \subset F\},$$
$$\hat{W}^s(F) = W^s(F) - F, \qquad \hat{W}^u(F) = W^u(F) - F,$$

where $\omega(y)$ and $\alpha(y)$ are the ω and α limit sets of y, respectively.

If $L^-(f)$ is finite, a j-cycle is a sequence of orbits $O(p_0), \ldots, O(p_j)$ such that $p_i \in L^-(f)$ for $0 \leq i \leq j$, $O(p_0) = O(p_j)$, and

$$\hat{W}^u(O(p_i)) \cap \hat{W}^s(O(p_{i+1})) \neq \emptyset \qquad \text{for} \quad 0 \leq i < j$$

(compare with [15]). In this chapter we prove Theorem (2.5) except in a certain situation where $L^-(\xi_{b_0})$ is hyperbolic and has only a 1-cycle. The proof in the remaining case will appear elsewhere.

Using some refinements of (2.2), we obtain

(2.7) THEOREM Let $\mathscr{A}^{k,r}$ be as in (2.6). There is a residual set $\mathscr{T}_3 \subset \mathscr{A}^{k,r}$ $k \geq 1$, $r \geq 5$, such that if $\xi \in \mathscr{A}^{k,r}$ and $L^-(\xi_{b_0})$ has no cycles, then there

is a neighborhood U of b_0 in I such that $\xi_t \in MS$ for $t \in U - B(\xi)$, and:

(1) If some periodic point of ξ_{b_0} has 1 as an eigenvalue, or if all the periodic points of ξ_{b_0} are hyperbolic, then $U \cap B(\xi)$ is at most countable;

(2) If some periodic point of ξ_{b_0} has -1 as an eigenvalue, then $U \cap B(\xi) = \{b_0\}$;

(3) If some periodic point of ξ_{b_0} has a nonreal eigenvalue of absolute value one, then $U \cap B(\xi)$ is zero-dimensional and contains a Cantor set K with $b_0 \in K$.

(2.8) REMARKS (1) Some of the local bifurcation theory of periodic points used in the proof of (2.7) has been known for some time and is contained in the work of Brunovsky [2], Sotomayor [19], Ruelle and Takens [14], and others. (Conversations with Sotomayor and Takens on that material have been very helpful.) We also refer the reader to the paper of Sotomayor [19] for related global questions in bifurcation theory.

(2) Some parts of (2.7) hold under weaker differentiability assumptions. (2.7.1) is true for a residual set for $k \geq 1$, $r \geq 2$, and (2.7.2) is true residually for $k \geq 1$, $r \geq 3$. We have stated the theorem as above in order to unify the results.

(3) The set of ξ in $\mathscr{F}_3$ satisfying the hypotheses of (2.7) is open in $\Phi^{k,r} = C^k(I, \mathscr{D}^r)$.

(4) The no-cycle assumption is not necessary under the hypotheses of (2.7.2) and (2.7.3). It can be shown using well-known arguments that the existence of a cycle under those hypotheses would imply the existence of transversal homoclinic points, and, hence, infinitely many periodic points for ξ_{b_0}.

(5) If ξ_{b_0} is as in the second half of (2.7.1), then ξ_{b_0} is Ω-stable.

From Theorem (2.7), it follows that to determine the structure of $B(\xi)$ near b_0 for most ξ with $L^-(\xi_{b_0})$ finite, it remains to consider the situation where $L^-(\xi_{b_0})$ has a cycle. We will study this case under some additional assumptions.

Recall that two diffeomorphisms $f, g \in \mathscr{D}^r$ are said to be topologically conjugate if there is a homeomorphism $h: M \to M$ such that $fh = hg$. $f \in \mathscr{D}^r$ is structurally stable if any $g \in \mathscr{D}^r$ which is C^1 close to f is topologically conjugate to f.

Our main result is the following:

(2.9) THEOREM Let $k \geq 1$, $r \geq 2$. There is a residual set $\mathscr{F}_4 \subset \mathscr{A}^{k,r}$ such that if $\xi \in \mathscr{F}_4$, $L^-(\xi_{b_0})$ is hyperbolic and has a cycle, and ξ_{b_0} satisfies (4.7) and (5.2), then there is a sequence $t_n \to b_0$ in I such that $\xi_{t_n} \in AS$, $\Omega(\xi_{t_n})$ is infinite and zero dimensional, and for $n \neq m$, ξ_{t_n} is not topologically conjugate to ξ_{t_m}.

We remark that for ξ as in (2.9), $B(\xi)$ is not countable. Also, since $r \geq 2$, each ξ_{t_n} in (2.9) is structurally stable by Robbin's theorem [13].

While (2.9) does not cover the general construction of a cycle even when $L^-(\xi_{b_0})$ is finite and hyperbolic, it does describe an open set of conditions which can be easily created from any Morse–Smale diffeomorphism (see Section 7). Hopefully, in the future, refinements of the present techniques will permit the improvement of Theorem 2.9 to the result in the following:

CONJECTURE For a residual set of $\xi: I \to \mathrm{Diff}(M)$, if $\xi_0 \in MS$, $L^-(\xi_{b_0})$ is finite, hyperbolic, and has a cycle, then there is a neighborhood U of b_0 in I such that $B(\xi) \cap U$ is zero dimensional and there are infinitely many topologically different structurally stable diffeomorphisms in $\xi(U)$.

REMARKS (1) In Thom's theory of mathematical biology, the set of sinks (or attractors) of a dynamical system plays a central role (see [20]). One can give additional assumptions to (2.7) and (2.9) to obtain that for $\varepsilon > 0$, there are t's with $b_0 < t < b_0 + \varepsilon$ such that the arc ξ_t has a sink far away from the sinks of ξ_s, $b_0 - \varepsilon < s \leq b_0$. We give a brief description of these hypotheses for the case in (2.9). Suppose $\xi_0 \in MS$, $L^-(\xi_{b_0})$ is finite and hyperbolic, and $O(p_1)$ and $O(p_2)$ are periodic points in a cycle of $L^-(\xi_{b_0})$ such that $W^u(O(p_2)) \cap W^s(O(p_1))$ consists of one orbit $O(x)$ of quasi-transversal intersections. Assume also that $\dim W^u(p_1) = \dim W^u(p_2) = 1$. Let $\lambda_1 < 1$ be the largest absolute value of the contracting eigenvalues of $T_{p_1} f$, and let $\lambda_2 > 1$ be the absolute value of the expanding eigenvalue of $T_{p_1} f$. Suppose $\lambda_1 \lambda_2 < 1$. Then, for most ξ, sinks will appear near x for some t's with $b_0 < t < b_0 + \varepsilon$.

It is interesting to ask whether for most ξ with $\xi_0 \in MS$ there is a neighborhood U of b_0 in I such that ξ_t has only finitely many sinks for $t \in U$. There are indications that this question has a positive answer. However, if one only assumes that $\xi_0 \in AS$, then ξ_t may have infinitely many sinks for most t near b_0 and greater than b_0 (see [7]).

(2) It will be shown later that the hypotheses of (2.9) guarantee that $\Omega(\xi_t)$ for $t \geq b_0$ is far from $\Omega(\xi_t)$ for $t < b_0$. That is, an Ω-explosion is created at $t = b_0$. Related Ω-explosions were studied in [9].

In view of Theorems (2.7) and (2.9), we pose the following problems.

(1) Is it true for most ξ with $\xi_0 \in MS$ that $L^-(\xi_{b_0})$ or $L^+(\xi_{b_0})$ is finite, and that $L(\xi_{b_0})$ has a finite number of orbits?

(2) Is $B(\xi) \cap U$ zero dimensional, where U is some neighborhood of b_0 in I for most ξ with $\xi_0 \in MS$?

3

Let U be a compact subset of a riemannian manifold M. Let $f: U \to M$ be a C^1 embedding. In this section we derive general conditions for $U_f = \bigcap_{n\in Z} f^n(U)$ to be a hyperbolic set for f. The techniques used are modifications of those in [5].

Given $x \in U$, $\varepsilon > 0$, and a direct sum decomposition $T_xM = E_1 \oplus E_2$, define the ε-sector about E_1 by E_2 to be

$$S_\varepsilon(E_1, E_2) = \{(v_1, v_2) \in E_1 \oplus E_2 : |v_2| \leq \varepsilon |v_1|\},$$

where $|\cdot|$ is the riemannian norm in T_xM. When the splitting of T_xM is understood, we will write $S_\varepsilon(E_1, E_2)$ as $S_\varepsilon(E_1)$. If S is a subset of T_xM, we will say that T_xf is a λ-expansion on S (or $T_xf|_S$ is a λ-expansion) if $|T_xf(v)| \geq \lambda |v|$, for all $v \in S$.

Assume $\bigcap_{n\in Z} f^n(U) \neq \emptyset$.

An almost hyperbolic splitting for f on U is a (not necessarily continuous) splitting $T_xM = E_{1x} \oplus E_{2x}$, for $x \in U$ such that dim E_1 and dim E_2 are constant and there exist bounded positive nowhere zero functions ε_1, ε_2 on U, a constant $\lambda > 1$, and an integer $m > 0$, such that

$$T_xf^m(S_{\varepsilon_2(x)}(E_{2x})) \subset S_{\varepsilon_2 f^m(x)}(E_{2f^m(x)}), \tag{1}$$

and $T_xf^m \mid S_{\varepsilon_2(x)}(E_{2x})$ is a λ-expansion for $x \in U \cap f^{-m}(U)$,

$$T_xf^{-m}(S_{\varepsilon_1(x)}(E_{1x})) \subset S_{\varepsilon_1 f^{-m}(x)}(E_{1f^{-m}(x)}), \tag{2}$$

and $T_xf^{-m} \mid S_{\varepsilon_1(x)}(E_{1x})$ is a λ-expansion for $x \in U \cap f^m(U)$.

Note that the existence of an almost hyperbolic splitting for f on U is independent of the riemannian metric on M by simply choosing different multiples of m if necessary.

(3.1) THEOREM If U has an almost hyperbolic splitting, then $U_f = \bigcap_{n\in Z} f^n(U)$ is a hyperbolic set for f.

REMARK Since there is at most one hyperbolic splitting for f on a given closed invariant subset, we see that all almost hyperbolic splittings for f on U give rise to the same hyperbolic structure on U_f.

To begin the proof of Theorem (3.1), we observe that it suffices to prove the result assuming $m = 1$. For $U_f \subset U_{f^m}$ and is f^m-invariant. Setting $g = f^m$, if U_g is hyperbolic for g, it is easy to show that U_f is hyperbolic for f.

Let $V = U \cap f(U) \cap f^{-1}(U)$, and define $\varepsilon_1'(x)$, $\varepsilon_2'(x)$ for $x \in V$ so that

$$S_{\varepsilon_1'(x)}(E_{1x}) = T_{f(x)} f^{-1}(S_{\varepsilon_1 f(x)}(E_{1f(x)})),$$
$$S_{\varepsilon_2'(x)}(E_{2x}) = T_{f^{-1}(x)} f(S_{\varepsilon_2 f^{-1}(x)}(E_{2f^{-1}(x)})).$$

Then Eqs. (1) and (2), and the uniform continuity of $T_x f$ and $T_x f^{-1}$ imply that

$$\sup_{x \in V} \varepsilon_1'(x) \cdot \varepsilon_2'(x) < 1.$$

Otherwise, there would be a nonzero tangent vector at some point of V which Tf would expand and contract. In a similar manner, one can show that the functions $\varepsilon_1'(x)$, $\varepsilon_2'(x)$ above can be chosen to be bounded away from zero. Thus, changing U, ε_1, and ε_2, if necessary, we may assume ε_1 and ε_2 are as in Eqs. (1) and (2),

$$\sup_{x \in U} \varepsilon_1(x) \cdot \varepsilon_2(x) < 1,$$

and ε_1 and ε_2 are bounded away from zero on U.

Let $\Lambda = U_f$. To prove Theorem (3.1) we will first show:

(3) For $x \in \Lambda$, there is a unique subspace $E_x^{\mathrm{u}} \subset S_{\varepsilon_2(x)}(E_{2x})$ such that

$$T_x f(E_x^{\mathrm{u}}) = E_{f(x)}^{\mathrm{u}}.$$

The same methods applied to f^{-1} will show there is a unique subspace $E_x^{\mathrm{s}} \subset S_{\varepsilon_1(x)}(E_{1x})$ such that $T_x f^{-1}(E_x^{\mathrm{s}}) = E_{f^{-1}(x)}^{\mathrm{s}}$. Then we will show that E_x^{u} and E_x^{s} are continuous in x. This, together with the expanding properties of $Tf \mid S_{\varepsilon_2(x)}(E_{2x})$ and $Tf^{-1} \mid S_{\varepsilon_1(x)}(E_{1x})$, will establish that $E^{\mathrm{s}} \oplus E^{\mathrm{u}}$ gives a hyperbolic splitting for Tf on Λ.

The first step of the proof is the following assertion.

(4) There is an integer $m > 0$ such that for each $x \in \Lambda$ there is a unique subspace $E_x^{\mathrm{u}} \subset S_{\varepsilon_2(x)}(E_{2x})$ such that

$$T_x f^m(E_x^{\mathrm{u}}) = E_{f^m(x)}^{\mathrm{u}}.$$

Note that (3) follows immediately from (4). We prove (4). For $m > 0$, let

$$F_{mx} = T_{f^m(x)} f^{-m}(E_{1f^m(x)}) \subset S_{\varepsilon_1(x)}(E_{1x}).$$

Write

$$T_x f^m = \begin{pmatrix} A_{mx} & B_{mx} \\ C_{mx} & D_{mx} \end{pmatrix}$$

with respect to the splitting $F_{mx} \oplus E_{2x} = T_x M$, $x \in \Lambda$, i.e.,

$$A_{mx}: F_{mx} \to F_{mf^m(x)}, \qquad C_{mx}: F_{mx} \to E_{2f^m(x)},$$
$$B_{mx}: E_{2x} \to F_{mf^m(x)}, \qquad D_{mx}: E_{2x} \to E_{2f^m(x)}.$$

Let $\mathscr{E}_m$ be the disjoint union $\bigcup_{x\in\Lambda} L(E_{2x}, F_{mx})$, where $L(E_{2x}, F_{mx})$ is the space of (bounded) linear maps from E_{2x} to F_{mx}, and let

$$\mathscr{E}_m' = \{P : \Lambda \to \mathscr{E}_m \mid P_x \in L(E_{2x}, F_{mx}) \text{ for all } x, \text{ and } \sup_x \| P_x \| < \infty\}.$$

$\mathscr{E}_m'$ is a Banach space with norm $\| P \| = \sup_{x\in\Lambda} \| P_x \|$. Let

$$\mathscr{G}_m = \{P \in \mathscr{E}_m' : \text{graph } P_x \subset S_{\varepsilon_2(x)}(E_{2x}, E_{1x})\}.$$

Since $\mathscr{G}_m$ is a closed subset of $\mathscr{E}_m'$, it is a complete metric space. Also, since

$$\sup_{x\in\Lambda} \varepsilon_1(x) \cdot \varepsilon_2(x) < 1 \quad \text{and} \quad F_{mx} \subset S_{\varepsilon_1(x)}(E_{1x}, E_{2x}) \quad \text{for all } x \in \Lambda,$$

there is a constant $k > 0$ (independent of x and m) such that

$$S_{\varepsilon_2(x)}(E_{2x}) \subset S_k(E_{2x}, F_{mx}), \qquad \text{for all } x \text{ and } m.$$

Thus, for any integer m, if $P \in \mathscr{G}_m$, then $\| P_x \| \leq k$, for all $x \in \Lambda$.

We seek $E_x^u = \text{graph } P_x$ for some $P \in \mathscr{G}_m$. The $T_x f$ invariance may be expressed as

$$\begin{pmatrix} A_{mx} & B_{mx} \\ C_{mx} & D_{mx} \end{pmatrix} \begin{pmatrix} P_x v \\ v \end{pmatrix} = \begin{pmatrix} P_{f(x)} u \\ u \end{pmatrix}$$

for any $v \in E_{2x}$, where the matrix represents $T_x f$ with respect to $F_m \oplus E_2$. This equation is equivalent to

$$A_{mx} P_x + B_{mx} = P_{f(x)}(C_{mx} P_x + D_{mx}).$$

It will be shown that

(5) for large m,

$$| C_{mx} | k | D_{mx}^{-1} | < 1 \text{ for all } x \in \Lambda.$$

Assuming this for the moment, one sees that $(C_{mx}P_x + D_{mx})^{-1}$ exists for m large. Letting

$$y = f^{-1}(x) \qquad \text{and} \qquad \Gamma_m(P)_x = (A_{my}P_y + B_{my})(C_{my}P_y + D_{my})^{-1},$$

we see that we want a fixed point P of Γ_m. Note that (1) implies that Γ_m maps $\mathscr{G}_m$ into itself since

$$\text{graph}(\Gamma_m(P)_x) = T_y f(\text{graph } P_y).$$

We will show that for large m, $\Gamma_m : \mathscr{G}_m \to \mathscr{G}_m$ is a contraction.

Imitating the proof of Lemma (4.3) of [5], we see that, for each $x \in \Lambda$,

$$\| \Gamma_m(P_1)_x - \Gamma_m(P_2)_x \| \leq s_m(y) \| P_{1y} - P_{2y} \|,$$

where

$$\begin{aligned} s_m(y) = {} & (| A_{my} | k | D_{my}^{-1} | \\ & + | B_{my}D_{my}^{-1} |)(1 - | C_{my} | k | D_{my}^{-1})^{-2} | C_{my} | | D_{my}^{-1} | \\ & + | A_{my} | | D_{my}^{-1} |(1 - | C_{my} | k | D_{my}^{-1} |)^{-1}. \end{aligned}$$

We claim that

(6) there is a constant $\nu < 1$ such that if m is large, then

$$\sup_{x \in \Lambda} s_m(x) \leq \nu.$$

Once this is shown, it follows that Γ_m is a contraction, for let $\varepsilon > 0$ be such that $\nu < \varepsilon < 1$, and choose $x_1 \in \Lambda$ such that

$$\| \Gamma_m(P_1)_{x_1} - \Gamma_m(P_2)_{x_1} \| > \varepsilon \| \Gamma_m(P_1) - \Gamma_m(P_2) \|.$$

Then, letting $y_1 = f^{-1}(x_1)$, we have

$$\begin{aligned} \| \Gamma_m(P_1) - \Gamma_m(P_2) \| &< \frac{1}{\varepsilon} \| \Gamma_m(P_1)_{x_1} - \Gamma_m(P_2)_{x_1} \| \\ &\leq \frac{s_m(y_1)}{\varepsilon} \| P_{1y_1} - P_{2y_1} \| \leq \frac{\nu}{\varepsilon} \| P_1 - P_2 \|. \end{aligned}$$

To prove (5) and (6), it suffices to show:

(7) $\quad |A_{my}||D_{my}^{-1}| \to 0 \quad$ uniformly in y as $m \to \infty$,

(8) $\quad |C_{my}||D_{my}^{-1}| \to 0 \quad$ uniformly in y as $m \to \infty$,

and

(9) there is a constant $k_1 > 0$, such that

$$|B_{my} D_{my}^{-1}| \leq k_1 \qquad \text{for all } m \text{ and all } y.$$

Since each $F_{my} \subset S_{\varepsilon_1(y)}(E_{1y}, E_{2y})$, the matrix H_{my} of the identity mapping $F_{my} \oplus E_{2y} \to E_{1y} \oplus E_{2y}$ has the form

$$\begin{pmatrix} \beta_{my}^1 & 0 \\ \beta_{my}^2 & I \end{pmatrix}$$

where I is the identity mapping

$$E_{2y} \to E_{2y}, \qquad \beta_{my}^1 : F_{my} \to E_{1y}, \qquad \beta_{my}^2 : F_{my} \to E_{2y},$$

and for $v \in F_{my}$,

$$|\beta_{my}^2(v)| \leq \varepsilon_1(y)\,|\beta_{my}^1(v)|.$$

Also

$$H_{my}^{-1} = \begin{pmatrix} \sigma_{my}^1 & 0 \\ \sigma_{my}^2 & I \end{pmatrix},$$

and for $v \in E_{1y}$,

$$|\sigma_{my}^2(v)| \leq \delta_1(y)\,|\sigma_{my}^1(v)|,$$

where $\delta_1(y)$ is a bounded function.

Note that since ε_1, ε_2 are bounded away from zero, $|\sigma_{my}^1|$ and $|\sigma_{my}^2|$ are uniformly bounded functions. Let $\sigma^1 = \sup_{y,m} |\sigma_{my}^1|$ and $\sigma^2 = \sup_{y,m} |\sigma_{my}^2|$.

Consider the following commutative diagram of matrices:

$$\begin{array}{ccc} F_{my} \oplus E_{2y} & \xrightarrow{\Phi_{my}} & F_{mf^m(y)} \oplus F_{2f^m(y)} \\ \downarrow H_{my} & \searrow^{\Phi'_{my}} & \uparrow H^{-1}_{mf^m(y)} \\ E_{1y} \oplus E_{2y} & \xrightarrow{\Phi''_{my}} & E_{1f^m(y)} \oplus E_{2f^m(y)} \end{array}$$

where

$$\Phi_{my} = \begin{pmatrix} A_{my} & B_{my} \\ C_{my} & D_{my} \end{pmatrix}, \qquad \Phi'_{my} = \begin{pmatrix} A'_{my} & B'_{my} \\ C'_{my} & D'_{my} \end{pmatrix}, \qquad \Phi''_{my} = \begin{pmatrix} A''_{my} & B''_{my} \\ C''_{my} & D''_{my} \end{pmatrix},$$

are matrices of $T_y f^m$ with respect to the indicated splittings, and H_{my}, $H^{-1}_{mf^m(y)}$ are as above. Thus

$$\begin{aligned} \begin{pmatrix} A_{my} & B_{my} \\ C_{my} & D_{my} \end{pmatrix} &= \begin{pmatrix} \sigma^1_{my} & 0 \\ \sigma^2_{my} & I \end{pmatrix} \begin{pmatrix} A'_{my} & B'_{my} \\ C'_{my} & D'_{my} \end{pmatrix} \\ &= \begin{pmatrix} \sigma^1_{my} A'_{my} & \sigma^1_{my} B'_{my} \\ \sigma^2_{my} A'_{my} + C'_{my} & \sigma^2_{my} B'_{my} + D'_{my} \end{pmatrix} \end{aligned} \tag{10}$$

So for (7) we need to show

$$| \sigma^1_{my} A'_{my} | \, | D^{-1}_{my} | \to 0.$$

Now $| \sigma^1_{my} A'_{my} | \leq \sigma^2 | A'_{my} | \leq \sigma^2 \lambda^{-m}$, by the definition of F_{my}. Thus to prove (7) it suffices to show that $| D^{-1}_{my} |$ is uniformly bounded. We actually prove $| D^{-1}_{my} | \to 0$ uniformly in y as $m \to \infty$.

We have

$$\begin{pmatrix} A_{my} & B_{my} \\ C_{my} & D_{my} \end{pmatrix} = \begin{pmatrix} \sigma^1_{my} & 0 \\ \sigma^2_{my} & I \end{pmatrix} \begin{pmatrix} A''_{my} & B''_{my} \\ C''_{my} & D''_{my} \end{pmatrix} \begin{pmatrix} \beta^1_{my} & 0 \\ \beta^2_{my} & I \end{pmatrix}. \tag{11}$$

Hence, $D_{my} = \sigma^2_{my} B''_{my} + D''_{my}$. Since σ^2_{my} is the projection of $E_{1f^m(y)}$ onto $E_{2f^m(y)}$ along $F_{mf^m(y)}$, and

$$F_{mf^m(y)} \subset S_{\varepsilon_1 f^m(y)}(E_{1f^m(y)}, E_{2f^m(y)}),$$

$| \sigma^2_{my} | \leq \varepsilon_1 f^m(y)$. Also, since

$$T_y f^m(E_{2y}) \subset S_{\varepsilon_2 f^m(y)}(E_{2f^m(y)}, E_{1f^m(y)}),$$

$| B''_{my} D''^{-1}_{my} | \leq \varepsilon_2 f^m(y)$.

Hence, letting $\nu_1 = \sup_{x \in A} \varepsilon_1(x) \cdot \varepsilon_2(x)$, we have

$$| \sigma^2_{my} B''_{my} D''^{-1}_{my} | \leq \varepsilon_1 f^m(y) \cdot \varepsilon_2 f^m(y) \leq \nu_1 < 1.$$

Now, let $v \in E_{2y}$ and $| v | = 1$. Then,

$$\begin{aligned} | D_{my}(v) | &= | (\sigma^2_{my} B''_{my} + D''_{my}) v | \\ &= | (\sigma^2_{my} B''_{my} D''^{-1}_{my} + I) D''_{my}(v) | \\ &\geq (1 - \nu_1) | D''_{my}(v) |, \end{aligned}$$

and, for $m \geq 0$,

$$\begin{aligned}\lambda^m \leq |\, T_y f^m(v)\,| &= |\, B''_{my}(v) + D''_{my}(v)\,| \\ &= |\, (B''_{my}\, D''^{-1}_{my} + I) D''_{my}(v)\,| \\ &\leq (1 + \sup_{y \in \Lambda} \varepsilon_2 f^m(y))\,|\, D''_{my}(v)\,|.\end{aligned}$$

Therefore, $|\, D''_{my}(v)\,| \to \infty$, and, hence, so does $|\, D_{my}(v)\,|$. This implies that $|\, D^{-1}_{my}\,| \to 0$, which proves (7). To prove (8), return to the expression (10). Note that $C'_{my} = 0$ since $T_y f^m(F_{my}) = E_{1f^m(y)}$. Hence $C_{my} = \sigma^2_{my} A'_{my}$. Thus

$$|\, C_{my}\,| \leq |\, \sigma^2_{my}\,|\,|\, A'_{my}\,| \leq \sigma^2\,|A'_{my}\,| \leq \sigma^2 \lambda^{-m} \to 0.$$

Then (8) follows, since $|\, D^{-1}_{my}\,| \to 0$.

For (9), returning to (11), we have

$$B_{my} = \sigma^1_{my} B''_{my}, \qquad D_{my} = \sigma^2_{my} B''_{my} + D''_{my} = (\sigma^2_{my} B''_{my}\, D''^{-1}_{my} + I) D''_{my}.$$

So

$$D^{-1}_{my} = D''^{-1}_{my}\,(I + \sigma^2_{my} B''_{my}\, D''^{-1}_{my}\,)^{-1}.$$

Hence,

$$\begin{aligned}|\, B_{my}\, D^{-1}_{my}\,| &= |\, \sigma^1_{my} B''_{my}\, D''^{-1}_{my} (I + \sigma^2_{my} B''_{my}\, D''^{-1}_{my})^{-1}\,| \\ &\leq \sigma^1 (\sup_{y \in \Lambda} \varepsilon_2 f^m(y))(1 - \nu_1)^{-1}.\end{aligned}$$

This proves (9).

As remarked earlier, the methods in the proof of (4) applied to f^{-1} will give a unique subspace $E^s_x \subset S_{\varepsilon_1(x)}(E_{1x}, E_{2x})$ for each $x \in \Lambda$ such that $T_x f^{-1}(E^s_x) = E^s_{f^{-1}(x)}$ or, equivalently, $T_x f(E^s_x) = E^s_{f(x)}$.

Thus we may assume there are an integer $m_0 > 0$ and a splitting $T_x M = E^s_x \oplus E^u_x$ for each $x \in \Lambda$ such that

$$\text{(12a)} \qquad \begin{aligned} &E^s_x \subset S_{\varepsilon_1(x)}(E_{1x}, E_{2x}), \qquad T_x f(E^s_x) = E^s_{f(x)}, \\ &|\, T_x f^{m_0}\,|\, E^s_x\,| \leq \lambda^{-1} < 1, \end{aligned}$$

and

$$\text{(12b)} \qquad \begin{aligned} &E^u_x \subset S_{\varepsilon_2(x)}(E_{2x}, E_{1x}), \qquad T_x f(E^u_x) = E^u_{f(x)}, \\ &|\, T_x f^{-m_0}\,|\, E^u_x\,| \leq \lambda^{-1} < 1. \end{aligned}$$

We now prove that E^s_x is continuous in x. A similar method works for E^u_x, so this will complete the proof of Theorem 4.1.

We first claim

(13) Given $\varepsilon > 0$ and $\lambda_1 > 1$, there is an integer $m_1 \geq 0$ such that if $m \geq m_1$, $x \in \Lambda$, and $v \notin S_\varepsilon(E_x^s, E_x^u)$, then

$$| T_x f^m(v) | \geq \lambda_1 | v |.$$

Let $\langle , \rangle$ denote the riemannian inner product on T_xM, $x \in \Lambda$. Write $v = (v_1, v_2)$ with $v_1 \in E_x^s$, $v_2 \in E_x^u$, and $| v_2 | > \varepsilon | v_1 |$, and write $T_x f^m(v) = A_m v_1 + D_m v_2$, where $A_m : E_x^s \to E_{f(x)}^s$ and $D_m : E_x^u \to E_{f(x)}^u$. Then

$$\begin{aligned} | T_x f^m(v) |^2 &= | D_m(v_2) |^2 + | A_m(v_1) |^2 + 2\langle D_m(v_2), A_m(v_1)\rangle \\ &\geq (| D_m(v_2) | - | A_m(v_1) |)^2 \\ &\geq (\lambda^m | v_2 | - \lambda^{-m} | v_1 |)^2 \end{aligned}$$

by (12). On the other hand,

$$| v |^2 \leq | v_1 |^2 + | v_2 |^2 + 2 | v_1 | | v_2 | \leq \frac{1}{\varepsilon^2} | v_2 |^2 + | v_2 |^2 + \frac{2}{\varepsilon} | v_2 |^2,$$

so

$$| v_2 |^2 \geq \left(1 + \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon}\right)^{-1} | v |^2.$$

Hence,

$$| T_x f^m(v) |^2 \geq \left(\lambda^m\left(1 - \frac{\lambda^{-2m}}{\varepsilon}\right)\right)^2 \left(1 + \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon}\right)^{-1} | v |^2,$$

from which (13) follows.

Now let d denote the metric induced by $\langle , \rangle$ on the Grassmann bundle of s-planes over U.

Let $x \in \Lambda$, and assume $x_n \to x$ in Λ as $n \to \infty$. If $E_{x_n}^x$ does not approach E_x^s, then there are a $\delta > 0$ and a subsequence (y_n) of (x_n) such that $d(E_{y_n}^s, E_x^s) > \delta$ for all n. Let (U, φ) be a coordinate chart about x in M, with $\varphi : U \to R^m$ a diffeomorphism such that $\varphi(x) = 0$. For $y \in U$, let ψ_y be the translation of R^m taking 0 to $\varphi(y)$, and let

$$F_y^u = T_x\varphi^{-1} \circ \psi_y \circ \varphi(E_x^u), \qquad F_y^s = T_x\varphi^{-1} \circ \psi_y \circ \varphi(E_x^s).$$

There are a positive integer n_0 and a positive real number $\varepsilon > 0$ such that for $n \geq n_0$, $y_n \in U$, and if $v_n \in E_{y_n}^s$, $v_n \neq 0$, then $v_n \notin S_{2\varepsilon}(F_{y_n}^s, F_{y_n}^u)$. Choose m_1 as in (13) for $\lambda_1 = 2$.

Let (v_{n_k}) be a subsequence of (v_n) such that $v_{n_k}/| v_{n_k} |$ converges to a unit vector w in T_xM as $k \to \infty$. Then $w \notin S_\varepsilon(E_x^s, E_x^u)$, so letting

$m_2 = m_0 m_1$ it follows that $| T_x f^{m_2}(w) | \geq 2$. Hence, if y_{n_k} is close to x,

$$| T_{y_{n_k}} f^{m_2}(v_{n_k}) | \geq \tfrac{3}{2} | v_{n_k} |.$$

This contradicts (12) since

$$v_{n_k} \in E^s_{y_{n_k}}$$

and completes the proof of the continuity of E^s_x.

4

Here we prove Proposition (2.4), Theorem (2.5), and Theorem (2.7). For the proof of Proposition (2.4), we note that the methods of [2] can be used to show there is a residual set $\mathscr{F} \subset \Phi^{k,r}$ such that for $\xi \in \mathscr{F}$, the periodic points of ξ_t lie in one-parameter families in $I \times M$. More precisely, for $\xi \in \mathscr{F}$, $n > 0$, the set

$$X_n(\xi) = \{(t, x) : x \text{ is a periodic point of } \xi_t \text{ of period } n\}$$

is a one-dimensional submanifold of $I \times M$.

Also:

(4.1) (1) if $X_n = X_n(\xi)$ is not closed in $I \times M$, then n is even, $\bar{X}_n - X_n \subset X_{n/2}$, and if $(t, x) \in \bar{X}_n - X_n$, then $T_x \xi_t^{n/2}$ has -1 as an eigenvalue;

(2) if $T_{(t,x)}\pi \,|_{T_{(t,x)}X_n(\xi)}$ fails to be surjective where $\pi : I \times M \to I$ is the natural projection, then 1 is an eigenvalue of $T_x \xi_t^n$;

(3) for any $t \in I$, there is at most one periodic point x of ξ_t with an eigenvalue of absolute value 1.

Now assume $\xi \in \mathscr{F}$ and $\xi_0 \in MS$. Let $O(p_1), \ldots, O(p_m)$ be the distinct periodic orbits of ξ_0, and let $X_{n_1}, \ldots, X_{n_m}$ be the one-dimensional submanifolds of $I \times M$ such that for $p \in O(p_i)$, $(t, p) \in X_{n_i}$, $i = 1, \ldots, m$. Then, (4.1), and the definition of $b_0(\xi)$, show that there is at most one of the X_{n_i} which fails to meet $b_0(\xi) \times M$. Also (4.1.2) shows if some X_j meets $b_0(\xi) \times M$ with $X_j \neq X_{n_i}$, $i = 1, \ldots, m$, then $X_j \cap b_0(\xi) \times M$ consists of one periodic orbit of $\xi_{b_0(\xi)}$, and it has 1 as an eigenvalue. This case corresponds to the introduction of a new periodic orbit far from the others. Thus, in fact, the number of periodic orbits of ξ_{b_0} differs from that of ξ_0 by at most one. The same argument proves the second statement of Proposition (2.4).

We prove (2.7) before (2.5) because its proof is shorter.

Let $\mathscr{F}_1$ be as in Theorem (2.2). Suppose $\xi \in \mathscr{F}_1$, $\xi_0 \in MS$, $L^-(\xi_{b_0})$ is finite, and $L^-(\xi_{b_0})$ has no cycles. Let $f = \xi_{b_0}$, and let $O(p_1), \ldots, O(p_n)$ be the distinct orbits of $L^-(f)$. Using obvious modifications of the methods in [8] (and, if necessary, relabeling the $O(p_i)$), one may construct a filtration $M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = \varnothing$ for f such that, for $0 < i \leq n$,

(4.2) (1) $f(M_i) \subset \operatorname{int} M_i$, $O(p_i) \subset \operatorname{int}(M_i - M_{i-1})$;

(2) $\bigcap_{j \in Z} f^j(M_i - M_{i-1}) = O(p_i)$.

It follows from (4.2) that $L^-(f) = L(f) = \Omega(f)$.

Proof of (2.7.1) First assume $L^-(f)$ is hyperbolic.

Then it follows that f is Ω-stable, for there is a neighborhood $\mathscr{N}$ of f in $\mathscr{D}^r$ such that for $g \in \mathscr{N}$, $\Omega(g)$ is finite, hyperbolic, and has no cycles. Also, $g \in MS$ if and only if $g \in KS$. Then, choosing U such that $t \in U$ implies that $\xi_t \in \mathscr{N}$, $U \cap B(\xi)$ is at most countable by Theorem (2.2).

Suppose now that $O(p_i)$ is a periodic orbit with 1 as an eigenvalue. Thus, if n_0 is the period of p_i, then 1 is an eigenvalue of $T_{p_i} f^{n_0}$. Applying results of Brunovsky and Sotomayor, there are neighborhoods V of $O(p_i)$ and U of b_0 such that for t in one component of $U - \{b_0\}$, ξ_t has no invariant subset in V, and for t in the other component of $U - \{b_0\}$, ξ_t has two hyperbolic orbits $O(q_1)$, $O(q_2)$ in V of the same period as p_i such that $\hat{W}^u(O(q_1)) \cap \hat{W}^s(O(q_2)) \neq \varnothing$, and the intersections are transversal. Now using the above filtration, one can show that if U is small enough, then $L^-(\xi_t)$ is finite and has no cycles for $t \in U$, and $L^-(\xi_t)$ is hyperbolic for $t \in U - \{b_0\}$. Hence,

$$U \cap B(\xi) = \{b_0\} \cup \{t \in U : \xi_t \notin KS\},$$

which is at most countable.

Proof of (2.7.2) Let $O(p_i)$ be the periodic orbit with -1 as an eigenvalue. Again by results of Brunovsky and Sotomayor, there are neighborhoods V of $O(p_i)$ and U of b_0 such that for t in one component of $U - \{b_0\}$, there is one hyperbolic periodic orbit $O(p_i(t))$ of ξ_t in V of the same period as p_i, and for t in the other component of $U - \{b_0\}$, there are two hyperbolic periodic orbits $O(q_1(t))$, $O(q_2(t))$ in V, one having the same period as p_i, the other having twice the period of p_i, such that

$$\hat{W}^u(O(q_1(t)) \cap \hat{W}^s(O(q_2(t))) \neq \varnothing.$$

In either of the above cases, the indicated periodic orbits will be the

only ξ_t-invariant subsets of V. In the latter case, the intersections of $\hat{W}^u(O(q_1(t)))$ and $\hat{W}^s(O(q_2(t)))$ are transversal.

Using the above filtration, one can show that for U small, $L^-(\xi_t)$ is finite and hyperbolic for $t \in U - \{b_0\}$. Moreover, since the stable and unstable manifolds of ξ_{b_0} are transverse, one can show $\xi_t \in MS$ for $t \in U - \{b_0\}$ with arguments similar to those used in [10] to prove that MS is open in $\mathscr{D}^r$.

Proof of (2.7.3) Let $\mathscr{F}_3 \subset \mathscr{F}_1$ be the set of $\xi \in \mathscr{F}_1$ such that for any t, no periodic point of ξ_t has a nonreal root of unity as an eigenvalue.

Assume $\xi \in \mathscr{F}_3$ and ξ_{b_0} has a periodic orbit $O(p_i)$ of period n_0, with a nonreal eigenvalue λ such that $|\lambda| = 1$. By the implicit function theorem, there is a neighborhood U of b_0 such that for each $t \in U$, there is a unique periodic orbit $O(p_i(t))$ near $O(p_i)$ of the same period as p_i, and there is a unique eigenvalue λ_t of $p_i(t)$ near $\lambda = \lambda_{b_0}$. Also, the mapping $t \mapsto \lambda_t$ is C^{r-1}, and we may assume it is transverse to the unit circle at $t = b_0$. We suppose that $|\lambda_t| < 1$ for $t < b_0$, $t \in U$. Then ξ_{b_0} is topologically conjugate to ξ_t for $t < b_0$, and $O(p_i(b_0))$ is a "weak" attractor of

$$\xi_{b_0} \mid W^s(O(p_t(b_0))).$$

The case $|\lambda_t| > 1$ for $t < b_0$ is handled similarly.

Now, as we verified with Takens, the methods of [14] can be generalized to show that for each $t > b_0$ near b_0, $\xi_t^{n_0}$ has a unique invariant circle Σ_t^i near $p_i(t_0)$, and the fixed point $p_i(\xi_t)$ is hyperbolic. Also, $\xi_t^{n_0}$ will be normally hyperbolic to Σ_t^i in the sense of [6]. Takens also told us that the differentiability assumption in [14], that $\xi: I \times M \to M$ is C^r, $r \geq 5$, can be weakened to $\xi: I \to \mathscr{D}^r$ is C^1, $r \geq 5$, which is exactly our setting.

We claim that if t is near b_0, $\xi_t \in \overline{MS}$. Indeed, an application of Peixoto's theorem shows that we may find a C^r approximation g of ξ_t such that $g(\bigcup \Sigma_t^i) = \bigcup \Sigma_t^i$, $g \mid \bigcup \Sigma_t^i$ is Morse–Smale, and $L(g) - \bigcup \Sigma_t^i = P(\xi_t)$. Since ξ_t is normally hyperbolic to $\bigcup \Sigma_t^i$, g can be chosen so that $L(g) \cap (\bigcup \Sigma_t^i)$ consists of a finite set of hyperbolic periodic points. Also, g will not have cycles, so it is Ω-stable. Now, by a small approximation, one can make the stable and unstable manifolds of g meet transversely.

Evidently, the facts above remain true for η in a neighborhood $\mathscr{N}$ of ξ in $\mathscr{F}_3$ using the same U. For $t > b_0(\eta)$, let $\Sigma_{\eta_t}^i$ denote the invariant circle of $\eta_t^{n_0}$ near Σ_t^i.

Now using a trick employed in [18], one may complete the proof of (2.7.3).

Choose a countable dense set $\{a_i\}$ in U, and let $\mathscr{B}_{a_i}(\xi) = \{\eta \in \mathscr{N} : \eta_{a_i} \in MS\}$. Then $\mathscr{B}_{a_i}(\xi)$ is dense and open in $\mathscr{N}$. Let $\mathscr{T}_3^{\mathscr{N}} = \bigcap_{a_i} \mathscr{B}_{a_i}(\xi)$. Then, for $\eta \in \mathscr{T}_3^{\mathscr{N}}$, let K_η be the closure of the set of t in U such that the rotation number of $\eta_t^{n_0} | x_{\eta_t}^i$ is irrational for some (and hence all) $i = 1, \ldots, n_0$. Clearly, K_η is a perfect, nowhere dense set, and $b_0(\eta) \in K_\eta$.

We have shown that there is a residual set $\mathscr{T}_3 \subset \Phi^{k,r}$ such that if $\xi \in \mathscr{T}_3$ and satisfies the hypothesis of (2.6.3), then there is a neighborhood $\mathscr{N}$ of ξ in $\mathscr{T}_3$ and a residual subset $\mathscr{T}_3^{\mathscr{N}} \subset \mathscr{N} \cap \mathscr{T}_3$ for which the conclusion or (2.7.3) holds. From this (2.7.3) follows easily.

We now begin the proof of Theorem (2.5).

Suppose $\xi \in \mathscr{T}$, $\xi_0 \in MS$, and $L^-(\xi_{b_0})$ is finite. More conditions, each true for a residual subset of $\Phi^{k,r}$ will be imposed on ξ_{b_0} until the desired conclusions are obtained.

Case 1 Some periodic point of ξ_{b_0} has an eigenvalue λ such that $|\lambda| = 1$ and $\lambda \neq 1$.

Let $O(p_i)$ be the periodic orbit having λ as an eigenvalue. Assume $|\lambda_t| < 1$ for $t < b_0$. The other case is similar. Then $\xi_{b_0} | W^s(O(p_i))$ is topologically conjugate to $\xi_t | W^s(O(p_i(t)))$ for $t < b_0$ (i.e., $O(p_i)$ is a "weak" attractor for $\xi_{b_0} | W^s(O(p_i))$ if $\lambda \neq -1$ or if $\lambda = -1$ and there is only one periodic orbit of ξ_t near $O(p_i)$ for $t < b_0$). If $\lambda = -1$ and, for $t < b_0$, there are two periodic orbits of ξ_t near $O(p_i)$, then $\xi_{b_0} | W^u(O(p_i))$ is topologically conjugate to $\xi_t | W^u(O(p_i(t)))$ for $t > b_0$ near b_0. In any event, using the strong transversality condition and the fact that $P(\xi_{b_0})$ is finite, one sees that $L^-(\xi_{b_0})$ has no cycles. Hence, one can construct a filtration for ξ_{b_0} as in the proof of (2.7). Now, it is easy to see that one can find an intrinsic neighborhood Q_1 of ξ_{b_0} in some Q^n, and a neighborhood $\mathscr{N}$ of ξ such that for $\eta \in \mathscr{N}$, $\Omega(\eta_t)$ remains finite and hyperbolic until η_t reaches Q_1, and $L^-(\eta_{t_0})$ is finite where $t_0 = \inf\{t : \eta_t \in Q_1\}$. Moreover, an appropriate version of the λ-lemma [10] will ensure that the strong transversality condition holds also until η_t reaches Q_1 which proves (2.5) for Case 1.

From now on, as remarked earlier, we only need $r \geq 2$.

Case 2 Some periodic orbit $O(p_i)$ of ξ_{b_0} has 1 as an eigenvalue.

First suppose $L^-(\xi_{b_0})$ has no cycles. Choose small neighborhoods U of b_0 and V of $O(p_i)$ as in the proof of (2.7.1). Using a filtration for ξ_{b_0}, one can find a neighborhood $\mathscr{N}$ of ξ and an intrinsic neighborhood Q_1

of ξ_{b_0} in some Q^n such that if $\eta \in \mathscr{N}$, $\Omega(\eta_t)$ is finite and hyperbolic for $t < t_0(\eta)$, where $t_0(\eta) = \inf\{t \in I : \eta_t \in Q_1\}$. Let U_1, U_2 be the two components of $U - \{b_0\}$ with $s_1 < s_2$ for $s_1 \in U_1$, $s_2 \in U_2$. If ξ_t has no periodic orbits for $t \in U_1$, then, assuming V is small enough, any stable and unstable manifolds of periodic points of ξ_t which meet V actually must have nonempty transversal intersections in V. If this were not true, there would exist t's in U_1 such that nontransversal intersections would occur between stable and unstable manifolds of ξ_t. This cannot happen by the definition of b_0. Now appropriate applications of the λ-lemma show that for η near ξ, η_t will satisfy the strong transversality condition for $t < t_0(\eta)$. If ξ_t has two periodic orbits in V for $t \in U_1$, it may happen that there are stable and unstable manifolds of periodic points of ξ_i, arbitrarily close to $O(p_i)$ which do not intersect. However, if n_0 is the period of p_i, and $O(q_1(t))$, $O(q_2(t))$ are the periodic orbits of ξ_t, a suitable version of the tubular family construction in [11] shows there are $\xi_t^{n_0}$-invariant codimension one topological disks Σ_{1t}^i, Σ_{2t}^i tangent to the hyperbolic parts of $T\xi_t^{n_0}$ at the points in the orbits of $q_1(t)$, $q_2(t)$, respectively. If V and U are small enough, for each i, both Σ_{1t}^i and Σ_{2t}^i will separate the component of V which contains them. Further, Σ_{1t}^i and Σ_{2t}^i can be chosen so that they will coalesce into one ξ_{b_0}-invariant manifold as t arrives at b_0. Such Σ_{1t}^i, Σ_{2t}^i will exist with similar properties for η near ξ. Using this, one sees for η near ξ that no new intersections of the stable and unstable manifolds can be created for $t < t_0(\eta)$ and, hence, as before, the strong transversality condition will hold for $t < t_0(\eta)$.

Now suppose $L^-(\xi_{b_0})$ has cycles. We first remark that if U and V are as above, then we may assume U_1 is the component of $U - \{b_0\}$ in which the periodic orbits $O(q_1(t))$, $O(q_2(t))$ appear. For if this were not the case, one could show that (restricting to a smaller residual set of ξ if necessary) arbitrarily near $O(p_i)$ there would exist periodic points not in $O(p_i)$ of some ξ_t's with $t \in U_1$. Since b_0 is the first bifurcation point of ξ, this would imply that $P(\xi_{b_0})$ is infinite, for none of these periodic points could disappear before t passes b_0. More details on this appearance of periodic points will appear elsewhere.

Since $L^-(\xi_{b_0})$ has cycles and all the stable and unstable manifolds meet transversely, it follows that every cycle contains $O(p_i)$. For, since all the other periodic orbits are hyperbolic and their stable and unstable manifolds meet transversely, the existence of a cycle not containing $O(p_i)$ would imply the existence of transversal homoclinic points which would contradict the assumption that $L^-(\xi_{b_0})$ is finite. The same argument

shows that if $O(q) \neq O(p_i)$ is in a cycle, then

$$W^s(O(q)) \cap \partial W^u(O(p_i)) = \emptyset \quad \text{and} \quad W^u(O(q)) \cap \partial W^s(O(p_i)) = \emptyset.$$

(Recall from Section 2 that $W^s(O(p_i))$ and $W^u(O(p_i))$ are manifolds with boundaries.) A standard argument will show that if

$$\overline{\hat{W}^u(O(q_1))} \cap \hat{W}^s(O(q_2)) \neq \emptyset$$

for some periodic orbits $O(q_1)$, $O(q_2)$, then there is a sequence $q_1', \ldots, q_s'$ of periodic points such that $q_1' \in O(q_1)$, $q_s' \in O(q_2)$, and $\hat{W}^u(q_i') \cap \hat{W}^s(q_{i+1}') \neq \emptyset$, for $i = 1, \ldots, s-1$. Since the intersections are transversal,

$$\hat{W}^u(O(q_1)) \cap \hat{W}^s(O(q_2)) \neq \emptyset.$$

Thus the c-loop class of $O(p_i)$ (defined as in [8]) consists of exactly the periodic orbits in the cycles containing $O(p_i)$, and the other c-loop classes are single periodic orbits. Now construct a filtration for ξ_{b_0},

$$M_n \supset M_{n-1} \supset \cdots \supset M_0 = \emptyset,$$

separating the c-loop classes of $L^-(\xi_{b_0})$. Let $O(q_1), \ldots, O(q_2)$ be the periodic points in the c-loop class of $O(p_i)$ with $O(q_1) = O(q_2) = O(p_i)$ and the other orbits distinct, and let $\Lambda = \bigcup_{i=1}^{s} O(q_i)$.

We first prove that $L(\xi_{b_0}) = L^-(\xi_{b_0})$. One can actually prove in this case that $\Omega(\xi_{b_0})$ is finite, but we do not need this. Assume $\Lambda \subset M_j - M_{j-1}$, and let $\Lambda' = \bigcap_n \xi_{b_0}^n(M_j - M_{j-1})$.

First suppose $x \in W^u(\Lambda) - W^s(\Lambda)$. We show that $\omega(x) \subset M_{j-1}$. Indeed, suppose, by way of contradiction, that $x \in W^u(q_{i_1}) - W^s(\Lambda)$ for some $q_{i_1} \in \Lambda$ and $\omega(x) \not\subset M_{j-1}$.

Then, $\omega(x) \subset W^u(\Lambda) \cap \Lambda'$ and there is a point $q_{i_2} \in \Lambda$ such that $\omega(x) \cap \hat{W}^u(q_{i_2}) \neq \emptyset$.

An easy version of [8, Lemma (3.9)] shows that there is a point $x_1 \in \omega(x) \cap \hat{W}^s(q_{i_2})$. Also, q_{i_2} cannot be in $O(p_i)$, for then one could find x_1 in $\partial W^s(q_{i_2}) - \{q_{i_2}\}$, and, since $\partial W^s(q_2) - \{q_2\} \cap W^u(\Lambda) = \emptyset$, this would give a contradiction. Let $q_{i_3} \in \Lambda$ be such that $x_1 \in \hat{W}^u(q_{i_3})$. As before, $\omega(x) \cap \hat{W}^s(q_{i_3}) \neq \emptyset$, so $q_{i_3} \notin O(p_i)$, and we may find a $q_{i_4} \in \Lambda$ such that $\omega(x) \cap \hat{W}^s(q_{i_3}) \cap \hat{W}^u(q_{i_4}) \neq \emptyset$. Continuing this process would eventually yield a cycle not involving $O(p_i)$ which is impossible. Thus $\omega(x) \subset M_{j-1}$.

Now if $x \in M$ and $\omega(x) \cap (M_j - M_{j-1}) = \emptyset$, then $x \in W^s(q)$ for some periodic point q, since the c-loop classes in $M - (M_j - M_{j-1})$

are single-periodic orbits. If $\omega(x) \cap (M_j - M_{j-1}) \neq \emptyset$, then $\omega(x) \subset M_j - M_{j-1}$, and we assert that $\omega(x) = O(q_i)$ for some $q_i \in M_j - M_{j-1}$. If this were not true, then there would exist a point x_1 in $\omega(x) \cap (W^u(\Lambda) - W^s(\Lambda))$. By the previous argument, $x_1 \in W^s(M_{j-1})$. But, since $W^s(M_{j-1})$ is a ξ_{b_0}-invariant open subset of M, this implies that $x \in W^s(M_{j-1})$, which is a contradiction.

Since $L(\xi_{b_0})$ is finite, it follows that

$$W^s(\Lambda) \cap W^u(\Lambda) = \Lambda' = \bigcap_n \xi_{b_0}^n(M_j - M_{j-1}).$$

For each $t \leq b_0$ near b_0, let $\Lambda(\xi_t)$ be the set of periodic points of ξ_t in $M_j - M_{j-1}$. One easily sees that

$$W^s(\Lambda(\xi_t)) \cap W^u(\Lambda(\xi_t)) = \bigcap_n \xi_t^n(M_j - M_{j-1}).$$

For η near ξ, let $t_0(\eta)$ be the real number such that $\eta_{t_0(\eta)}$ has a quasi-hyperbolic periodic orbit $O(p_i(\eta))$ near $O(p_i)$. Using tubular family techniques, one can show that for η near ξ,

$$W^s(\Lambda(\eta_t)) \cap W^u(\Lambda(\eta_t)) = \bigcap_n \eta_t^n(M_j - M_{j-1}) \qquad \text{for} \quad t \leq t_0(\eta),$$

where $\Lambda(\eta_t)$ is the set of η_t-periodic points in $M_j - M_{j-1}$, $L(\eta_t)$ is finite and hyperbolic for $t < t_0(\eta)$, and the strong transversality condition holds for $t \leq t_0(\eta)$. Thus $t_0(\eta) = b_0(\eta)$, which proves (2.5) for this case.

REMARK Using tubular family techniques one can actually show that ξ_{b_0} is semistable in the following sense. For $t < b_0$, there is a continuous mapping $h_t: M \to M$ such that $\xi_{b_0} h_t = h_t \xi_t$. This mapping h_t can be taken to be a homeomorphism off the orbit of a small neighborhood of $O(p_i)$.

Case 3 $L^-(\xi_{b_0})$ is hyperbolic.

This case will require some more notation and some preliminary results. Let $O(p_1)$, $O(p_2)$ be the periodic orbits of ξ_b such that $\hat{W}^u(O(p_2)) \cap \hat{W}^s(O(o_1))$ has an orbit of quasi-transversal intersections. In what follows, here as well as in Section 5 and 6, we make the assumption that p_1 and p_2 are fixed points. This simplifies notation and makes the essential ideas more transparent. The general case may be obtained with straightforward modifications.

Let $x \in \hat{W}^u(p_2) \cap \hat{W}^s(p_1)$ be a point of quasi-transversal intersection, and let

$$s_1 = \dim W^s(p_1), \qquad s_2 = \dim W^s(p_2),$$
$$u_1 = \dim W^u(p_1), \qquad \text{and} \qquad u_2 = \dim W^u(p_2).$$

Let D^n be the closed unit n-disk in the euclidean space R^n. We note that p_1 may equal p_2.

We may choose neighborhoods U_1, U_2 of p_1, p_2 such that, for $i = 1, 2$, there are diffeomorphisms φ_i mapping U_i onto $D^{s_i} \times D^{u_i}$ such that

$$\varphi_i(p_i) = 0, \qquad \varphi_i^{-1}(D^{s_i}) \subset W^s(p_i), \qquad \varphi_i^{-1}(D^{u_i}) \subset W^u(p_i),$$

and

$$x \in U_1 \cap \hat{W}^u(p_2) \cap \hat{W}^s(p_1).$$

Let $\xi_{b_0} = f$. We will identify $\varphi_i f \varphi_i^{-1}$ with $f \mid U_i$ and U_i with $D^{s_i} \times D^{u_i}$. Write

$$D_i^s = \varphi_i^{-1}(D^{s_i} \times 0) \qquad \text{and} \qquad D_i^u = \varphi_i^{-1}(0 \times D^{u_i})$$

so that, with our identifications, $U_i = D_i^s \times D_i^u$, $i = 1, 2$, Let π_i^σ denote the projection

$$U_i = D_i^s \times D_i^u \to D_i^\sigma \qquad \text{for} \quad i = 1, 2, \quad \sigma = s, u.$$

If we choose x close enough to p_1, then, since x is a quasi-transversal intersection of $W^u(p_2)$, $W^s(p_1)$,

$$\dim(T_x \pi_1^u(T_x W^u(p_2))) = u_1 - 1.$$

Similarly, for m large, $f^{-m}(x) \in U_2$ and

$$\dim(T_{f^{-m}(x)} \pi_2^s(T_{f^{-m}(x)}(W^s(p_1))) = s_2 - 1.$$

Let λ be the weakest expanding eigenvalue of $T_{p_1} f$, and let μ be the weakest contracting eigenvalue of $T_{p_2} f$. Thus if $\lambda_1(\mu_1)$ is an eigenvalue of $T_{p_1} f \mid_{T_{p_1} D_1^u}(T_{p_2} f \mid_{T_{p_2} D_2^s})$ such that if $\lambda_1 \neq \lambda, \bar{\lambda}$ $(\mu_1 \neq \mu, \bar{\mu})$, then $\mid \lambda_1 \mid > \mid \lambda \mid$ ($\mid \mu_1 \mid < \mid \mu \mid$). Write $T_{p_1} D_1^u = H_1 \oplus J_1$, and $T_{p_2} D_2^s = H_2 \oplus J_2$, where H_1 (H_2) is the eigenspace corresponding to the eigenvalue $\lambda(\mu)$, and J_1 (J_2) is the sum of the other eigenspaces of $T_{p_1} f \mid_{T_{p_1} D_1^u}(T_{p_2} f \mid_{T_{p_2} D_2^s})$. For $y \in U_1$, denote by $T_y D_1^u = H_{1y} \oplus J_{1y}$, the translate of $H_1 \oplus J_1$ to y via φ_1, and for $z \in U_2$, denote the translate of $H_2 \oplus J_2$ to z via φ_2 by $H_{2z} \oplus J_{2z}$. Using invariant manifold theory [6], one sees that for

$y \in D_1^u$, $z \in D_2^s$, the coordinates

$$\varphi_1(y) = (y_1, y_2) \in \mathbb{R}^{\dim H_1} \times \mathbb{R}^{u_1 - \dim H_1},$$
$$\varphi_2(z) = (z_1, z_2) \in \mathbb{R}^{\dim H_2} \times \mathbb{R}^{s_2 - \dim H_2}$$

may be chosen so that $\varphi_1^{-1}((y_2 = 0))$, $\varphi_1^{-1}((y_1 = 0))$ are f^{-1}-invariant and tangent at p_1 to H_1, J_1, respectively, and $\varphi_2^{-1}((z_2 = 0))$, $\varphi_2^{-1}((z_1 = 0))$ are f-invariant and tangent at p_2 to H_2, J_2, respectively. Here it may only be possible to choose the coordinates φ_1, φ_2 to be C^1. Previously, they could have been taken to be C^r. When convenient, we identify H_1 with $\varphi_1^{-1}((y_2 = 0))$, J_1 with $\varphi_1^{-1}((y_1 = 0))$, etc.

Let

$$\pi_{11}: D_1^u \to H_1, \qquad \pi_{12}: D_1^u \to J_1, \qquad \pi_{21}: D_2^s \to H_2, \qquad \pi_{22}: D_2^s \to J_2,$$

be the natural projections. Note that if $D_1^u = H_1$, then π_{11} is the identity and $J_1 = 0$. A similar statement holds for π_{21} and J_2.

Assume M is furnished with a riemannian metric whose norm reduces to the usual euclidean norms on U_1 and U_2 via the coordinates φ_1 and φ_2.

If $\dim H_1 = 1$, let $H_{1+} = \{y \in H_1 : y_1 \geq 0\}$ and define a half-sector S_{1+} about H_{1+} in D_1^u by

$$S_{1+} = \{y \in D_1^u : |y_2| \leq \varepsilon y_1\}, \qquad \text{if} \quad u_1 > 1,$$
$$S_{1+} = H_{1+}, \qquad \text{if} \quad u_1 = 1.$$

Similarly, let

$$H_{1-} = \{y \in H_1 : y_1 \leq 0\}, \quad S_{1-} = \{y \in D_1^u : |y_2| \leq -\varepsilon y_1\}, \quad \text{if } u_1 > 1$$
$$S_{1-} = H_{1-}, \qquad \text{if} \quad u_1 = 1.$$

Let

$$D_{1+}^u = \pi_{11}^{-1}(H_{1+}) \qquad \text{and} \qquad D_{1-}^u = \pi_{11}^{-1}(H_{1-}).$$

Define H_{2+}, H_{2-}, S_{2+}, S_{2-}, D_{2+}, D_{2-} in D_2^s in an analogous manner if $\dim H_2 = 1$. For instance, $H_{2+} = \{z \in H_2 : z_1 \geq 0\}$, $D_{2+}^s = \pi_{21}^{-1}(H_{2+})$, etc.

Figure 4.1 gives a typical picture of these structures in a three-dimensional manifold with $\dim W^s(p_1) = \dim W^u(p_2) = 1$. The following observation will be of fundamental importance.

(4.3) For g near f, we may define all of the structures $\varphi_{ig}, p_i(g), \lambda_g$, $H_i(g)$, $x(g)$, etc., so that the new structures satisfy properties analogous to those of the structures defined for f. Further, these structures will vary continuously with g in the naturally associated topologies.

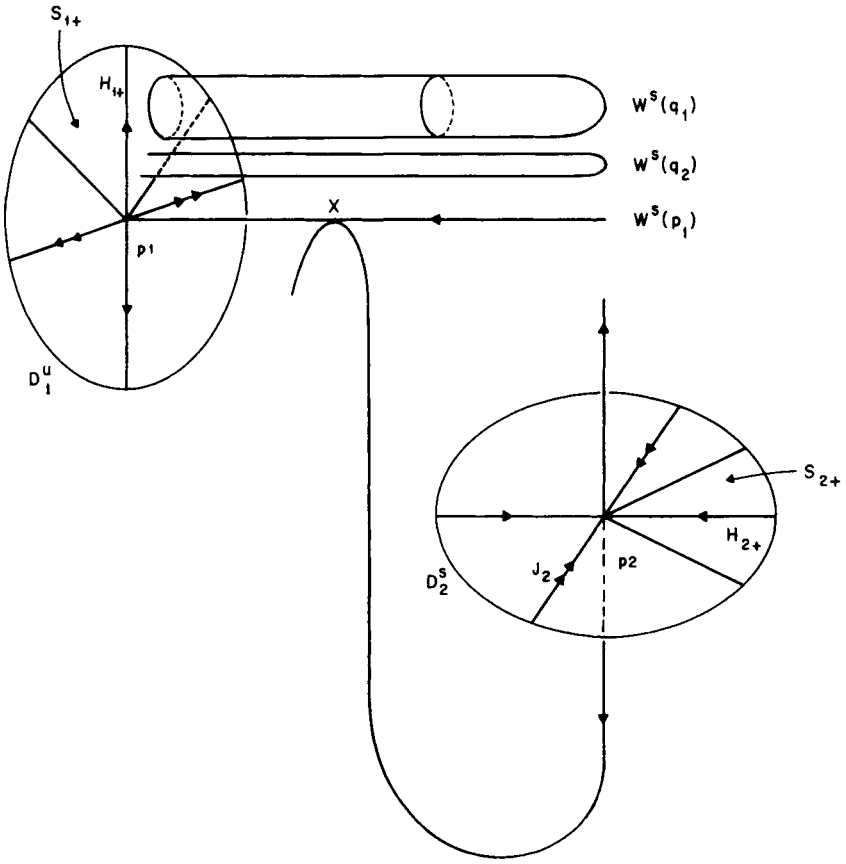


Figure 4.1

When convenient we will refer to the above structures for g as g-structures, and we will occasionally suppress notation and use the same letters for g-structures as for f-structures.

As in Section 3, given a direct sum decomposition of a normed vector space $F = E_1 \oplus E_2$ and a constant $\varepsilon > 0$, let

$$S_\varepsilon(E_1) = S_\varepsilon(E_1, E_2) = \{(v_1, v_2) \in F : |\, v_2 \,| \leq \varepsilon \,|\, v_1 \,|\}.$$

Let $k > 0$ be a constant. Define

$C_k(\varphi_1) = \{y \in D_1{}^u$: there is a periodic point $q \neq p_1$ of f such that $y \in W^s(q) \cap D_1{}^u$ and $T_y(W^s(q) \cap D_1{}^u) \cap S_k(H_{1y}, J_{1y})$ does not contain a subspace with the same dimension as $H_1\}$,

and define

$C_k(\varphi_2) = \{y \in D_2^s$: there is a periodic point $q \neq p_2$ of f such that $y \in W^u(q) \cap D_2^s$ and $T_y(W^u(q) \cap D_2^s) \cap S_k(H_{2y}, J_{2y})$ does not contain a subspace with the same dimension as $H_2\}$.

For a subset $G \subset M$, let

$$\partial_1 G = \bigcap \{\bar{F} : F \subset G \text{ and } G - F \text{ is compact}\}.$$

(4.4) LEMMA There is a residual subset $\mathscr{F}_5 \subset \mathscr{F}_4$ such that if $\xi \in \mathscr{F}_5$ and $\xi_{b_0} = f$ is as above, the coordinates φ_1, φ_2 may be chosen so that the following statements are true.

For m large:

(1) the $(u_1 - 1)$-dimensional subspace

$$T_{fm(x)}\pi_1^u(T_{fm(x)}W^u(p_2))$$

of $T_{p_1}D_1^u$ contains a subspace arbitrarily close to J_1; and

(2) the $(s_2 - 1)$-dimensional subspace

$$T_{f-m(x)}\pi_2^s(T_{f-m(x)}W^s(p_1))$$

of $T_{p_2}D_2^s$ contains a subspace arbitrarily close to J_2.

Also, there is a constant $k > 0$ such that

(3) if $C_k(\varphi_1) \neq \emptyset$, then

(a) λ is real and positive,

(b) there is a half-sector S_{1+} about H_{1+} in D_1^u such that $C_k(\varphi_1) \subset S_{1+}$ and

$$W^u(p_2) \cap \pi_1^{u-1}(S_{1+}) = O(x) \cap D_1^s,$$

(c) $\partial_1 W^u(p_2) \cap D_1^u \subset D_{1-}$;

(4) if $C_k(\varphi_2) \neq \emptyset$, then

(a) μ is real and positive,

(b) there is a half-sector S_{2+} about H_{2+} in D_2^s such that $C_k(\varphi_2) \subset S_{2+}$ and

$$W^s(p_1) \cap \pi_2^{s-1}(S_{2+}) = O(x) \cap D_2^u,$$

(c) $\partial_1 W^s(p_1) \cap D_2^s \subset D_{2-}^s$.

Proof Choose an intrinsic neighborhood Q_1 of f in some Q^n and a neighborhood $\mathscr{N}$ of ξ in $\Phi^{k,r}$ such that if $\eta \in \mathscr{N}$, then there is a unique number t_0 near b_0 such that $\eta_{t_0} \in Q_1$, η_{t_0} has a quasi-transversal intersection of stable and unstable manifolds near x, and η is transverse to Q_1 at t_0. To prove (4.4), it is sufficient to find a residual subset $\mathscr{B}_1 \subset Q_1$ such that if $\eta \in \mathscr{N}$ and $\eta_{b_0(\eta)} = g \in \mathscr{B}_1$, then g satisfies the conclusions in (4.4).

Since λ and μ are the weakest eigenvalues, it follows easily that if the conclusions of (4.4) hold for $f = \xi_{b_0}$, the will hold for η near ξ with η_{b_0} near f. Thus to prove (4.4), it remains to show that the properties may be obtained densely.

For density of (4.4.1), by Sternberg's linearization theorem [3], given $g_1 \in Q_1$, we may find a C^∞ approximation g of g_1 in Q_1 such that g is linear in some coordinates φ_{1g} near $p_1(g)$, the point $x(g)$ of quasi-transversal intersection for g is in the domain of φ_{1g}, and

$$T_{x(g)}\pi^{\mathrm{u}}_{1g} \mid T_{x(g)}W^{\mathrm{u}}(p_2(g))$$

is transverse to H_{1g}. Then (4.4.1) for g follows immediately using the linearity of $\varphi_{1g} \circ g \circ \varphi_{1g}^{-1}$.

The proof of (4.4.2) is the same.

To prove density of (4.4.3), one shows again that it is true for most η with $g = \eta_{b_0} \in Q_1$ and linear in some C^∞ coordinates φ_{1g}.

Supposing $g = \eta_{b_0}$ is C^∞ and linear in the coordinates φ_{1g} on U_{1g}, let $C_\infty(\varphi_{1g})$ be the set of points y in D^{u}_{1g} such that the linear map

$$T_y\pi_{11g} \mid_{T_y(W^{\mathrm{s}}(y) \cap D^{\mathrm{u}}_{1g})}.$$

is not onto H_{1g}. If $C_\infty(\varphi_{1g}) = \varnothing$, by linearity, there is a constant $k = k(\varphi_{1g}) > 0$, such that $C_{k_1}(\varphi_{1g}) = \varnothing$ for $k \leq k_1$.

Thus we may suppose $C_\infty(\varphi_{1g}) \neq \varnothing$. Assume, by may of contradiction, that $\dim H_{1g} = 2$. By approximating, we may assume $\lambda_g = \varrho e^{i\Theta}$ where $\Theta/2\pi$ is not rational.

For each t near $b_0(\eta)$, there is a unique critical point of

$$\pi_{11\eta_t}\pi^{\mathrm{u}}_{1\eta_t} \mid_{W^{\mathrm{u}}(p_2(\eta_t))}$$

near the point $x(g)$ of quasi-transversal intersection of $W^{\mathrm{u}}(p_2(g))$ and $W^{\mathrm{s}}(p_1(g))$. The coordinates φ_{1g} induce complex coordinates on the two-dimensional subspace H_{1g} in which $g \mid_{H_{1g}}$ has the form $z \mapsto \lambda_g z = \varrho e^{i\Theta} z$. We may assume that $T_{x(g)}\pi^{\mathrm{u}}_{1g} \mid T_{x(g)}W^{\mathrm{u}}(p_2(g))$ is transverse to H_{1g}. Now since $\Theta/2\pi$ is irrational and $C_\infty(\varphi_{1g}) \neq \varnothing$, there exist t's less than b_0

for which η_t has a nontransversal intersection of stable and unstable manifolds of periodic points. This contradicts the assumption that b_0 was the first bifurcation point of η. Continuing with the assumption that $C_\infty(\varphi_{1g}) \neq \emptyset$, a similar argument will show that λ_g is real and positive.

Using more approximations, we may arrange for the critical points in $C_\infty(\varphi_{1g})$ to be nondegenerate, and then that $C_\infty(\varphi_{1g}) \cap J_{1g} = \emptyset$.

Now it is clear that one can find a half-sector about a half-line in H_{1g} which contains $C_\infty(\varphi_{1g})$. If not, the definition of b_0 would again be contradicted. Also, the coordinates may be chosen so that $C_\infty(\varphi_{1g}) \subset S_{1+}$. By linearity, there is a constant $k > 0$ such that $C_k(\varphi_{1g}) \subset S_{1+}$. Using the definition of b_0 again, one sees that the remaining properties in (4.4.3) will also hold for g.

The proof of (4.4.4) is similar.

We now prove (2.5) when $L^-(\xi_{b_0})$ is finite, hyperbolic, and has no cycles.

Construct a filtration separating the periodic orbits of ξ_{b_0} as in the previous cases. Using this, one sees that ξ_{b_0} is Ω-stable and each g near ξ_{b_0} has $\Omega(g)$ finite hyperbolic with no cycles. Assume $\xi \in \mathscr{T}_5$, where $\mathscr{T}_5$ is as in (4.4). Choose an intrinsic neighborhood $\mathscr{N}$ of ξ in $\Phi^{k,r}$ such that for $\eta \in \mathscr{N}$, $t_0(\eta) \equiv \inf\{t : \eta_t \in Q_1\}$ is the first point where $W^u(p_2(\eta_t))$ has a quasi-transversal intersection with $W^s(p_1(\eta_t))$ near x. Then, using tubular family techniques and the λ-lemma, one sees that for $\mathscr{N}$ small and t close to b_0, the only stable and unstable manifolds of η_t which could have nontransversal intersections are those which have transversal intersections with $W^u(p_1(\eta_t))$ and $W^s(p_2(\eta_t))$, respectively. Also, the orbit of a nontransversal intersection has to have some point near x.

Now, there are neighborhoods U_1 of D_1^s and U_2 of D_2^u such that for any periodic points $q_1(\eta_t)$, $q_2(\eta_t)$ of η_t:

(4.5) (1) If $y \in W^s(q_1(\eta_t)) \cap U_1$, then $W^s(q_1(\eta_t))$ contains an s_1-disk through y near D_1^s; and

(2) If $y \in W^u(q_2(\eta_t)) \cap U_2$, then $W^u(q_2(\eta_t))$ contains a u_2-disk through y near D_2^u.

Fix k as in (4.4). Using (4.4.1), choose integers n_1, n_2 such that

$$T_{f^{n_1(x)}}\pi_1^u(T_{f^{n_1(x)}}W^u(p_2))$$

has a subspace $J_1' \subset S_{k/2}(J_1, H_1)$ such that $\dim_1' = \dim J_1$, and

$$T_{f^{-n_2(x)}}\pi_2^s(T_{f^{-n_2(x)}}W^s(p_1))$$

has a subspace $J_2' \subset S_{k/2}(J_2, H_2)$. If t is close to b_0, and y is near x, $\eta_t^{n_1}(y)$ and $\eta_t^{-n_2}(y)$ will have similar properties.

Suppose $C_k(\varphi_1) = \emptyset$. Then, by (4.5.1), and (4.5.2), if $y \in W^u(q_2(\eta_t)) \cap W^s(q_1(\eta_t))$ is near $\eta_t^{n_1}(x)$, then $W^u(q_2(\eta_t))$ is transverse to $W^s(q_1(\eta_t))$ at y. Similarly, if $C_k(\varphi_2) = \emptyset$ and $y \in W^u(q_2(\eta_t)) \cap W^s(q_1(\eta_t))$ is near $\eta_t^{-n_2}(x)$, then $W^u(q_2(\eta_t))$ is transverse to $W^s(q_1(\eta_t))$ at y.

Now suppose $C_k(\varphi_1) \neq \emptyset$ and $C_k(\varphi_2) \neq \emptyset$. Let S_{1+} and S_{2+} be as in (4.4.3) and (4.4.4). If y is a point of nontransversal intersection of $W^s(q_1(\eta_t))$ and $W^u(q_2(\eta_t))$ near $\eta_t^{n_1}(x)$, the previous arguments show that there exist an s_1-disk $\Sigma_y^{s_1} \subset W^s(q_1(\eta_t))$ a u_2-disk $\Sigma_y^{u_2}$ in $W^u(q_2(\eta_t))$ such that $y \in \Sigma_y^{s_1} \cap \Sigma_y^{u_2}$ and such that $\Sigma_y^{s_1} \cap S_{1+} \neq \emptyset$, $\Sigma_y^{s_1}$ is C^1 near $D_{1\eta_t}^{s_1}$, $\Sigma_y^{u_2} \cap S_{2+} \neq \emptyset$, and $\Sigma_y^{u_2}$ is C^1 near $\eta_t^{n_1+n_2}(D_{2\eta_t}^{u_2})$.

It follows from (4.4.3b) and (4.4.4b) that if $\mathcal{N}$ is small enough, and t is close enough to b_0, the existence of such a y implies that $t \geq t_0(\eta)$. This completes the proof.

We proceed to prove (2.5) when $L^-(\xi_{b_0})$ is finite, hyperbolic, and has a ν_0-cycle, $\nu_0 > 1$. It will be indicated that the proof also works if there is a 1-cycle and Condition (4.7) below is satisfied. The proof of (2.5.) in the remaining 1-cycle cases will appear elsewhere.

It is interesting to note that the assumptions made in the cases considered here actually imply that the limit set $L(\xi_{b_0})$ will be finite and hyperbolic. However, we will give an example at the end of Section 7 to show that there are open sets of ξ's for which $L^-(\xi_{b_0})$ is finite, hyperbolic, and has a 1-cycle, but $L^+(\xi_{b_0})$ is infinite and not hyperbolic.

Assume $\mathscr{F}_5$ is as in (4.4), $\xi \in \mathscr{F}_5$, $\xi_{b_0} = f$, and:

(4.6) $\xi_{b_0} = f$ and $L^-(f)$ is finite, hyperbolic, and has a ν_0-cycle, $\nu_0 > 1$.

We continue with the previous notation so that $W^u(p_2)$ has a quasi-transversal intersection x with $W^s(p_1)$ in U_1, p_1 and p_2 are fixed, etc. As before, one may show that the only c-loop class of $L^-(f)$ which has more than one orbit consists of cycles containing p_1 and p_2. Moreover, letting $q_1 = p_1$, $q_2, \ldots, q_\nu = p_2$ be the periodic points in the cycles, and $q_i(t)$ be the corresponding periodic points for t near b_0, one has that

$$\hat{W}^u(p_1(t)) \cap \hat{W}^s(q_i(t)) \neq \emptyset \qquad \text{for} \quad i > 1,$$
$$\hat{W}^u(q_i(t)) \cap \hat{W}^s(p_2(t)) \neq \emptyset \qquad \text{for} \quad i < \nu.$$

Also, these intersections are transversal.

It follows from (4.4) that if f is as in (4.6), then:

(4.7) (a) λ and μ are real and positive;
(b) $\partial_1(W^u(p_2)) \cap D_1^u \subset D_{1-}^u$, and $\partial_1(W^s(p_1)) \cap D_2^s \subset D_{2-}^s$;

(4.8) the coordinates φ_1, φ_2 may be chosen such that

(1) (a) the $(u_1 - 1)$-dimensional subspaces $T_{fm(x)}\pi_1^u(T_{fm(x)}W^u(p_2))$ of $T_{p_1}D_1^u$ converge to J_1 as $m \to \infty$,
(b) the $(s_2 - 1)$-dimensional subspaces $T_{f-m(x)}\pi_2^s(T_{f-m(x)}W^s(p_1))$ of $T_{p_2}D_2^s$ converge to J_2 as $m \to \infty$.

(2) there are a constant $k > 0$ and half-sectors S_{1+}, S_{2+} as in Lemma 4.4, parts 3 and 4 such that
(a) for any q_i in a cycle of f,

$$W^s(q_i) \cap D_1^u \subset S_{1+} \quad \text{and} \quad W^u(q_i) \cap D_2^s \subset S_{2+},$$

(b) $\partial_1(\bigcup_{n\geq 0} f^{-n}\pi_1^{u-1}(S_{1+})) \cap D_2^s \subset D_{2-}^s$,
(c) $\partial_1(\bigcup_{n\geq 0} f^{n}\pi_2^{s-1}(S_{2+})) \cap D_1^u \subset D_{1-}^u$.

To prove this, first note that, since $\hat{W}^u(p_1)$ has a nonempty transversal intersection with $\hat{W}^s(p_2)$, and x is a point of quasi-transversal intersection of $W^u(p_2)$ and $W^s(p_1)$, either $u_2 = u_1$ or $u_2 = u_1 - 1$. Thus, $u_1 + s_2 \leq \dim M + 1$, and, by (4.4.3) and (4.4.4), this implies $\dim H_1 = \dim H_2 = 1$.

Now (4.8.1) follows immediately from (4.4.1) and (4.4.2). If $u_2 = u_1$, (4.7) and (4.8.2a) follow immediately from (4.4.3) and (4.4.4) since

$$\dim(W^u(p_2) \cap W^s(p_1)) = 0.$$

If $u_2 = u_1 - 1$ and either of (4.7) and (4.8.2a) were not true, (4.4.3) would imply the existence of transversal intersections of $W^u(p_2)$ and $W^s(p_1)$ in $\pi_1^{u-1}(D_{1-}^u)$ which would contradict the assumption that $L^-(f)$ is finite.

It remains to prove (4.8.2b) and (4.8.2c). Observe that shrinking U_1 and U_2, the half-sectors S_{1+} and S_{2+} in (4.4) may be taken arbitrarily close to H_{1+} and H_{2+}, respectively. Also it follows from (4.4) that for U_1 small,

$$\partial_1\Big(\bigcup_{n\geq 0} f^{-n}(H_{1+})\Big) \cap D_2^s \subset D_{2-}^s, \quad \text{and} \quad \partial_1\Big(\bigcup_{n\geq 0} f^{n}(H_{2+})\Big) \cap D_1^u \subset D_{1-}^u.$$

Now (4.8.2b) and (4.8.2c) are obtained by choosing S_{1+} near H_{1+} and S_{2+} near H_{2+}.

As before, let $M_n \supset \cdots \supset M_0 = \emptyset$ be a filtration for f, and for t near b_0, let $O(q_i(t))$ be the periodic orbit near $O(q_i)$, for $i = 1, \ldots, \nu$. Choose j such that $O(q_i) \subset M_j - M_{j-1}$. Let

$$\Lambda = \bigcup_{i=1}^{\nu} O(q_i). \qquad \text{and} \qquad \Lambda' = \bigcap_n f^n(M_j - M_{j-1}).$$

At this point, one can prove [assuming f is as in (4.6)] that $L(f)$ is finite and $\Omega(f) = L(f) \cup O(x)$. Also, there is a neighborhood V of x in M such that if there are integers $n_1 < n_2$ and a point $y \in M$ such that $f^{n_1}(y)$ and $f^{n_2}(y)$ are in V, then $f^{n_1}(y) \in \pi_1^{u-1}(S_{1+})$ and there is an integer n_1' between n_1 and n_2 such that $f^{n_1'}(y) \in \pi_2^{s-1}(S_{2+})$. The argument to be used is a slight modification of the one used to prove that $L(f)$ is finite if $L^-(f)$ is finite, has a cycle, and has some periodic point with one as an eigenvalue. For instance, to prove $L^+(f)$ is finite, one first shows that if $y \in W^u(\Lambda) - W^s(\Lambda)$, then $\omega(y) \subset M_{j-1}$. This is proved as follows: Assuming the contrary, one obtains a point in

$$\omega(y) \cap W^u(p_1) \cap W^s(\Lambda) \cap (\pi_1^{u-1}(S_{1+}) - D_1^s).$$

Using (4.8.2b) one then gets a point in

$$\omega(y) \cap W^u(\Lambda) \cap (D_2^s - S_{2+})$$

which contradicts (4.8.2a). Then, using the fact that $W^s(M_{j-1})$ is open, it follows as before that $L^+(f)$ is finite. The other assertions at the beginning of this paragraph are proved similarly.

Now if η is near ξ and $t_0(\eta)$ is the first time for which η_t has a quasi-transversal intersection of $W^u(p_2(\eta_t))$ and $W^s(p_1(\eta_t))$ near x, then for $t < t_0(\eta)$,

$$W^u(p_2(\eta_t)) \subset \pi_{1\eta_t}^{u-1}(D_{1-}^u) - D_1^s \subset W^s(M_{j-1}).$$

It follows that $L(\eta_t)$ is finite and hyperbolic for $t \leq t_0(\eta)$ and $L(\eta_t)$ has no cycles for $t < t_0(\eta)$. On the other hand, it follows from (4.8) for η that the strong transversality condition also holds for $t < t_0(\eta)$. This proves (2.5) when f satisfies (4.6).

Continuing with the assumptions that $\xi \in \mathscr{F}_5$ and $\xi_{b_0} = f$, suppose $L^-(f)$ is finite hyperbolic, has only a 1-cycle, and (4.7) holds with $p_1 = p_2$. Then it can be shown that (4.8) also holds. Moreover, arguments similar to the ones just given will prove (2.5) for this case as well. One proves that $L(f)$ is finite, $\Omega(f) = L(f) \cup O(x)$, etc.

This completes the proof of (2.5) except when $L^-(f)$ is finite hyperbolic, has a 1-cycle, and fails to satisfy (4.7). As has been mentioned, the proof in this remaining case will appear elsewhere.

5

In this section, we prove Theorem (2.9). We also show that the assumptions in this theorem are satisfied by an open set of ξ in $\Phi^{k,r}$, $k \geq 1$, $r \geq 2$.

Continuing with the notation of Section 4, let $\mathscr{F}_5$ be as in (4.4), and suppose $\xi \in \mathscr{F}_5$ and $\xi_{b_0} = f$. Assume also that:

(5.1) $L^-(f)$ is finite, hyperbolic, has a cycle, and f satisfies (4.7).

Then, the results of Section 4 show that f satisfies (4.8) as well, and that $L(f)$ is finite and hyperbolic. The assumption that f satisfies (4.7) will be needed, since we are not necessarily assuming there is a j-cycle with $j > 1$.

Let λ_0 be the minimum of the absolute values of the eigenvalues of $T_{p_1}f \mid T_{p_1}D_1^{\mathrm{u}}$ which are different from λ, and let μ_0 be the maximum of the absolute values of the eigenvalues of $T_{p_2}f \mid T_{p_2}D_2^{\mathrm{s}}$ which are different from μ.

The main assumption we need is

(5.2) (1) $\lambda^2 < \lambda_0$, $\mu^2 > \mu_0$;

(2) there is a constant $k > 0$ such that given $\varepsilon > 0$, there are intervals $B_1 \subset H_{1+}$, $B_2 \subset H_{2+}$ such that

(a) $\operatorname{dist}(B_1, p_1) < \varepsilon$, $\operatorname{dist}(B_2, p_2) < \varepsilon$,

(b) $C_k(\varphi_1) \cap \pi_{11}^{-1}(B_1) = \varnothing$,

(c) $C_k(\varphi_2) \cap \pi_{21}^{-1}(B_2) = \varnothing$,

(d) $$\frac{l(B_1)}{\operatorname{dist}(B_1, p_1)} \cdot \frac{l(B_2)}{\operatorname{dist}(B_2, p_2)} \geq (1 + k) \min\{\lambda, \mu^{-1}\},$$

where $l(B_i)$ is the length of B_i, $i = 1, 2$;

(3) $\dim W^{\mathrm{u}}(p_1) = \dim W^{\mathrm{u}}(p_2)$.

To give some feeling for the meaning of (5.2.2), we mention that if there are periodic points q', q'' of f such that $\hat{W}^{\mathrm{u}}(p_1) \subset W^{\mathrm{s}}(q')$, $\dim W^{\mathrm{s}}(q') = \dim M$, $\hat{W}^{\mathrm{s}}(p_2) \subset W^{\mathrm{u}}(q'')$, and $\dim W^{\mathrm{u}}(q'') = \dim M$, then (5.2.2) is implied by the condition $\mu + \lambda^{-1} < 1$.

It follows from (5.2.1) and the invariant manifold theory in [6] that we may assume the coordinates φ_1 and φ_2 chosen such that they satisfy all the properties of Section 4, are of class C^2, and such that

$$f(\pi_1^{u-1}(H_1) \cap f^{-1}(U_1)) \subset \pi_1^{u-1}(H_1),$$
$$f^{-1}(\pi_2^{s-1}(H_2) \cap f(U_2)) \subset \pi_2^{s-1}(H_2)).$$

Moreover, (4.3) is still valid using the C^2 topology for φ_1 and φ_2. The present technique seems to require these facts for (2.9) to hold. If one moves to C^∞ maps with the C^∞ topology, then using linearization theorems, one can get these properties to be satisfied for a dense open set of f's, so assumption (5.2.1) can be omitted. On the other hand, it may be possible to obtain Theorem (2.9) even in the C^2 topology without assuming (5.2.1).

(5.3) PROPOSITION The set of ξ in $\mathscr{F}_5$ such that ξ_{b_0} satisfies (5.1) and (5.2) is open in $\Phi^{k,r}$, $k \geq 1$, $r \geq 2$.

Proof The openness of (5.1) is contained in Section 4, so we need only consider (5.2). We show (5.2) is an open condition on f in $\mathscr{D}^r$, from which (5.3) follows using Theorem (2.5).

It is clear that (5.2.1) and (5.2.3) are open conditions.

Now note that the methods of Section 4 show that if f is close to g which is C^∞ and φ_i is near a linearization of g, then (5.2.2b) and (5.2.2c) are true for f with some constant, $k > 0$. Therefore, it remains to examine (5.2.2d).

Let B_1 and B_2 satisfy (5.2.2d). Recall that for g near f, B_{1g} and B_{2g} will be defined as in (4.3).

We claim that if B_1, B_2 are close enough to p_1, p_2, there is a neighborhood $\mathscr{N}$ of f in $\mathscr{D}^r$ such that for $g \in \mathscr{N}$ and any integers $m \geq 0$, $n \geq 0$, (5.2.2d) holds for g with B_{1g} and B_{2g} replaced by $g^{-n}(B_{1g})$ and $g^m(B_{2g})$, respectively. From this, since $g^{-n}(B_{1g}) \to p_1$ and $g^m(B_{2g}) \to p_2$ as $\min(m, n) \to \infty$, (5.2.2d) for g follows. Hence (5.3) will be proved.

To prove the claim, we first show that if B_1 is close to p_1, then for $n > 0$,

(1) $$\frac{l(f^{-n}(B_1))}{\operatorname{dist}(f^{-n}(B_1), p_1)} \quad \text{is close to} \quad \frac{l(B_1)}{\operatorname{dist}(B_1, p_1)}.$$

Since the estimates in the proof will be preserved under small perturbation, it will follow that the analogous statement for g near f and B_{1g} will be true. Then, similar methods will yield the corresponding results for g near f and B_{2g}.

Let B_{11} be the interval in H_{1+} from p_1 to the point of B_1 nearest p_1. First note that, for $n > 0$,

$$\frac{l(f^{-n}(B_1))}{\operatorname{dist}(f^{-n}(B_1), p_1)} = \frac{l(f^{-n}(B_1))}{l(f^{-n}(B_{11}))} = \frac{\alpha_n}{\beta_n} \frac{l(f^{-(n-1)}(B_1))}{l(f^{-(n-1)}(B_{11}))},$$

where α_n is the derivative of $f^{-1}|_{H_1}$ at some point in $f^{-(n-1)}(B_1)$ and β_n is the derivative of $f^{-1}|_{H_1}$ at some point in $f^{-(n-1)}(B_{11})$.

Thus,

$$\frac{l(f^{-n}(B_1))}{\operatorname{dist}(f^{-n}(B_1), p_1)} = \frac{\alpha_n}{\beta_n} \cdot \frac{\alpha_{n-1}}{\beta_{n-1}} \cdots \frac{\alpha_1}{\beta_1} \frac{l(B_1)}{l(B_{11})},$$

and (1) will follow if we show that B_1 close to p_1 implies $\prod_{j=1}^{n} \alpha_j/\beta_j$ is close to 1 for any $n > 0$.

Let

$$k_1 = \sup_{y \in D_1^u \cap H_1} |T_y^2 f^{-1}| \qquad \text{and} \qquad \zeta = l(B_1 \cup B_{11})$$

The mean value theorem gives that for $1 \leq j \leq n$,

$$|\alpha_j - \beta_j| \leq k_1 l(f^{-(j-1)}(B_1 \cup B_{11})).$$

If $\varepsilon > 0$ is small, and B_1 is close to p_1, then

$$l(f^{-(j-1)}(B_1 \cup B_{11})) \leq (\lambda^{-1} + \varepsilon)^{j-1}\zeta, \qquad \text{and} \qquad \beta_j \geq (\lambda^{-1} - \varepsilon) \quad \text{for} \quad 1 \leq j \leq n.$$

Thus

$$|\alpha_j - \beta_j| \leq k_1 \zeta (\lambda^{-1} + \varepsilon)^{j-1}$$

$$\frac{|\alpha_j|}{|\beta_j|} \leq 1 + \frac{|\alpha_j - \beta_j|}{|\beta_j|}$$

$$\leq 1 + k_2 \zeta (\lambda^{-1} + \varepsilon)^{j-1}, \qquad \text{where} \quad k_2 = k_1/(\lambda^{-1} - \varepsilon).$$

(2) Therefore,

$$\prod_{j=1}^{n} \frac{\alpha_j}{\beta_j} \leq \prod_{j=1}^{n} (1 + k_2 \zeta (\lambda^{-1} + \varepsilon)^{j-1}).$$

Now $\zeta \to 0$ as $B_1 \to p_1$, so given $\varepsilon_1 > 0$ we may find ε small enough and B_1 close enough to p_1 so that

$$\sum_{j=1}^{\infty} k_2 \zeta (\lambda^{-1} + \varepsilon)^{j-1} < \varepsilon_1. \tag{3}$$

Further, for $\varepsilon_2 > 0$, we may choose ε_1 small enough so that (2) and (3) imply that for any $n > 0$, $\prod_{j=1}^{n} \alpha_j/\beta_j \leq 1 + \varepsilon_2$. Similarly, one shows for B_1 close to p_1 and $n > 0$, $\prod_{j=1}^{n} \beta_j/\alpha_j \leq 1 + \varepsilon_2$, which proves $\prod_{j=1}^{n} \alpha_i/\beta_j$ is close to 1, and hence proves (1).

The proof of Theorem (2.9) is contained in the next result.

(5.4) There is an open dense set $\mathscr{F}_6 \subset \mathscr{A}^{k,r}$, $k \geq 1$, $r \geq 2$, such that if $\xi \in \mathscr{F}_6$, and ξ_{b_0} satisfies (5.1) and (5.2), then given $\varepsilon > 0$, there is a t_1 with $b_0 < t_1 < b_0 + \varepsilon$ such that $\xi_{t_1} \in AS$ and $\Omega(\xi_{t_1})$ is infinite and zero-dimensional. Moreover, the minimum period of the periodic points which are not isolated points of $\Omega(\xi_{t_1})$ goes to infinity as ε approaches zero.

REMARK Recall that a basic set for ξ_{t_1} is a maximal closed invariant subset of $\Omega(\xi_{t_1})$ with a dense orbit. It will be shown that the $\Omega(\xi_{t_1})$ in (5.4) has one infinite basic set. Under assumptions (5.1), (5.2.1), and (5.2.2), if one replaces (5.2.3) with the assumption that $\dim W^u(p_2) = \dim W^u(p_1) + 1$ (the only other possibility), then appropriate modifications of the proof of (5.4) show that for $\varepsilon > 0$, there is a t_1 such that $b_0 < t_1 < b_0 + \varepsilon$ and $\xi_{t_1} \in AS$. This t_1 is to be chosen so that the property analogous to the one in statement (2) of the proof of Lemma (6.4) below holds. However, for this t_1, $\Omega(\xi_{t_1})$ will not necessarily be infinite, and, even if it is, there may be two infinite basic sets. More precisely, let Λ be the set of periodic orbits in the cycles of f containing p_1 and p_2. If

$$W^s(\Lambda) \cap \pi_{11}^{-1}(B_1) = \varnothing \qquad \text{and} \qquad W^u(\Lambda) \cap \pi_{21}^{-1}(B_2) = \varnothing,$$

then $\xi_{t_1} \in MS$. If

$$W^s(\Lambda) \cap \pi_{11}^{-1}(B_1) = \varnothing, \qquad \text{but} \qquad W^u(\Lambda) \cap \pi_{21}^{-1}(B_2) \neq \varnothing,$$

then $\Omega(\xi_{t_1})$ will have one infinite basic set containing $p_1(\xi_{t_1})$. If

$$W^s(\Lambda) \cap \pi_{11}^{-1}(B_1) \neq \varnothing, \qquad \text{but} \qquad W^u(\Lambda) \cap \pi_{21}^{-1}(B_2) = \varnothing,$$

$\Omega(\xi_{t_1})$ will have one infinite basic set containing $p_2(\xi_{t_1})$. Finally, if

$$W^s(\Lambda) \cap \pi_{11}^{-1}(B_1) \neq \varnothing, \qquad \text{and} \qquad W^u(\Lambda) \cap \pi_{21}^{-1}(B_2) \neq \varnothing,$$

then $\Omega(\xi_{t_1})$ will have two disjoint infinite basic sets one of which contains $p_1(\xi_{t_1})$ and the other of which contains $p_2(\xi_{t_1})$. (See Example 4 of Section 7.)

Before continuing, we state the final open dense conditions needed for (5.4). Notice that by (4.4.1), the subspace $T_x\pi_1{}^u(T_xW^u(p_2))$ may be taken

to be transverse to H_1, so the manifold $\pi_1^{u-1}(H_1)$ is transverse to $W^u(p_2)$ at x. Thus, since $\dim W^u(p_2) = \dim W^u(p_1)$, $\pi_1^{u-1}(H_1) \cap W^u(p_2)$ is a one-dimensional manifold near x. The first additional condition needed for (5.4) is the following:

(5.5) If v is a nonzero tangent vector in $T_xW^u(p_2) \cap T_xW^s(p_1)$, there is a C^2 curve

$$\gamma\colon [-1, 1] \to W^u(p_2) \cap \pi_1^{u-1}(H_1)$$

such that $\gamma(0) = x$, $\gamma'(0) = v$, and $\pi_1{}^u\gamma$ has a nonzero second derivative at zero.

Now let $\bar{n}$ be the least positive integer such that for $n \geq \bar{n}, f^{-n}(x) \in U_2$. We may assume $f^{-\bar{n}}(x) \in \operatorname{int} U_2$. For each $y \in U_i$, let D^u_{iy} be the coordinate u-disk in U_i parallel to $D_i{}^u$ which contains $y, i = 1, 2$. Define D^s_{iy} similarly. If $z \in D^u_{2y}$ for $y \in f^{-\bar{n}}(U_1) \cap U_2$, set $\tilde{D}^u_{2z} = f^{\bar{n}}(D^u_{2y})$. By (5.5), if z is close to p_2,

$$\pi_1{}^u \mid_{\tilde{D}^u_{2z} \cap \pi_1^{u-1}(H_1)}$$

has one nondegenerate critical point $\varrho(z)$ near x. Standard results about nondegenerate critical points give that ϱ is a C^1 function on a neighborhood of p_2 in M.

The final condition needed for (5.4) is

(5.6) The function $\pi_1{}^u\varrho$ is a C^1 diffeomorphism from a neighborhood of p_2 in H_2 onto a neighborhood of p_1 in H_1.

Observe that the methods in the proof of (4.4) show that if γ is as in (5.5), the second derivative of $\pi_1{}^u\gamma$ at zero is negative, and if ϱ is as in (5.6), the mapping $\pi_1{}^u\varrho$ is orientation reversing on the small neighborhood of p_2 in H_2. Also (5.5) and (5.6) are open dense conditions on $f = \xi_{b_0}$.

We begin the proof of (5.4).

Continuing with the notation above, assume $\xi_{b_0} = f$ satisfies (5.1), (5.2), (5.5), and (5.6), and that all the necessary auxiliary structures are defined as before. In particular, $p_1 = q_1, \ldots, q_\nu = p_2$ are the periodic points in the cycles, p_1 and p_2 are fixed, $\hat{W}^u(p_2) \cap \hat{W}^s(p_1) = O(x)$ in the cycles, p_1 and p_2 are fixed, $\hat{W}^u(p_2) \cap \hat{W}^s(p_1) = O(x)$ is an orbit of quasi-transversal intersections, $x \in U_1$, and if $p_1 \neq p_2$, $\bigcup_{i=1}^{\nu-1} W^u(q_i)$ is transverse to $\bigcup_{i=1}^{\nu} W^s(q_1)$.

Let

$$\Lambda_1 = \left(\bigcup_{i=1}^{\nu} W^u(q_i)\right) \cap \left(\bigcup_{i=1}^{\nu} W^s(q_i)\right) - O(x),$$

$$\Lambda_2 = W^s(\Lambda_1) \cap W^u(\Lambda_1) = \Lambda_1 \cup O(x).$$

Using filtration techniques, we may find compact neighborhoods V_1 of Λ_1 and V_3 of x such that

(5.7) (1) $V_1 \cap V_3 = \emptyset$, $V_3 \subset (f^{-1}(V_1) - V_1) \cap (f(V_1) - V_1)$;
(2) $\Lambda_2 = \bigcap_n f^n(V_1 \cup V_3) = \Omega(f) \cap (V_1 \cup V_3)$;
(3) $V_1 \supset (\bigcup_{i=1}^{\nu} W^s(q_i)) \cap U_1$ and $V_1 \supset (\bigcup_{i=1}^{\nu} W^u(q_i)) \cap U_2$;
(4) $\bigcap_{n\geq 0} f^n(V_1) \subset W^u(\Lambda_1)$, $\bigcap_{n\geq 0} f^{-n}(V_1) \subset W^s(\Lambda_1)$, $\bigcap_{-\infty<n<\infty} f^n(V_1) = \Lambda_1$.

Put $V_2 = V_1 \cup V_3$. Then, if g is close to f, $\Omega(g) \subset \Omega_1(g) \cup \Lambda_2(g)$, where $\Omega_1(g)$ is a finite hyperbolic set near $P(f) - V_2$ and $\Lambda_2(g) = \bigcap_n g^n(V_2)$.

We will prove

(5.8) There are t_1's arbitrarily close to b_0 such that:

(1) $\Lambda_2(\xi_{t_1})$ is a hyperbolic set;

(2) ξ_{t_1} satisfies the strong transversality condition.

Once (5.8) is proved, one shows with direct arguments similar to those in [16] that $\dim \Lambda_2(\xi_{t_1}) = 0$ and the homoclinic points of p_1 are dense in $\Lambda_2(\xi_{t_1})$. Then it follows from [8] that $\Lambda_2(\xi_{t_1})$ is topologically transitive with periodic points dense. Further, it will follow from the proof of (5.8) that the minimum period of the periodic points in $\Lambda_2(\xi_{t_1})$ goes to infinity as t_1 approaches b_0. This will prove (5.4).

We remark in passing that it follows immediately from (5.8) and the results in [8] that $\xi_{t_1} \in AS$, so if one is not interested in the zero-dimensionality of $\Omega(\xi_{t_1})$, the corresponding arguments in [16] may be avoided.

We proceed to prove (5.8.1).

Assume $\dim W^u(p_1) = \dim W^u(p_2) = u$, and $\dim W^s(p_1) = s$. Setting

$$E_z^s = T_z W^s(z), \qquad E_z^u = T_z W^u(z) \qquad \text{for} \quad z \in \Lambda_1,$$

one can show easily that Λ_1 is a hyperbolic set for f. Assume U_1 and U_2 and the riemannian metric on M are chosen so that the metric is adapted to

Λ_1 (see [5] for the definition). When convenient, we will write

$$T_yD_i^u = T_yD_{iy}^u \quad \text{and} \quad T_yD_i^s = T_yD_{iy}^s,$$

where D_{iy}^s and D_{iy}^u are the coordinate disks defined before (5.6), $i = 1, 2$. Given a splitting $T_yM = E_y^s \oplus E_y^u$ and $\beta > 0$, the sectors $S_\beta(E_y^s)$, $S_\beta(E_y^u)$ will be defined as in Section 3.

We may assume U_1 and U_2 are small enough so that there are constants $\beta_1, \beta_2 > 0$, $\lambda' > 1$ such that

(5.9) (1) $\beta_1\beta_2 < 1$;

(2) For $y \in U_i \cap f^{-1}(U_i)$,

$$T_yf(S_{\beta_2}(T_yD_i^u)) \subset S_{\beta_2}(T_{f(y)}D_i^u), \quad \text{and} \quad T_yf|_{S_{\beta_2}(T_yD_i^u)}$$

is a λ'-expansion, $i = 1, 2$;

(3) For $y \in U_i \cap f(U_i)$,

$$T_yf^{-1}(S_{\beta_1}(T_yD_i^s)) \subset S_{\beta_1}(T_{f^{-1}(y)}D_i^s), \quad \text{and} \quad T_yf^{-1}|_{S_{\beta_1}(T_yD_i^s)}$$

is a λ'-expansion, $i = 1, 2$.

Further, we may assume V_1 is chosen small enough such that the splitting $E^s \oplus E^u$ of $T_{\Lambda_1}M$ extends to a (not necessarily invariant) splitting $E_y^s \oplus E_y^u$, for $y \in V_1$, so that the following is true. If we define

$$E_{1y} = E_y^s \quad \text{for} \quad y \in V_1 - (U_1 \cup U_2),$$
$$E_{1y} = T_yD_i^s \quad \text{for} \quad y \in U_i, \quad i = 1, 2,$$

and

$$E_{2y} = E_y^u \quad \text{for} \quad y \in V_1 - (U_1 \cup U_2),$$
$$E_{2y} = T_yD_i^u \quad \text{for} \quad y \in U_i, \quad i = 1, 2,$$

then $E_{1y} \oplus E_{2y}$ is a splitting of T_yM, for $y \in V_1 \cup U_1 \cup U_2$, and there are positive functions α_1, α_2 defined on $V_1 \cup U_1 \cup U_2$ and a constant $\lambda_1 > 1$ such that

(5.10) (1) For $i = 1, 2$, α_i is continuous and bounded away from zero on $V_1 - (U_1 \cup U_2)$, and $\alpha_i(y) = \beta_i$ for $y \in U_i$;

(2) (a) For $y \in V_1 \cap f^{-1}(V_1)$,

$$T_yf(S_{\alpha_2(y)}(E_{2y})) \subset S_{\alpha_2 f(y)}(E_{2f(y)}) \quad \text{and} \quad T_yf|_{S_{\alpha_2(y)}(E_{2y})}$$

is a λ_1-expansion,

(b) For $y \in V_1 \cap f(V_1)$,

$$T_y f^{-1}(S_{\alpha_1(y)}(E_{1y})) \subset S_{\alpha_1 f^{-1}(y)}(E_{1f^{-1}(y)}) \text{ and } T_y f^{-1}|_{S_{\alpha_1(y)}(E_{1y})}$$

is a λ_1-expansion.

One sees this by first obtaining the splitting $E_y^s \oplus E_y^u$ on V_1 and continuous functions α_i' on V_1 satisfying properties similar to (5.10.2), and then defining $\alpha_i(y)$ to be $\alpha_i'(y)$ or β_i at the appropriate points.

Remark (4.3) applies equally well to all the structures in (5.10). We will occasionally suppress notation and use the same letters for structures associated to g near f.

Next we state the fundamental lemma needed for the proof of (5.8.1).

(5.11) LEMMA For $\varepsilon > 0$ and $\lambda_2 > \lambda_1 > 1$, there exist t_1 with $b_0 < t_1 < b_0 + \varepsilon$, integers n_0, n_0', n_1, n_2, with $n_0 > n_0' > n_1 + n_2$, and a real number $e > 0$ such that if $g = \xi_{t_1}$ and $y \in V_3 \cap \bigcap_{-n_0 \le j \le n_0} g^{-j}(V_2)$, then:

(1) $g^n(y) \notin V_3$, for $0 < |n| < n_0'$;
(2) $T_y g^{n_2}(S_e(E_{2y})) \subset S_{\alpha_2 g^{n_2}(y)}(E_{2g^{n_2}(y)})$;
(3) $T_y g^{n_2}$ is a λ_2-expansion on $S_e(E_{2y})$;
(4) $T_y g^{-n_1}(T_y M - S_e(E_{2y})) \subset S_{\alpha_1 g^{-n_1}(y)}(E_{1g^{-n_1}(y)})$;
(5) $T_y g^{-n_1}$ is a λ_2-expansion on $T_y M - S_e(E_{2y})$.

The proof of Lemma (5.11) will be deferred to Section 6. Note that (5.11.1) implies that $g^j(y) \in V_1$ for $-n_1 \le j \le n_2$.

Let

$$V_3^{n_0} = V_3 \cap \bigcap_{-n_0 \le j \le n_0} g^{-j}(V_2).$$

For

$$y \in V_2 - \bigcup_{-n_1 < j < n_2} g^j(V_3^{n_0})$$

let $\varepsilon_i(y) = \alpha_i(y)$, $i = 1, 2$. For $y \in g^{j-n_1}(V_3^{n_0})$, define $\varepsilon_i(y)$ so that

$$S_{\varepsilon_2(y)}(E_{2y}) = T_{g^{-j}(y)} g^j(S_{\varepsilon_2 g^{-j}(y)}(E_{2g^{-j}(y)})),$$

and

$$S_{\varepsilon_1(y)}(E_{1y}) = T_{g^{r_1}(y)} g^{-r_1}(S_{\varepsilon_1 g^{r_1}(y)}(E_{1g^{r_1}(y)})),$$

where $r_1 = n_1 + n_2 - j$.

With these definitions, we claim

(5.12) There is an integer $n_3 > n_0$ such that if $y \in \bigcap_{-n_3 \le j \le n_3} g^{-j}(V_2)$, then

(1) $T_y g^{n_3}(S_{\varepsilon_2(y)}(E_{2y})) \subset S_{\varepsilon_2 g^{n_3}(y)}(E_{2g^{n_3}(y)})$;

(2) $T_y g^{n_3}$ is a λ_1-expansion on $S_{\varepsilon_2(y)}(E_{2y})$;

(3) $T_y g^{-n_3}(S_{\varepsilon_1(y)}(E_{1y})) \subset S_{\varepsilon_1 g^{-n_3}(y)}(E_{1g^{-n_3}(y)})$;

(4) $T_y g^{-n_3}$ is a λ_1-expansion on $S_{\varepsilon_1(y)}(E_{1y})$.

Once (5.12) is proved, the hyperbolicity Theorem (3.1) applies to prove (5.8.1). We only prove (5.12.1) and (5.12.2) since the other proofs are similar.

Given an integer n, a point $y \in M$, and a subset $F \subset T_y M$, define

$$m(T_y g^n \mid F) = \inf\{ \mid T_y g^n(v) \mid / \mid v \mid : v \in F, v \neq 0\}.$$

Let

$$k_1 = \inf\{m(T_y g^n \mid S_{\varepsilon_2(y)}(E_{2y})) : y \in \bigcap_{-n_1 \le j \le n_2} g^{-j}(V_2),\ -n_1 \le n \le n_2\},$$

and let $k = \frac{1}{2}\min(\lambda_1, k_1)$.

For $m \ge n_0$, define

$$V_m = \bigcap_{-m \le j \le m} g^{-j}(V_2) = \bigcap_{-m \le j \le m} g^{-j}(V_1 \cup V_3),$$

and for $y \in V_m$, write $S_{2y} = S_{\varepsilon_2(y)}(E_{2y})$.

Note that if

$$y \in V_m - \Big(\bigcup_{0 \le j < n_2} g^j(V_3^{n_0})\Big), \qquad m > n_2,$$

then $T_y g \mid S_{2y}$ is a λ_1-expansion since $\varepsilon_2(y) \le \alpha_2(y)$ except on $\bigcup_{0 \le j < n_2} g^j(V_3^{n_0})$.

Let $\nu_1 > 0$ be a positive integer such that $\lambda_1^{\nu_1} k^2 > \lambda_1$, and let $n_3 > n_0 + \max(n_1, n_2)$ be large enough so that $(n_3 - \nu_1)/n_2 - 1 > \nu_1$. We claim that this n_3 satisfies the conditions of (5.12).

Let $y \in V_{n_3}$. We make some observations.

(5.13) (1) If $s \ge 0$ is such that $n_3 - s n_2 < \nu_1$, then $s - 1 > \nu_1$,

(2) If $g^j(y) \notin V_3^{n_0}$, for $0 < j < n \le n_3$, then

(a) $T_y g^n(S_{2y}) \subset S_{2g^n(y)}$, and

(b) $m(T_y g^n \mid S_{2y}) \ge \lambda_1^{n - n_2} k$;

(3) If $y \in V_3^{n_0}$ and $g^j(y) \notin V_3^{n_0}$ for $0 < j < n$, $n_2 < n < n_3$, then

$$m(T_y g^n \mid S_{2y}) \geq \lambda_1^{n-n_2}\lambda_2 \geq \lambda_1^{n-n_2+1}.$$

(5.13.1) follows from the definition of n_3. For (5.13.2a), first note that since $n_3 > n_0 + \max(n_1, n_2)$,

$$V_{n_3} \cap \Big(\bigcup_{-n_1 \leq j \leq n_2} g^j(V_3)\Big) = V_{n_3} \cap \Big(\bigcup_{-n_1 \leq j \leq n_2} g^j(V_3^{n_0})\Big).$$

Now suppose $y \in V_{n_3}$ and $g^j(y) \notin V_3^{n_0}$ for $0 < j < n \leq n_3$. If

$$y \in \bigcup_{-n_1 \leq j < 0} g^j(V_3) = \bigcup_{-n_1 \leq j < 0} g^j(V_3^{n_0}),$$

then (5.13.2a) follows easily from the definition of ε_2. Observe that, by (5.11.4), the fact that $S_{\alpha_1(z)}(E_{1z}) \cap S_{\alpha_2(z)}(E_{2z}) = (0)$, for $z \in g^{-n_1}(V_3^{n_0})$, and the definition of ε_2, it follows that $S_{\varepsilon_2(z)}(E_{2z}) \subset S_e(E_{2z})$, for $z \in V_3^{n_0}$. Thus, if $y \in V_3^{n_0}$, then

$$T_y g^{n_2}(S_{2y}) \subset S_{\alpha_2(y)}(E_{2g^{n_2}(y)}) = S_{2g^{n_2}(y)},$$

by (5.11.2). Also, by the definition of ε_2, if

$$y \in \bigcup_{0 \leq j \leq n_2} g^j(V_3) = \bigcup_{0 \leq j \leq n_2} g^j(V_3^{n_0}),$$

then (5.13.2a) holds. Assume now that $y \notin \bigcup_{-n_1 \leq j \leq n_2} g^j(V_3)$. If

$$g^n(y) \notin \bigcup_{-n_1 < j \leq 0} g^j(V_3^{n_0}),$$

then

$$\varepsilon_2 g^j(y) = \alpha_2 g^j(y) \qquad \text{for} \quad 0 \leq j \leq n,$$

so (5.13.2a) holds. On the other hand, if there is a first integer $m < n$ such that $g^m(y) \in g^{-n_1}(V_3^{n_0})$, then $T_y g^m(S_{2y}) \subset S_{2g^m(y)}$ as in the previous case, and

$$T_y g^n(S_{2y}) = T_{g^m(y)} g^{n-m} \circ T_y g^m(S_{2y}) \subset T_{g^m(y)} g^{n-m}(S_{2g^m(y)}) \subset S_{2g^n(y)}$$

by the definition of $\varepsilon_2(y)$.

To prove (5.13.2b), first recall that if $y \notin \bigcup_{0 \leq j \leq n_2} g^j(V_3^{n_0})$, then $m(T_y g \mid S_{2y}) \geq \lambda_1$, and if $y \in g^j(V_3^{n_0})$, for some $0 \leq j < n_2$, then $m(T_y g^{n_2-j} \mid S_{2y}) \geq k$. Thus, under the assumption of (5.13.2), if

$y \in g^j(V_3^{n_0})$ for some $0 \leq j < n_2$, then

$$m(T_y g^n \mid S_{2y}) \geq m(T_{g^{n_2-j}(y)} g^{n-(n_2-j)} \mid S_{2g^{n_2-j}(y)}) \cdot m(T_y g^{n_2-j} \mid S_{2y})$$
$$\geq \lambda_1^{n-(n_2-j)} k \geq \lambda_1^{n-n_2} k.$$

On the other hand, if $y \notin \bigcup_{0 \leq j < n_2} g^j(V_3^{n_0})$, then $\varepsilon_2 g^j(y) = \alpha_2 g^j(y)$, for $0 \leq j < n$, so

$$m(T_y g^n \mid S_{2y}) \geq \lambda_1^n > \lambda_1^{n-n_2} k.$$

(5.13.3) is obvious using (5.11.3), and the above mentioned facts that for $y \in V_3^{n_0}$,

$$S_{2y} \subset S_e(E_{2y}) \quad \text{and} \quad m(T_z g \mid S_{2z}) \geq \lambda_1 \quad \text{for} \quad z \notin \bigcup_{0 \leq j < n_2} g^j(V_3^{n_0}).$$

Now we indicate how (5.12.1) and (5.12.2) follow from (5.13). For $y \in V_{n_3}$, if $g^m(y) \notin V_3^{n_0}$, for $0 < m \leq n_3$, let $m_1 = n_3$. Otherwise, let $0 = m_0 < m_1 < \cdots < m_s \leq n_3$ be the successive integers such that $g^{m_i}(y) \in V_3^{n_0}$. Then $g^n(y) \notin V_3^{n_0}$, for $m_i < n < m_{i+1}$, and (5.13.2a) gives that for $i \geq 1$,

$$T_{g^{m_i(y)}}(S_{2g^{m_i(y)}}) \subset S_{2g^{m_{i+1}(y)}},$$

from which (5.12.1) follows easily.

Now, if $y \in V_3^{n_0} \cap V_{n_3}$, then $g^n(y) \notin V_3^{n_0}$, for $0 < n < n_0'$ and $n_0' > n_2$, so, for $i \geq 2$, $m_i - m_{i-1} > n_2$. Thus, by (5.13.2b) and (5.13.3),

$$m(T_y g^{n_3} \mid S_{2y}) \geq \lambda_1^{m_1-n_2} k \lambda_1^{m_2-m_1-n_2+1} \cdots \lambda_1^{m_s-m_{s-1}-n_2+1} \lambda_1^{n_3-m_s} k$$
$$= \lambda_1^{n_3-sn_2+s-1} k^2.$$

Now, $n_3 - sn_2 \geq (s-1)n_2 \geq 0$ and $s - 1 \geq 0$, so

$$\lambda_1^{n_3-sn_2+s-1} k^2 \geq \max\{\lambda_1^{n_3-sn_2} k^2, \lambda_1^{s-1} k^2\} \geq \lambda_1^{r_1} k^2 > \lambda_1.$$

This proves (5.12.2) and hence (5.12).

Thus, to complete the proof of (5.8.1), it remains to prove Lemma (5.11). This and (5.8.2) will be proved in Section 6.

6

Here the technical Lemma (5.11) and the strong transversality condition of (5.8.2) are proved. This completes the proof of (5.8).

Continuing with the notation of Section 5, we may assume V_3 is such

that $f^{-\bar{n}}(V_3) \subset \operatorname{int} U_2$, that the critical point $\varrho(z)$ of

$$\pi_1^{\mathrm{u}} \mid \tilde{D}_{2z}^{\mathrm{u}} \cap \pi_1^{\mathrm{u}-1}(H_1)$$

is in V_3, and that the u-disk $\tilde{D}_{2z}^{\mathrm{u}}$ contains $\tilde{D}_{2z}^{\mathrm{u}} \cap V_3$ in its interior.

Using the techniques in [11], we may construct surjective invariant tubular family retractions $r_1: U_{11} \to D_1^{\mathrm{u}}$, $r_2: U_{21} \to D_2^{\mathrm{s}}$ on compact neighborhoods U_{11} of $f^{-1}(D_1^{\mathrm{s}})$ and U_{21} of $f(D_2^{\mathrm{u}})$, so that

(1) (a) for $y \in r_1(U_{11})$, $r_1^{-1}(y)$ is a C^r s-disk,
(b) $r_1 f^{-1} = f^{-1} r_1$ on a neighborhood $U_{12} \subset U_{11}$ of D_1^{s},
(c) the map $z \mapsto T_z r_1^{-1}(r_1(z))$ is continuous into the Grassmann bundle of s-planes over U_{11}; and

(2) (a) for $y \in r_2(U_{21})$, $r_2^{-1}(y)$ is a C^r u-disk,
(b) $r_2 f = f r_2$ on a neighborhood $U_{22} \subset U_{21}$ of D_2^{u},
(c) the map $z \mapsto T_z r_2^{-1}(r_2(z))$ is continuous into the Grassmann bundle of u-planes over U_{21}.

Also, r_1 may be taken so that $r_1^{-1}(H_1) \cap U_1 = \pi_1^{\mathrm{u}-1}(H_1)$. Note that $D_1^{\mathrm{s}} = \pi_1^{\mathrm{u}-1}(p_1) = r_1^{-1}(p_1) \cap U_1$. Recall that $\pi_{11}: D_1^{\mathrm{u}} \to H_1$ and $\psi_{21}: D_2^{\mathrm{s}} \to H_2$ are the natural projections. Let

$$\psi_{11} = \pi_{11} r_1 \mid U_1 : U_1 \to H_1 \qquad \text{and} \qquad \psi_{21} = \pi_{21} r_2 \mid U_2 : U_2 \to H_2.$$

A pair (A, B) of closed intervals in H_1 will be called *linked pair* in H_1 if A lies to the left of B in H_1 (with respect to the real coordinates defined by φ_1) and $A \cap B$ is a single point. Similarly, one defines linked pairs in H_2. A pair of intervals $(I_1, I_2) \subset H_1$ will be called *overlapping* if $I_1 \cap I_2$ is an interval of positive length whose endpoints belong to $\operatorname{int} I_1 \cup \operatorname{int} I_2$. If A is a closed interval in H_1 (H_2), $\operatorname{int} A$ means its interior in H_1 (H_2). Let k be as in (4.4). Define

$$W_{\mathrm{s},k} = \{y \in U_1 : T_y W^{\mathrm{s}}(y) \subset S_{k/2}(J_{1y} \oplus T_y D_1^{\mathrm{s}}, H_{1y})\},$$
$$W_{\mathrm{u},k} = \{y \in U_2 : T_y W^{\mathrm{u}}(y) \subset S_{k/2}(J_{2y} \oplus T_y D_2^{\mathrm{u}}, H_{2y})\}.$$

(6.1) LEMMA If U_1 and U_2 are small enough, there are constants $0 < c_1 < 1$, $c_2 > 0$, $c_3 > 0$; linked pairs $(B_{11}, B_1) \subset H_1$, $(B_{21}, B_2) \subset H_2$; a neighborhood U_1' of $B_{11} \cup B_1$ in U_1, and a neighborhood U_2' of $B_{21} \cup B_2$, with the following property: Given $\varepsilon > 0$ and $n_4 > 0$, there are integers $n', m' > n_4$ and a pair of overlapping intervals $(I_2, I_{21}) \subset H_1$ such that

(1) B_1 and B_2 satisfy (5.2.2);

(2) $p_1 \in \operatorname{int}(f^{-n'}(B_{11}) \cap I_{21})$, $p_2 \in \operatorname{int}(f^{m'}(B_{21}))$;

(3) $l(I_{21}) < c_1 l(f^{-n'}(B_1)),\ l(f^{-n'}(B_{11})) < c_1 l(I_2 - I_{21})$;

(4) $l(f^{-n'}(B_1)) > c_2(\lambda - \varepsilon)^{-n'}$;

(5) $l(f^{m'}(B_2)) > c_2(\mu - \varepsilon)^{m'}$;

(6) $l(f^{-n'}(B_{11})) < c_3 l(f^{-n'}(B_1)),\ l(f^{m'}(B_{21})) < c_3 l(f^{m'}(B_2))$;

(7) $l(f^{-n'}(B_1))^{-1/2} < c_3 l(f^{m'}(B_2))^{-1/2}$;

(8) $r_1\varrho(\pi_{21}^{-1}(f^{m'}(B_{21})) \cap f^{m'}(U_2')) \subset I_{21},\ r_1\varrho(f^{m'}(U_2')) \supset I_1 \cup I_{12}$;

(9) (a) $\operatorname{dist}(W_{s,k} \cap \psi_{11}^{-1}(f^{-n'}(B_{11} \cup B_1)), \psi_{11}^{-1}(f^{-n'}(B_1)) > c_2 l(f^{-n'}(B_1))$,

(b) $\operatorname{dist}(W_{u,k} \cap \psi_{21}^{-1}(f^{m'}(B_{21} \cup B_2)), \psi_{21}^{-1}(f^{m'}(B_2)) > c_2 l(f^{m'}(B_2))$;

(10) (a) $\psi_{11}^{-1}(f^{-n'}(B_{11} \cup B_1)) \cap W_{s,k} \subset f^{-n'}(U_1')$,

(b) $\psi_{21}^{-1}(f^{m'}(B_{21} \cup B_2)) \cap W_{u,k} \subset f^{m'}(U_2')$.

Proof Choose B_1 and B_2 as in (5.2.2). At first, we let B_{11} be the interval in H_{1+} from p_1 to the point of B_1 nearest p_1 (i.e., the connected component of $H_{1+} - \operatorname{int} B_1$ containing p_1), and let B_{21} be the interval in H_{2+} from p_2 to the point of B_2 nearest p_2. This is convenient for the estimates below. At the end of the proof, we will stretch B_{11} and B_{21} slightly so that (6.1.2) will hold. By the proof of (5.3), if B_1 and B_2 are close enough to p_1 and p_2, respectively, then (5.2.2) is still satisfied if one replaces B_1 by $f^{-n}(B_1)$ and B_2 by $f^m(B_2)$, for any $n, m \geq 0$.

Let $\bar{\varrho} = r_1\varrho$. We will find integers $n', m' > n_4$ so that, setting $I_2 = \bar{\varrho} f^{m'}(B_2)$, $I_{21} = \bar{\varrho} f^{m'}(B_{21})$, (6.1.2)–(6.1.7) and (6.1.9) are satisfied. Then we will move the lower endpoint of I_{21} into I_2 so that (6.1.3) is still satisfied. After this, U_1' and U_2' can be chosen so that (6.1.8) and (6.1.10) will hold.

Let $F_{1y} = T_y r_1^{-1}(r_1(y))$, for $y \in U_{11}$. Since H_1 is the weakest expanding eigenspace of $T_{p_1} f$ and $\dim H_1 = 1$, one can use a modification of the proof of Theorem (6.3b) of [5] and methods in [4] to show that, for U_1 small, $F_1 \mid U_1$ is partially C^1 in the following sense. There is a real number $\delta > 0$ such that if $\gamma\colon [0, 1] \to U_1$ is a C^1 curve such that

$$\gamma'(s) \in S_\delta(H_{1\gamma(s)} \oplus T_{\gamma(s)} D_1^s, J_{1\gamma(s)}) \qquad \text{for all} \quad s,$$

then the curve $F_1\gamma$ is C^1.

For $\delta > 0$, call an arc $\gamma\colon [0, 1] \to D_1^u$ a *δ-arc* if

$$\gamma'(s) \in S_\delta(H_{1\gamma(s)}, J_{1\gamma(s)}) \qquad \text{for all} \quad s,$$

Since F_1 is partially C^1, $r_1 \mid_{\pi_1^{u-1}(H_1)}$ is a C^1 fibration, and there are constants $\delta, \tau > 0$ such that

(6.2) (1) If γ is a δ-arc in D_1^u, then

$$\operatorname{dist}(r_1^{-1}\gamma(0), r_1^{-1}\gamma(1)) \geq \tau \operatorname{dist}(\gamma(0), \gamma(1));$$

(2) if $y, z \in U_1$ are such that $r_1(y)$ can be joined to $r_1(z)$ by a δ-arc, then

$$\operatorname{dist}(\pi_{11}\pi_1^u y, \pi_{11}\pi_1^u z) \geq \tau \operatorname{dist}(\pi_{11} r_1 y, \pi_{11} r_1 z);$$

(3) if y and z are such that $z \in D_{1y}^u$ (i.e., $\pi_1^s(y) = \pi_1^s(z)$) and $r_1(y)$ can be joined to $r_1(z)$ by a δ-arc, then

$$\operatorname{dist}(F_{1y}, F_{1z}) \leq \tau^{-1} \operatorname{dist}(\pi_{11}\pi_1^u y, \pi_{11}\pi_1^u z),$$

where the first distance is in the Grassmann bundle of s-planes over U_1.

Of course $H_{1\gamma(s)}$ above is to be understood as the one-dimensional subspace of $T_{\gamma(s)}U_1$ parallel to H_1.

Assume $\min(\lambda, \mu^{-1}) = \mu^{-1}$. The other case may be treated replacing f by f^{-1}.

Now pick $0 < c_1 < 1$, ε small, and B_1, B_2 as in (5.2.2) such that:

(1a) $$c_1^2 \frac{l(f^{-n}(B_1))}{\operatorname{dist}(f^{-n}(B_1), p_1)} \cdot \frac{l(f^m(B_2))}{\operatorname{dist}(f^m(B_2), p_2)} \geq \mu^{-1} + 2\varepsilon$$
for $n, m \geq 0$,

(1b) $$\lambda - \varepsilon \leq |\, T_y f|_{H_1} \,| \leq \lambda + \varepsilon \qquad \text{for } y \in H_1 \cap U_1,$$

(1c) $$\mu^{-1} - \frac{\varepsilon}{2} \leq |\, T_y f^{-1}|_{H_2} \,| \leq \mu^{-1} + \frac{\varepsilon}{2} \qquad \text{for } y \in H_2 \cap U_2.$$

Now, $\bar{\varrho}(p_2) = p_1$, and by (5.9.2), and the fact that $r_1 \mid \pi_1^{u-1}(H_1)$ is a C^1 fibration, one has that $\bar{\varrho} \mid H_{2+}$ is a C^1 diffeomorphism of a neighborhood of p_2 in H_2 onto a neighborhood of p_1 in H_1. Thus for m large,

$$\frac{l(\bar{\varrho} f^m(B_2))}{\operatorname{dist}(\bar{\varrho} f^m(B_2), p_1)} = \frac{l(\bar{\varrho} f^m(B_2))}{l(\bar{\varrho} f^m(B_{21}))} \quad \text{is near} \quad \frac{l(f^m(B_2))}{l(f^m(B_{21}))}$$

Therefore, we may choose m_0 large enough so that

(2) $$c_1^2 \frac{l(f^{-n}(B_1))}{\operatorname{dist}(f^{-n}(B_1), p_1)} \cdot \frac{l(\bar{\varrho} f^m(B_2))}{\operatorname{dist}(\bar{\varrho} f^m(B_2), p_1)} \geq \mu^{-1} + \varepsilon,$$
for $n \geq 0$, $m \geq m_0$.

Choose n such that

$$c_1 l(f^{-n}(B_1)) < \operatorname{dist}(\bar{\varrho} f^{m_0}(B_2), p_1),$$

and let $m > m_0$ be the least integer such that

$$\operatorname{dist}(\bar{\varrho} f^{m}(B_2), p_1) < c_1 l(f^{-n}(B_1)). \tag{3}$$

Then,

$$\begin{aligned} c_1 l(f^{-n}(B_1)) &\leq \operatorname{dist}(\bar{\varrho} f^{m-1}(B_2), p_1) \\ &\leq \operatorname{dist}(\bar{\varrho} f^{m}(B_2), p_1) k_1 k_2 (\mu^{-1} + \varepsilon/2), \end{aligned}$$

where $k_1 = |\, T_{y_1}\bar{\varrho}\,|_{H_2}|$ and $k_2 = |\, T_{y_2}\bar{\varrho}^{-1}\,|_{H_1}|$, for some points $y_1 \in f^m(B_2)$, $y_2 \in \bar{\varrho} f^m(B_2)$.

If $m > m_0$ is large, $k_1 k_2$ is near 1, so we may assume (taking n larger also) that

$$\operatorname{dist}(\bar{\varrho} f^{m-1}(B_2), p_1) < (\mu^{-1} + \varepsilon) \operatorname{dist}(\bar{\varrho} f^{m}(B_2), p_1).$$

Hence,

$$\frac{c_1 l(f^{-n}(B_1))}{(\mu^{-1} + \varepsilon) \operatorname{dist}(\bar{\varrho} f^{m}(B_2), p_1)} < 1,$$

so

$$\begin{aligned} \frac{c_1 l(\bar{\varrho} f^{m}(B_2))}{\operatorname{dist}(f^{-n}(B_1), p_1)} &> \frac{c_1 l(\bar{\varrho} f^{m}(B_2))}{\operatorname{dist}(f^{-n}(B_1), p_1)} \\ &\quad \cdot \frac{c_1 l(f^{-n}(B_1))}{(\mu^{-1} + \varepsilon) \operatorname{dist}(\bar{\varrho} f^{m}(B_2), p_1)} \geq 1, \end{aligned} \tag{4}$$

by (2).

Now (3) and (4) give (6.1.3) for $I_{21} = \bar{\varrho} f^m(B_{21})$, $I_2 = \bar{\varrho} f^m(B_2)$, and $n' = n$, $m' = m$.

Also, (6.1.4) and (6.1.5) follow from (1b) and (1c).

(6.1.6) follows from an argument similar to one used in the proof of (5.3). One shows that

$$\frac{l(f^{-n'}(B_{11}))}{l(f^{-n'}(B_1))} \leq \prod_{i=1}^{\infty} \left(1 + \frac{k_3}{\lambda^{-1} - \varepsilon} (\lambda^{-1} + \varepsilon)^i\right) < \infty$$

and

$$\frac{l(f^{m'}(B_2))}{l(f^{m'}(B_{21}))} \leq \prod_{i=1}^{\infty} \left(1 + \frac{k_3}{\mu - \varepsilon} (\mu + \varepsilon)^i\right) < \infty$$

for some constant k_3 and ε small enough for the product to converge.

For (6.1.7), note that the method of proof of (6.1.6) will also show $l(f^{m'}(B_{21})) > c_2 l(f^{m'}(B_2))$ for c_2 small, since $f \mid H_2$ is a diffeomorphism. Thus,

$$l(f^{-n'}(B_1)) > \frac{1}{c_1} l(I_{21})$$

$$= \frac{1}{c_1} l(\bar{\varrho} f^{m'}(B_{21})) \geq \frac{1}{c_1} k l(f^{m'}(B_{21})) > \frac{k c_2}{c_1} l(f^{m'}(B_2))$$

where $k = \inf\{ \mid T_y \bar{\varrho}^{-1} \mid : y \text{ is in the domain of } \bar{\varrho}^{-1}\}$.

Observe that the n' and m' needed for the above properties can be chosen arbitrarily large. This will be true below also, so we can satisfy the condition $n', m' > n_4$.

To prove (6.1.8), enlarge I_{21} slightly by setting it equal to $\bar{\varrho}(f^{m'}(B'_{21}))$, where $B'_{21} \supset B_{21}$ is obtained by sliding the left endpoint of B_{21} into B_2. This can be done so that (6.1.3) still holds. Now choosing U_2' so that $f^m(U_2') \cap D_2^s$ is very close to $f^m(B_{21} \cup B_2)$, for all $m \geq 0$, one may obtain (6.1.8).

We proceed to prove (6.1.9a) and (6.1.10a). The other proofs are similar.

Fix k as in (4.4) and (5.2).

Then $\operatorname{dist}(\pi_{11}^{-1}(B_1), W_{s,k} \cap D_1^u) > 0$. Also, by (4.4(3b)), $W_{s,k} \cap D_1^u$ is in a half-sector about H_{1+} in D_1^u, so $f^{-j}(W_{s,k} \cap D_1^u)$ approaches H_{1+} as $j \to \infty$ (i.e., $f^{-j}(W_{s,k} \cap D_1^u)$ gets in arbitrarily small half-sectors about H_{1+}).

Assuming U_1 is small enough, the methods of the proof of (5.3) can be used to show there are neighborhoods $\tilde{U}$ of B_1, $\tilde{V}$ of $W_{s,k} \cap \pi_{11}^{-1}(B_1 \cup B_{11})$ in M such that $\tilde{U} \cap \tilde{V} = \emptyset$, and

$$\operatorname{dist}(f^{-j}(\tilde{U} \cap D_1^u), f^{-j}(\tilde{V} \cap D_1^u)) > c l(f^{-j}(B_1)) \qquad \text{for} \quad j \geq 0,$$

where c is a constant independent of $l(f^{-j}(B_1))$.

Also, we may pick $\tilde{U}$, $\tilde{V}$ so that $r_1(\tilde{U}) = \tilde{U} \cap D_1^u$ and $r_1(\tilde{V}) = \tilde{V} \cap D_1^u$. Then,

(a) $f^{-j}(\tilde{U} \cap D_1^u) = f^{-j}(\tilde{U}) \cap D_1^u = r_1 f^{-j}(\tilde{U})$, and

(b) $r_1^{-1} f^{-j}(\tilde{V} \cap D_1^u) = r_1^{-1} f^{-j} r_1(\tilde{V})$

$$= r_1^{-1}(r_1(U_{11} \cap f^{-j}(\tilde{V}))) = U_{11} \cap f^{-j}(\tilde{V}).$$

Since $f^{-j}(W_{s,k} \cap D_1^u)$ approaches H_{1+}, we may assume

(c) $\operatorname{dist}(W_{s,k} \cap \psi_{11}^{-1} f^{-j}(B_1 \cup B_{11}), \psi_{11}^{-1} f^{-j}(B_1))$

$$\geq \operatorname{dist}(W_{s,k} \cap \psi_{11}^{-1}(f^{-j}(B_1 \cup B_{11})), r_1^{-1}[\pi_{11}^{-1} f^{-j}(B_1) \cap f^{-j}(\tilde{U})]).$$

By (6.2.1), there is a constant $c_2 > 0$ so that

$$\operatorname{dist}(r_1^{-1}f^{-j}(\tilde{U} \cap D_1{}^{u}), r_1^{-1}f^{-j}(\tilde{V} \cap D_1{}^{u})) > c_2 l(f^{-j}(B_1)) \quad \text{for } j \geq 0.$$

Thus, by (a) and (b), we have

(d) $\operatorname{dist}(r_1^{-1}(f^{-j}(\tilde{U}) \cap D_1{}^{u}), U_{11} \cap f^{-j}(\tilde{V})) > c_2 l(f^{-j}(B_1))$, for $j \geq 0$.

It is easily checked that for U_1 small and j large,

$$U_{11} \cap f^{-j}(W_{s,k} \cap \tilde{V}) \supset W_{s,k} \cap r_1^{-1}\pi_{11}^{-1} f^{-j}(B_1 \cap B_{11}),$$

so,

(e) $U_{11} \cap f^{-j}(\tilde{V}) \supset W_{s,k} \cap r_1^{-1}\pi_{11}^{-1} f^{-j}(B_1 \cup B_{11})$.

Now, using (c)–(e), and the obvious fact that

$$\pi_{11}^{-1}(f^{-j}(B_1)) \cap f^{-j}(\tilde{U}) \subset f^{-j}(\tilde{U}) \cap D_1{}^{u},$$

we get

$$\operatorname{dist}(W_{s,k} \cap r_1^{-1}\pi_{11}^{-1} f^{-j}(B_1 \cup B_{11}), r_1^{-1}\pi_{11}^{-1} f^{-j}(B_1)) > c_2 l(f^{-j}(B_1)) \quad \text{for large } j,$$

which is (6.1.9a). Similarly, it is clear from (e) that U_1' may be chosen to satisfy (6.1.10a).

Recall that $V_2 = V_1 \cup V_3$ is a neighborhood of Λ_2 such that $\bigcap_n f^n(V_2) = \Lambda_2$.

Let $M = M_{r_1} \supset M_{r_1-1} \supset \cdots \supset M_1 \supset \emptyset$ be a filtration for f such that $V_2 \subset \operatorname{int}(M_{i_0} - M_{i_0-1})$. Note that $f(M_{i_0-1}) \subset \operatorname{int} M_{i_0-1}$ and $f^{-1}(M - M_{i_0}) \subset \operatorname{int}(M - M_{i_0})$.

(6.3) LEMMA If U_1 and U_2 are small enough, and U_1' and U_2' are as in (6.1), there is an integer $m_1 > 0$ such that for $n \geq 0$,

(1a) $f^{m_1+n}\psi_{11}^{-1} f^{-n}(B_1 \cup (H_{1-} - B_{11})) \subset \operatorname{int} M_{i_0-1}$,

(1b) $f^{m_1+n}r_1^{-1}(\pi_{11}^{-1} f^{-n}(B_1 \cup B_{11}) \cap f^{-n}(U_1')) \subset \operatorname{int} M_{i_0-1}$;

and

(2a) $f^{-m_1-n}\psi_{21}^{-1} f^{-n}(B_2 \cup (H_{2-} - B_{21})) \subset \operatorname{int}(M - M_{i_0})$,

(2b) $f^{-m_1-n}r_2^{-1}(\pi_{21}^{-1} f^{n}(B_2 \cup B_{21}) \cap f^{n}(U_2')) \subset \operatorname{int}(M - M_{i_0})$.

Proof We prove the first part of (6.3.1a); that is, there is an integer $m_1 > 0$ such that for $n \geq 0$,

$$f^{m_1+n}\psi_{11}^{-1} f^{-n}(B_1) \subset \operatorname{int} M_{i_0-1}.$$

The rest of the proof is similar.

Let $W_1^s = \bigcup_{i=2}^{\nu} W^s(q_i)$. Since $\dim W^u(p_1) = \dim W^u(p_2)$, it follows that $\dim W^s(q_i) = \dim W^s(p_1)$ for $i = 1, \ldots, \nu$. By the λ-lemma, if U_1 is small, $W_1^s \cap U_1 \subset W_{s,k}$.

Using the methods of the proof of (6.1.9), one can show that if B_1 and B_{11} are close enough to p_1 and U_1' is small enough, the following is true. There is an interval $B_3 \subset H_{1+}$ such that $B_3 \supset B_1$ and

(1) for $n \geq 0$, $\quad f^n(\psi_{11}^{-1} f^{-n}(B_1) \cap f^{-n}(U_1')) \subset \psi_{11}^{-1}(B_3) \cap U_1'$,

(2) for $0 \leq j \leq n$, $\quad f^j \psi_{11}^{-1} f^{-n}(B_1) \subset \psi_{11}^{-1} f^{j-n}(B_3) \subset \psi_{11}^{-1}(U_1' \cap H_1)$,

(3) $[\psi_{11}^{-1}(B_3) \cup (\psi_{11}^{-1}(U_1' \cap H_1) - U_1')] \cap W_1^s = \emptyset$.

Now recall that a proper fundamental domain for $W^u(p_1)$ is a compact set $\tilde{G} \subset W^u(p_1)$ such that $p_1 \notin \tilde{G}, f^2(\tilde{G}) \cap \tilde{G} = \emptyset$, and $\{p_1\} \cup \bigcup_{n \in \mathbb{Z}} f^n(\tilde{G}) = W^u(p_1)$. Given such a $\tilde{G}$, let G be a compact neighborhood of $\tilde{G}$ in M such that $f^2(G) \cap G = \emptyset$ (G is a proper fundamental neighborhood for $W^u(p_1)$). We may assume $B_1 \subset \operatorname{int} G$, $W_1^s \cap G \subset U_1'$, and $G \supset f^{-m_0}(D_1^u - f^{-1}(D_1^u))$, for some integer m_0. Let G' be a compact subset of G such that $G' \cap W_1^s = \emptyset$, and $\psi_{11}^{-1}(B_3) \cup (\psi_{11}^{-1}(U_1' \cap H_1) - U_1') \subset G'$. For each $y \in G'$, the ω-limit set $\omega(y) \subset M_{i_0-1}$, so there is an integer $m_1 > 0$ such that $f^n(G') \subset \operatorname{int}(M_{i_0-1})$, for $n \geq m_1$.

Now the choice of G' and Eq. (1) show that for $n \geq 0$,

$$f^n(\psi_{11}^{-1} f^{-n}(B_1) \cap f^{-n}(U_1')) \subset \psi_{11}^{-1}(B_3) \cup U_1' \subset \psi_{11}^{-1}(B_3) \subset G'.$$

Suppose $y \in \psi_{11}^{-1} f^{-n}(B_1) - f^{-n}(U_1')$, and let $0 < j \leq n$ be such that $f^{j-1}(y) \in U_1'$, but $f^j(y) \notin U_1'$. Then

$$f^j(y) \in f^j \psi_{11}^{-1} f^{-n}(B_1) - U_1' \subset \psi_{11}^{-1}(U_1' \cap H_1) - U_1',$$

by (2), so $f^j(y) \in G'$. Hence $f^{m_1+n}(y) \in \operatorname{int} M_{i_0-1}$, for $n \geq 0$.

Therefore, for any $n \geq 0$,

$$f^{m_1+n} \psi_{11}^{-1} f^{-n}(B_1) = f^{m_1+n}(\psi_{11}^{-1} f^{-n}(B_1) \cap f^{-n}(U_1')) \cup f^{m_1+n}(\psi_{11}^{-1} f^{-n}(B_1) - f^{-n}(U_1')) \subset \operatorname{int} M_{i_0-1}.$$

For $n > 0$ and g near f, put

$$V_3^n(g) = V_3 \cap \bigcap_{-n \leq j \leq n} g^j(V_2).$$

Recall that $f^{-n}(x) \in \operatorname{int} U_2$, for $n \geq \bar{n}$. The structures in Lemma (6.4) are to be understood as the g-structures corresponding to the associated f-structures as indicated in (4.3).

(6.4) LEMMA Let c_2, c_3 be as in (6.1). There are two more constants $c_4, c_5 > 0$ such that for $\varepsilon > 0$ and $n_4 > 0$, there exist $b_0 < t_1 < b_0 + \varepsilon$ and integers n_0, n_0', n', m' such that, putting $g = \xi_{t_1}$,

(1a) $n_0 > \max(n_0', n' + m')$, $\min(n_0', n', m') > n_4$,

(1b) $\operatorname{dist}(B_1, p_1) < \varepsilon$, $\operatorname{dist}(B_2, p_2) < \varepsilon$,

(2) $l(g^{-n'}(B_1)) > c_2(\lambda - \varepsilon)^{-n'}$, $l(g^{m'}(B_2)) > c_2(\mu - \varepsilon)^{m'}$,

(3) $l(g^{-n'}(B_{11})) < c_3 l(g^{-n'}(B_1))$, $l(g^{m'}(B_{21})) < c_3 l(g^{m'}(B_2))$,

(4) if $y \in V_3^{n_0}(g)$, then

(a) $g^j(y) \notin V_3$ for $0 < |j| < n_0'$,

(b) $T_y(\tilde{D}^{\mathrm{u}}_{2y}) \subset S_{c_5 l(g^{-n'}(B_1))^{-1/2}}(E_{2y})$,

(c) $T_y g^{-\bar{n}}(T_y M - S_{2c_5 l(g^{-n'}(B_1))^{-1/2}}(E_{2y})) \subset S_{c_6 l(g^{m'}(B_2))^{-1/2}}(E_{1g^{-\bar{n}}(y)})$,

(d) $y \in \bigcap_{0 \le j \le n'} g^{-j}(U_1)$, $g^{-\bar{n}}(y) \in \bigcap_{0 \le j \le m'} g^j(U_2)$.

Proof Fix M_{i0}, M_{i_0-1}, and m_1 as in (6.3). Examining the proofs of (6.1) and (6.3), one sees that for g near f, the facts in (6.1) and (6.3) remain true for the corresponding g-structures. Thus, given $\varepsilon > 0$ and $n_4 > m_1$, if we choose n' and m' as in (6.1), then (6.3.1b), (6.3.2), and (6.3.3) are true for $\xi_{t_1} = g$ near f.

From (4.8.1a), it follows that there are a neighborhood V_4 of x and an integer $m_0 > 0$ such that if $z \in V_4$ and g is close to f, $T_z\tilde{D}^{\mathrm{u}}_{2z}$ contains a subspace J'_{1z} of dimension $u - 1$ such that

$$T_z g^{m_0}(J'_{1z}) \subset S_{\alpha_2 g^{m_0}(z)}(E_{2y}). \tag{1}$$

It follows from [19] that, since ξ is transverse to the codimension-one submanifolds $Q^n \subset \mathscr{D}^r$ at b_0, the parameter t moves $W^{\mathrm{u}}(p_2)$ through $W^{\mathrm{s}}(p_1)$ at x so that, for $b_0 < t < b_0 + \varepsilon$, $W^{\mathrm{u}}(p_2(\xi_t))$ has two points of transversal intersection with $W^{\mathrm{s}}(p_1(\xi_t))$ near x, and $r_{1t}\varrho_t(\tilde{D}_{2p_2}(\xi_t)) > 0$ in the real coordinate in $H_1(\xi_t)$.

Thus, we may choose t_1 with $b_0 < t_1 < b_0 + \varepsilon$ such that (1) holds and the intervals I_2, I_{21} (for g) are such that

$$\begin{gathered} I_{21} \subset \operatorname{int} g^{-n'}(B_1), \qquad g^{-n'}(B_{11}) \subset \operatorname{int}(I_2 - I_{21}), \\ \operatorname{dist}(I_{21}, g^{-n'}(B_{11})) > \frac{1 - c_1}{2} l(g^{-n'}(B_1)). \end{gathered} \tag{2}$$

Suppose $n_0 > m_1 + m' + n'$. It follows from Lemma (6.3) for g that

$$V_3^{n_0} \subset r_1^{-1}(\pi_{11}^{-1} g^{-n'}(B_{11}) \cap g^{-n'}(U_1')), \tag{3a}$$

$$g^{-\bar{n}}(V_3^{n_0}) \subset r_2^{-1}(\pi_{21}^{-1} g^{m'}(B_{21}) \cap g^{m'}(U_2')). \tag{3b}$$

Now from (2), (3), and (6.1.8) for g, one sees that there is a constant $c > 0$ such that for $y \in V_3^{n_0}$, there is a C^2 curve $\gamma_y\colon [0, 1] \to \tilde{D}_{2y}^{u}$ near $\tilde{D}_{2y}^{u} \cap \pi_1^{u-1}(H_1)$ such that:

(4a) the second derivative of $\pi_{11}\pi_1{}^u\gamma_y$ at s is less than $-c$ for all s in the domain of γ_y;

(4b) $\pi_1{}^u\gamma_y$ is a δ-arc connecting $r_1(y)$ with $r_1\varrho(y)$, where δ is as in Condition (6.2);

(4c) $\operatorname{dist}(\pi_{11}r_1(y), r_1\varrho(y)) > cl(g^{-n'}(B_1))$;

(4d) $T_y\gamma_y \oplus J'_{1y} = T_y\tilde{D}_{2y}^{u}$.

From (4b), (4c), and (6.2.2) for g, it follows that

(5) $\operatorname{dist}(\pi_{11}\pi_1{}^u(y), \pi_1{}^u\varrho(y)) > \tau cl(g^{-n'}(B_1))$.

Now (4a), (4d), and (5) imply that there is a constant $c_5 > 0$ such that (6.4.4b) will be satisfied for $y \in V_3^{n_0}$ and $n_0 > m_1 + m' + n'$. Further, (6.4.4c) follows from (6.4.4b) using (6.1.7) for g, and (6.4.4d) follows from (3). Finally, one chooses $n_0{}'$ to be the smallest integer such that either $g^{n_0'}.(V_3^{n_0}) \cap V_3 \neq \emptyset$ or $g^{-n_0'}(V_3^{n_0}) \cap V_3 \neq \emptyset$. It is clear that one may arrange for $n_0{}'$ to be larger than n_4. This proves (6.4).

Using Lemmas (6.3) and (6.4), we complete the proof of Lemma (5.11) by proving

(6.5) Given $\varepsilon > 0$ and $\lambda_2 > \lambda_1 > 1$, we may choose n_0, n', m' and $g = \xi_{t_1}$ as in (6.4) so that, for $y \in V_3^{n_0} = V_3^{n_0}(g)$,

(1a) $$T_y g^{n'}(S_{2c_5 l(g^{-n'}(B_1))^{-1/2}}(E_{2y})) \subset S_{\alpha_2 g^{n'}(y)}(E_{2g^{n'}(y)});$$

(1b) $$T_y g^{n'} \,|_{S_{2c_5 l(g^{-n'}(B_1))^{-1/2}}(E_{2y})}$$

is a λ_2-expansion;

(2a) $$T_{g^{-\bar{n}}(y)} g^{-m'}(S_{c_6 l(g^{m'}(B_2))^{-1/2}}(E_{1g^{-\bar{n}}(y)})) \subset S_{\alpha_1 g^{-\bar{n}-m'}(y)}(E_{1g^{-\bar{n}-m'}(y)});$$

(2b) $$T_{g^{-\bar{n}}(y)} g^{-m'} \,|_{S_{c_6 l(g^{m'}(B_2))^{-1/2}}(E_{1g^{-\bar{n}}(y)})},$$

is a λ_2-expansion.

Once this is proved, (5.11) follows taking $n_1 = n'$, $n_2 = \bar{n} + m'$, and $e = 2c_5 l(g^{-n'}(B_1))^{-1/2}$.

We prove (6.5.1) and leave the similar proof of (6.5.2) to the reader.

Let U_1', $r_1\colon U_{11} \to D_1{}^u$, and $F_{1y} = T_y r_1^{-1}(r_1(y))$ be the g-structures corresponding to those in the proof of (6.1). Let $y \in V_3^{n_0}$. Then

$$y \in r_1^{-1}(\pi_{11}^{-1} g^{-n'}(B_{11}) \cap g^{-n'}(U_1'))$$

by (3a) in the proof of (6.4). If n' is large, (6.2.3) and (6.1.6) for g imply there is a constant $c_7 > 0$ such that

$$\begin{aligned} F_{1y} &\subset S_{c_7 \operatorname{dist}(g^{-n'}(B_1), p_1)}(E_{1y}, E_{2y}) \\ &\subset S_{c_7 l(g^{-n'}(B_{11}))}(E_{1y}, E_{2y}) \\ &\subset S_{c_7 c_3 l(g^{-n'}(B_1))}(E_{1y}, E_{2y}). \end{aligned}$$

Hence, for some constant $c_8 > 0$,

$$S_{2c_5 l(g^{-n'}(B_1))^{-1/2}}(E_{2y}, E_{1y}) \subset S_{c_8 l(g^{-n'}(B_1))^{-1/2}}(E_{2y}, F_{1y}).$$

So, if

$$v \in S_{2c_5 l(g^{-n'}(B_1))^{-1/2}}(E_{2y}, E_{1y}),$$

we may write $v = v_1 + v_2$, with $v_1 \in F_{1y}$, $v_2 \in E_{2y}$, and

$$| v_1 | \leq c_8 l(g^{-n'}(B_1))^{-1/2} | v_2 | \leq c_9 (\lambda - \varepsilon)^{n'/2} | v_2 |,$$

for a constant c_9.

Since $y \in V_3^{n_0}$, $g^j(y) \in U_1$, for $0 \leq j \leq n'$, by (6.4.4d).

Thus, for n' large, the λ-lemma and the fact that the function α_2 is bounded away from zero give

$$T_y g^{n'}(v_2) \in S_{\alpha_2 g^{n'}(y)/2}(E_{2g^{n'}(y)}, E_{1g^{n'}(y)}).$$

Also, since $\lambda = \lambda_g$ is the weakest expanding eigenvalue of $T_{p_1} g \mid D_1^{\mathrm{u}}$, $| T_y g^{n'}(v_2) | \geq c(\lambda - \varepsilon)^{n'} | v_2 |$ for some constant c.

Now, let $\mu_1 = \sup_{z \in U_1} | T_z g \mid F_{1z} | < 1$. Then,

$$T_y g^{n'}(v) = T_y g^{n'}(v_1) + T_y g^{n'}(v_2),$$

and

$$\begin{aligned} | T_y g^{n'}(v_1) | &\leq \mu_1^{n'} c_9 (\lambda - \varepsilon)^{n'/2} | v_2 | \\ &\leq \mu_1^{n'} \frac{c_9}{c} (\lambda - \varepsilon)^{n'/2} (\lambda - \varepsilon)^{-n'} | T_y g^{n'}(v_2) | \\ &\leq \mu_1^{n'} \frac{c_9}{c} (\lambda - \varepsilon)^{-n'/2} | T_y g^{n'}(v_2) |. \end{aligned}$$

Hence, for n' large,

$$T_y g^{n'}(v) \subset S_{\alpha_2 g^{n'}(y)}(E_{2g^{n'}(y)}, E_{1g^{n'}(y)}),$$

as required since F_1 remains bounded away from E_2.

Similarly, for constants $c, c_9 > 0$ as above,

$$\begin{aligned} | T_y g^{n'}(v) | &= | T_y g^{n'}(v_1) + T_y g^{n'}(v_2) | \\ &\geq c(\lambda - \varepsilon)^{n'} | v_2 | - \mu_1^{n'} c_9 (\lambda - \varepsilon)^{n'/2} | v_2 | \\ &\geq c_9 (\lambda - \varepsilon)^{n'/2} | v_2 | \left(\frac{c}{c_9} (\lambda - \varepsilon)^{n'/2} - \mu_1^{n'} \right) \\ &\geq 2\lambda_2 | v_2 | \qquad \text{for } n' \text{ large.} \end{aligned}$$

But for n' large, $| v_1 | \leq \frac{1}{2} | v_2 |$, so $| v_2 | \geq \frac{1}{2} | v |$, and, hence, $| T_y g^{n'}(v) | \geq \lambda_2 | v |$. This proves (6.5.1).

Proof of (5.8.2) Let t_1, n', and m' be as in (6.4). We first claim if q_0, q_1 are periodic points of $g = \xi_{t_1}$ outside of V_2 such that $\hat{W}^u(q_0) \cap \hat{W}^s(p_2) \neq \emptyset$ and $\hat{W}^u(p_1) \cap \hat{W}^s(q_1) \neq \emptyset$, then

(a) $W^u(q_0)$ is transverse to $W^s(q_1)$ in $\psi_{11}^{-1} g^{-n'}(B_{11} \cap B_1)$;

(b) $W^u(q_0)$ is transverse to D_{1y}^s, for y in $\psi_{11}^{-1} g^{-n'}(B_{11})$;

(c) $W^s(q_1)$ is transverse to D_{2y}^u, for y in $\psi_{21}^{-1} g^{m'}(B_{21})$.

This follows from (6.1.8)–(6.1.10) and the proof of (6.4).

Now for t_1 near b_0, each manifold in $W^s(\Lambda_2(g)) \cap \psi_{11}^{-1} g^{-n'}(B_{11} \cup B_1)$ is sufficiently near some D_{1y}^s, for y in $\psi_{11}^{-1} g^{-n'}(B_{11})$, that if q_0 is as above, then $W^u(q_0)$ is transverse to $W^s(\Lambda_2(g))$ in $\psi_{11}^{-1} g^{-n'}(B_{11} \cup B_1)$. Hence $W^u(q_0)$ is transverse to $W^s(\Lambda_2(g))$ everywhere, since if $y \in W^u(q_0) \cap W^s(\Lambda_2(g))$, the orbit of y eventually gets in $\psi_{11}^{-1} g^{-n'}(B_{11} \cup B_1)$. Similarly, if q_1 is as above, $W^s(q_1)$ is transverse to each manifold in $W^u(\Lambda_2(g))$. Further, since $W^u(\Lambda_2(g)) \cap W^s(\Lambda_2(g)) = \Lambda_2(g)$, these intersections are transverse. If V_2 is taken small enough and t_1 is near b_0, then any point y whose orbit enters V_2 will be in the above-mentioned stable and unstable manifolds, so $W^u(y)$ is transverse to $W^s(y)$ at y. On the other hand, tubular family techniques show, for t near b_0, points whose orbits never enter V_2 satisfy the condition as well. Hence (5.8.2) is proved.

7

In this section some examples will be presented to illustrate Theorems (2.7) and (2.9). Also, it will be shown that the conditions in Theorem (2.9) can be obtained by the evolution of any Morse–Smale diffeomorphism through a one-parameter family after some simple preliminary bifurcations. Finally, examples will be given of open subsets of $\Phi^{k,r}$

each of whose elements are such that their first bifurcation maps have finitely many α-limit points but infinitely many ω-limit points.

(7.1) Start with the diffeomorphism ξ_0 on S^2 pictured in Figure 7.1.a. There are two sources q_0 and q_2, two sinks q_1 and q_3, and two saddle points p_1 and p_2. Also $W^u(p_1) \cap W^s(p_2) \neq \emptyset$. One can choose a point

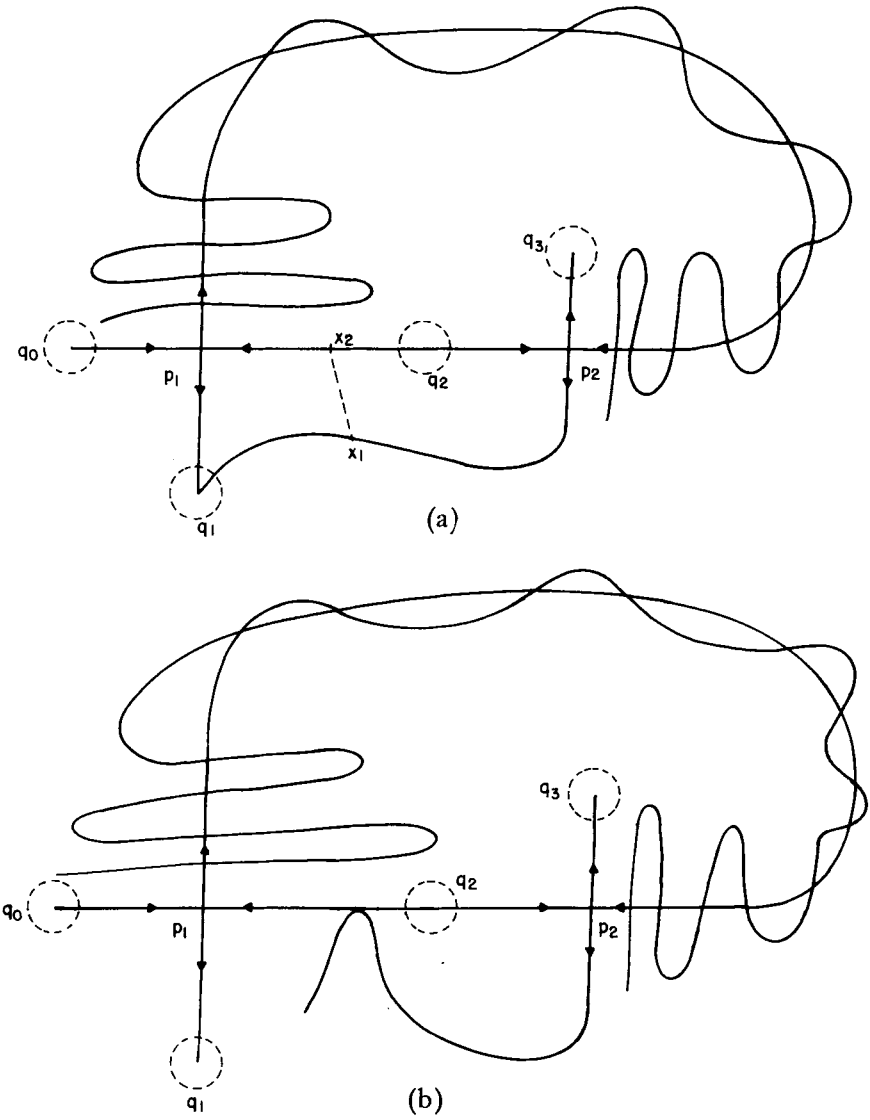


Figure 7.1

x_1 in one component of $W^u(p_2)$ and a point x_2 in one component of $W^s(p_1)$ and connect them with an arc $\varrho : [0, 1] \to S^2$ such that $\varrho(s) \in W^s(q_1)$, for $0 \leq s < 1$, and $\varrho(s) \in W^u(q_2)$, for $0 < s \leq 1$, as in Figure 7.1.a.

Then one can find a one parameter family ξ_t, $0 \leq t \leq 1$ such that $\xi_t = \xi_0$ off a small neighborhood of $\xi_0^{-1}(\varrho)$ and $\xi_{1/2}(\xi_0^{-1}(x_1)) = x_2$, as in Figure 7.1b. Then $b_0(\xi) = \frac{1}{2}$, and $L(\xi_{b_0})$ is finite, hyperbolic, and has a 2-cycle.

(7.2) This is an example with $L(\xi_{b_0})$ finite hyperbolic having a 1-cycle on the projective plane. Start with one source, one sink, and one saddle point p_1 as indicated in Figure 7.2.a.

Then connect a point x_1 in one component of $W^u(p_1)$ with a point x_2 in one component of $W^s(p_1)$ as indicated and perform an isotopy as in Example 1. The final diffeomorphism is depicted in Figure 7.2.b.

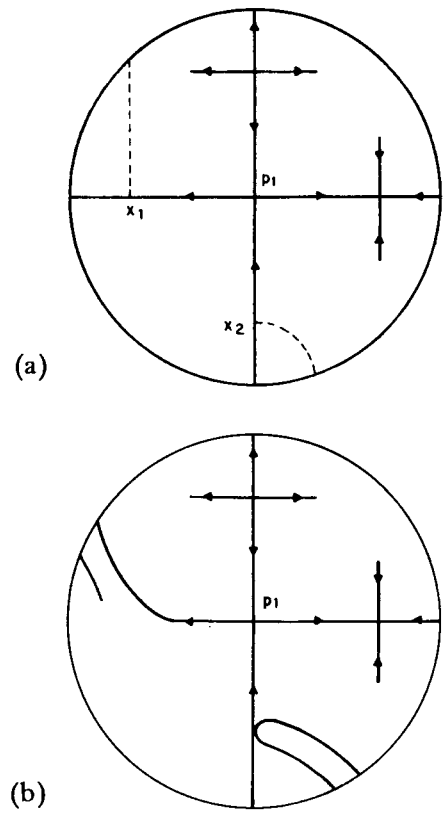


Figure 7.2

(7.3) Let $f_1(x)$ be a diffeomorphism of R^1 such that $f_1(x) = x - x^2$, for $x \leq \frac{1}{4}$, and $f_1(x) = x$, for $x \geq \frac{3}{8}$. The one parameter family $\xi_t(x) = f_1(x) + t - \frac{1}{2}$ is such that $b_0(\xi) = \frac{1}{2}$, ξ_t has no fixed point for $t < b_0$, ξ_{b_0} has zero as a fixed point with eigenvalue 1, and ξ_t has two hyperbolic fixed points for $t > b_0$.

Now, let (x, y, z) denote coordinates on R^{1+s+u}, and consider the one parameter family of diffeomorphisms

$$\xi_t(x, y, z) = (f_1(x) + t - \tfrac{1}{2}, Ay, Bz),$$

where A is an $s \times s$ matrix whose eigenvalues have absolute value less than 1, and B is a $u \times u$ matrix whose eigenvalues have absolute value bigger than 1. It is easily checked that $b_0(\xi) = \frac{1}{2}$, ξ_t has no fixed point for $t < \frac{1}{2}$, $\xi_{1/2}$ has the origin as a fixed point with 1 as an eigenvalue, and for $t > \frac{1}{2}$, ξ_t has two hyperbolic fixed points $p_1(t)$, $p_2(t)$, such that $W^u(p_1(t)) \cap W^s(p_2(t)) \neq \emptyset$. Also,

$$\dim W^u(p_1(t)) = u + 1 = 1 + \dim W^u(p_2(t)).$$

This is a local bifurcation corresponding to the second part of (2.7.1).

The reader will be able to easily construct examples of local bifurcations corresponding to (2.7.2) and (2.7.3).

(7.4) On the n-sphere S^n, let f be a Morse–Smale diffeomorphism such that $\Omega(f)$ consists of two fixed points one of which is a source q_0 (i.e., $\dim W^u(q_0) = n$) and the other of which is a sink q_1 (i.e., $\dim W^s(q_1) = n$). Choose a disk $D \subset S^n$ such that $f(D) \cap D \neq \emptyset$ but $f^n(D) \cap D = \emptyset$, for $n \geq 2$, and let $\gamma\colon [0, 1] \to \operatorname{int} D$ be a smooth embedded arc such that $f\gamma(0) = \gamma(1)$. Now given $0 < u < n$, one can take an isotopy φ_t of S^n such that:

(1) $\varphi_0(x) = x$, $x \in S^n$;

(2) $\varphi_t(x) = x$, $x \in S^n - D$, $0 \leq t \leq 1$;

(3) $\varphi_{1/2}(\gamma(1)) = \gamma(0)$;

(4) If $\xi_t = \varphi_t f$, then

(a) $L(\xi_t) = \{q_0, q_1\}$, for $t < \frac{1}{2}$,

(b) $L(\xi_{1/2})$ has $\gamma(0)$ as a fixed point with 1 as an eigenvalue and $L(\xi_{1/2}) = \{\gamma(0)\} \cup \{q_0, q_1\}$,

(c) for $\frac{1}{2} < t \leq 1$, ξ_t has two hyperbolic fixed points $p_1(t)$, $p_2(t)$ near $\gamma(0)$ such that

$$W^u(p_1(t)) \cap W^s(p_2(t)) \neq \emptyset,$$
$$L(\xi_t) = \{p_1(t), p_2(t), q_0, q_1\},$$
$$\dim W^u(p_1(t)) = u + 1 = 1 + \dim W^u(p_2(t)).$$

Observe that for $\frac{1}{2} < t \leq 1$, $s = n - u - 1$, $\Sigma^s = W^s(p_1(t)) \cup \{q_0\}$ is an s-sphere, $\Sigma^u = W^u(p_2(t)) \cup \{q_1\}$ is a u-sphere, and Σ^s generates the s-homology $H_s(S^n - \Sigma^u)$. Figure 7.3 indicates the above situation on S^3, with $u = 1$.

Now, if $n > 2$ and $1 \leq u \leq n - 2$, it is clear that one can create several types of cycles from the diffeomorphism ξ_1. Let $x_1 \in \hat{W}^u(p_1)$ and $x_2 \in \hat{W}^s(p_1)$. Connect x_1 to x_2 by an arc $\varrho\colon [0, 1] \to S^n$ such that $\varrho(s) \in W^s(q_1)$ for $0 \leq s < 1$ and $\varrho(s) \in W^u(q_0)$ for $0 < s \leq 1$. Then it is possible to take an isotopy ξ_t, $t > 1$, with $\xi_t = \xi_1$ outside a small neighborhood of ϱ for all t such that $\xi_{2/3}(x_1) = \xi_1(x_2)$. This produces a 1-cycle. ξ_t, $0 \leq t \leq 2$, may be chosen so that $\xi_{3/2}$ satisfies the conditions in Section 5, so (5.4) holds. Alternatively, if $x_1 \in \hat{W}^u(p_2)$ and $x_2 \in \hat{W}^s(p_1)$, one may choose ϱ connecting x_1 and x_2 in an appropriate way, and, by isotoping ξ_1, obtain a 2-cycle. In this case it also can be arranged for (5.4) to hold using the methods indicated in the remark following (5.4).

Proceeding similarly, one can create periodic points of period n from $\xi_0 = f$ above as follows: Choose a disk D such that $f^n(D) \cap D \neq \emptyset$, but $f^j(D) \cap D = \emptyset$, for $j \neq 0, n$. Then define φ_t as above satisfying (1)–(3), so that $\xi_t = \varphi_t f^n$ has the properties in (4). Then $\bar{\xi}_t = \varphi_t f$ will have similar properties, except that the periodic points near $\gamma(0)$ will have period n, since

$$L(\varphi_t f) = \bigcup_{0 \leq k < n} f^k(L(\varphi_t f^n)) \qquad \text{for all } t.$$

Now let M be any compact manifold with dimension greater than one, and let $f \in MS(M)$. Let q_0 and q_1 be periodic points of f such that $\hat{W}^u(q_0) \cap \hat{W}^s(q_1) \neq \emptyset$ and $\dim W^u(q_0) = \dim M = \dim W^s(q_1)$. Let n be the period of q_1. Choose a disk $D \subset \hat{W}^u(q_0) \cap \hat{W}^s(q_1)$ such that $f^n(D) \cap D) \neq \emptyset$ and $f^j(D) \cap D = \emptyset$, for $j \neq 0, n$.

If $\dim M > 2$, using the previously described methods, one may find a one-parameter family of diffeomorphisms of M, φ_t, $0 \leq t \leq \frac{1}{2}$, such that $\varphi_0(x) = x$ for all x, $\varphi_t(x) = x$ for $x \in M - D$ and all t, and $\xi_t = \varphi_t f$ has the following properties. There is a number t_1, $0 < t_1 < \frac{1}{2}$, such that $\xi_t \in MS$ for $t \neq t_1$, $L(\xi_{t_1}) = L(f) \cup \{O(p_1)\}$, where p_1 is a periodic

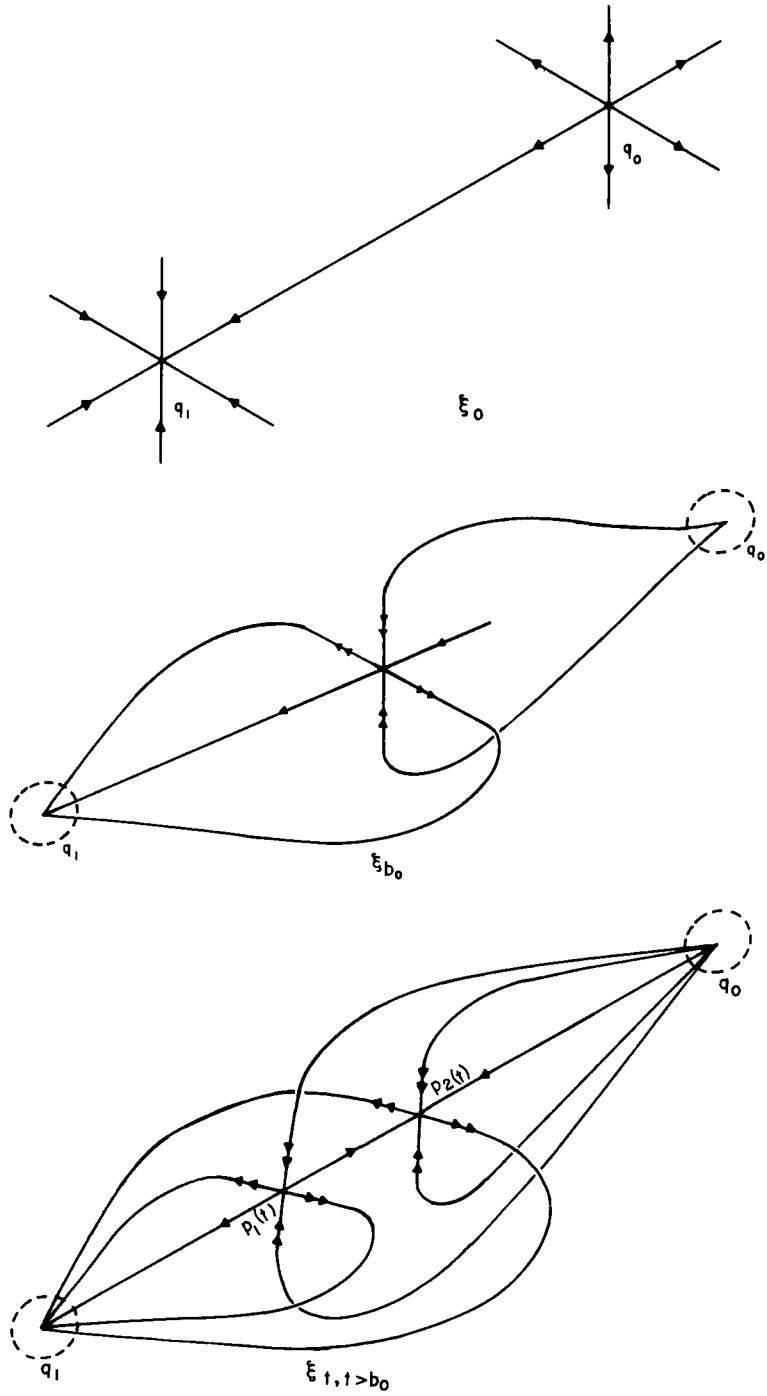


Figure 7.3

point of period n having 1 as an eigenvalue, and for $t > t_1$,

$$L(\xi_t) = L(f) \cup \{O(p_1(t))\} \cup \{O(p_2(t))\},$$

where $p_1(t), p_2(t)$ are periodic points of period n such that $W^u(p_1(t)) \cap W^s(p_2(t)) \neq \varnothing$ and $0 < \dim W^u(p_1(t)) = \dim W^u(p_2(t)) + 1 < \dim M$. Then one can create cycles, as was done in Example 4.

Now suppose $\dim M = 2$. One first creates a saddle point p_1 and a sink p_2 as in Figure 7.4a. Next one introduces a saddle p_3 and a source p_4

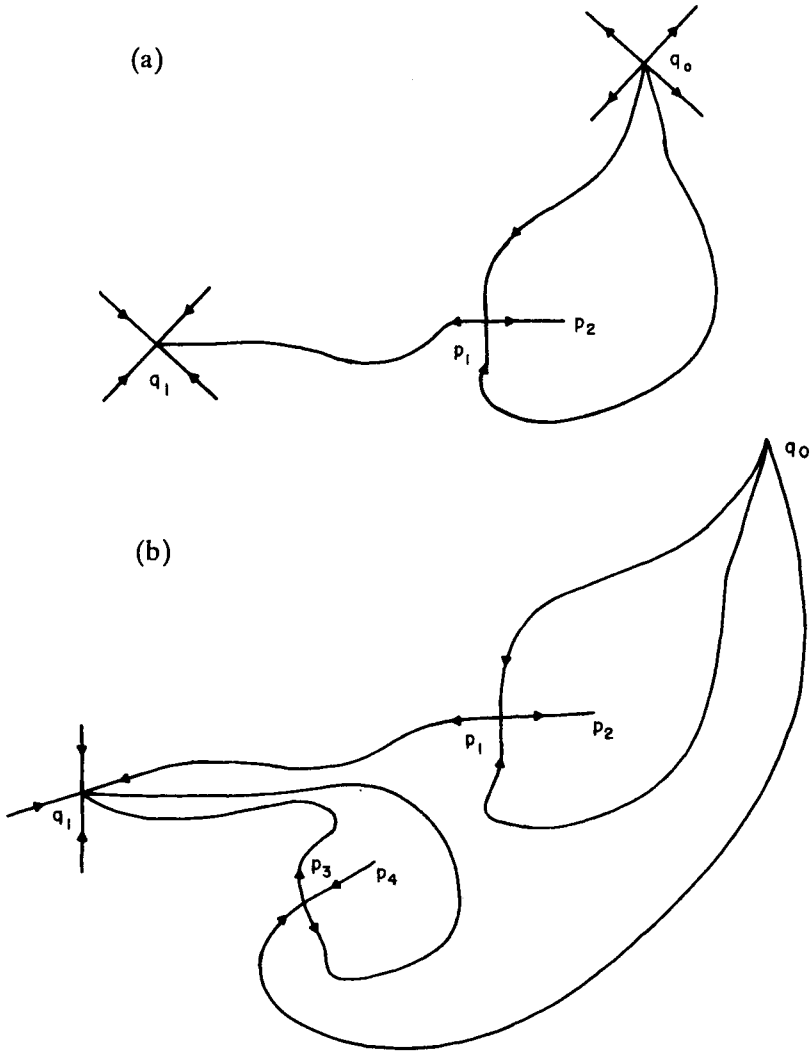


Figure 7.4

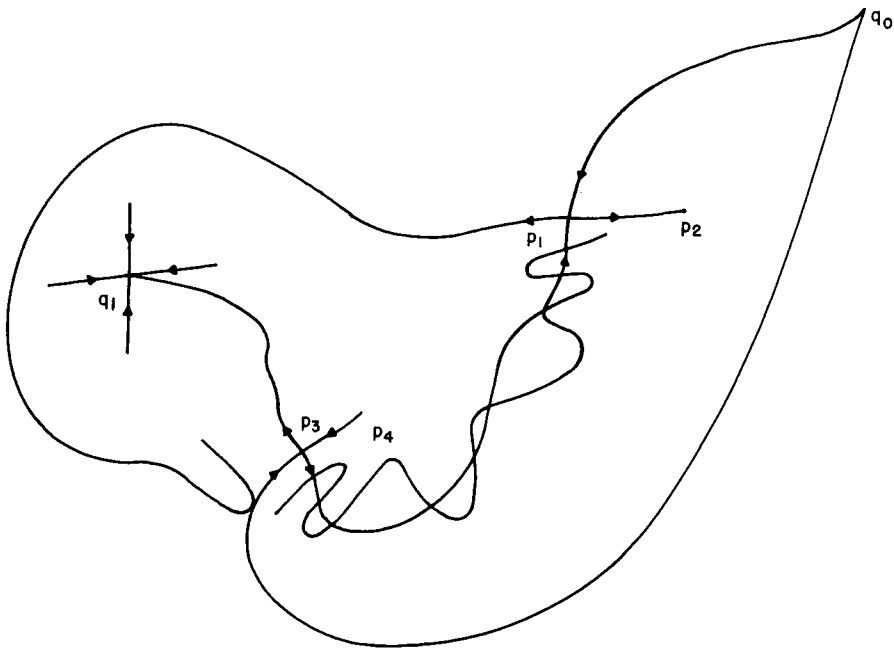


Figure 7.4 (*continued*)

as in Figure 7.4b. Then one introduces a transversal intersection between one component of $W^u(p_3)$ and one component of $W^s(p_1)$ and a quasi-transversal intersection between the other components to produce a 2-cycle as in Figure 7.4c.

Now we give a simple example of an open subset $\mathscr{N}$ of $\Phi^{k,r}(S^2)$ such that for $\xi \in \mathscr{N}$, $L^-(\xi_{b_0})$ is finite, hyperbolic, and has a 1-cycle, but $L^+(\xi_{b_0})$ is infinite. This phenomenon can be obtained on any manifold using the previously explained methods.

Let $\xi_0 \in MS$ be the diffeomorphism on S^2 in Figure 7.5a. There are four fixed points. p_1 and p_2 are sources, p_3 is a saddle, and p_4 is a sink. Also $\hat{W}^u(p_1) \subset \hat{W}^s(p_4)$. Let λ_1, λ_2 be the eigenvalues of $T_{p_3}\xi_0$, with $\lambda_1 < 1 < \lambda_2$. Assume $\lambda_1\lambda_2 < 1$.

Now bring $\hat{W}^s(p_3) \cap \hat{W}^u(p_2)$ around p_2 with ξ_t so that an orbit $O(x)$ of quasi-transversal intersections with $W^u(p_3)$ is introduced. (See Figure 7.5b.) Using the fact that $\lambda_1\lambda_2 < 1$, it can be seen that this can be done so that

$$L^-(\xi_{b_0}) = \{p_1, p_2, p_3, p_4\}.$$

We leave it to the reader to verify the following assertions. If $y \in W^u(p_3) - W^s(p_3)$, then $\omega(y) = p_4$. Also, $\overline{W^s(p_3)} - W^s(p_3)$ is nonempty. If

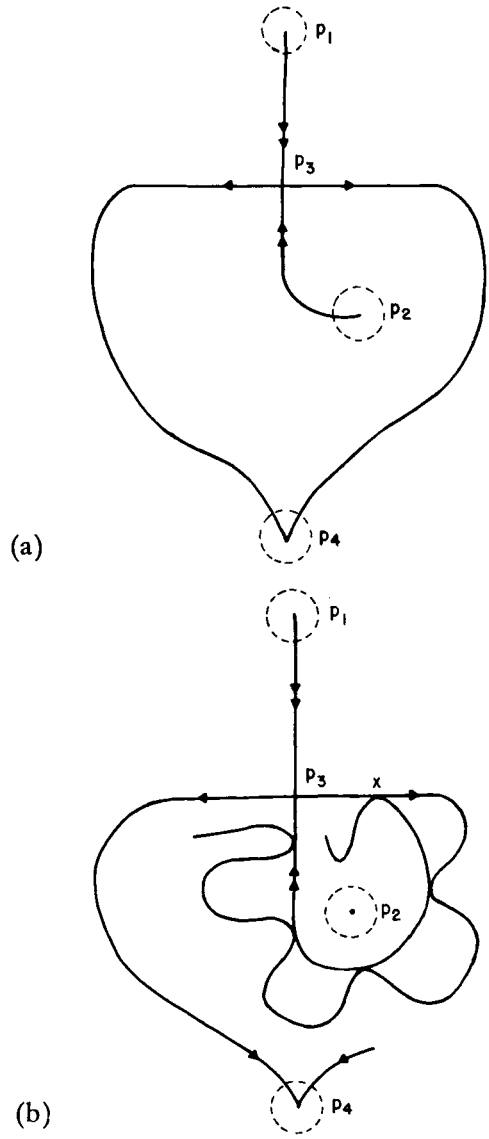


Figure 7.5

$y \in \overline{W^s(p_3)} - W^s(p_3)$, then

$$\omega(y) \subset W^u(p_3) \cap W^s(p_3) = O(x) \cup \{p_3\}.$$

$\omega(y) \neq \{p_3\}$, since $y \notin W^s(p_3)$, so $\omega(y) = O(x) \cup \{p_3\}$. Moreover, if one perturbs ξ in $\Phi^{k,r}$, $k \geq 1, r \geq 2$, the same phenomena persist.

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