Two dimensional systems

We will apply some topology of the Euclidean plane to obtain information about two dimensional planar autonomous systems.

**Definition.** Let $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ be the unit circle in the plane $\mathbb{R}^2$. A Jordan curve in the plane $\mathbb{R}^2$ (or a simple closed curve) in the plane $\mathbb{R}^2$ is the image of a 1-1 continuous map $h : S^1 \rightarrow \mathbb{R}^2$.

**Theorem (Jordan Curve Theorem).** Let $\gamma$ be a Jordan curve in the plane $\mathbb{R}^2$. Then, $\mathbb{R}^2 \setminus \gamma$ is the union of two disjoint open connected sets $S_1, S_2$ each of which have $\gamma$ as boundary. Precisely one of the regions $S_1, S_2$ is bounded.

**Remark.** The book refers to the sets $S_i$ as being arcwise connected. But open connected sets in the plane are arcwise connected.

We will not prove this theorem here, instead referring to a course in topology. The bounded region of $\mathbb{R}^2 \setminus \gamma$ is frequently referred to as the interior of $\gamma$ although it is not the interior in the sense of topology.

Unless otherwise stated, we will assume that $f$ is a $C^1$ vector field defined in the plane $\mathbb{R}^2$ and, for each $x$, the solution $\phi(t, x)$ is defined for all $t \in \mathbb{R}$.

Let $p$ be a regular point of the vector field $f$; i.e., $f(p) \neq 0$. Let $L$ be a closed tranversal to $f$ at $p$. This means that there is a $C^1$ diffeomorphism $h : [-1, 1] \rightarrow L$ such that $h(0) = p, h'(t) \neq 0$, and, for each $t \in [-1, 1], h'(t)$ is not a multiple of $f(h(t))$. Let $L_0$ be the interior of $L$; i.e., $L_0 = \{ h(s) : -1 < s < 1 \}$.

Let $V = \{ q \in L_0 : \text{there is a } t_q > 0 \text{ with } \phi(t_q, q) \in L_0 \text{ and } \phi(t, q) \in \mathbb{R}^2 \setminus L_0 \text{ for } 0 < t < t_q \}$. The set $V$ is the set of points in $L_0$ whose positive orbits return to $L_0$. Let $V' = \{ \phi(t_q, q) : q \in V \}$. Set $W = h^{-1} V, W' = h^{-1} V'$ so that $W$ and $W'$ are subsets of $(-1, 1)$.

Define $g : W \rightarrow W'$ by $g(w) = h^{-1} \phi(t_{h(w)}, h(w))$.

See Figure 1

**Lemma.** The set $W$ is open in $(-1, 1)$ and the function $g$ is continuous on $W$. For any $z \in W$ for which the iterates $z, g(z), \ldots, g^n(z)$ is defined, the sequence $z, g(z), \ldots, g^n(z)$ is monotone in $(-1, 1)$.

**Proof.**

$W$ is open:

Fix $w \in W$, we want to see that if $z$ is near $w$ in $(-1, 1)$, then $z \in W$ also. We can find a flow-box $B$ centered at $\phi(t_{h(w)}, h(w))$ such that each connected component of an orbit in $B$ is an arc which only meets $L$ in one point. Moreover, there is an $\epsilon_1 > 0$ such that if $u \in B$, then there is a $\eta(u) \in$
\[ -\epsilon_1, \epsilon_1 \) such that \( \phi(\eta(u), u) \in L \). Moreover, in the flow-box coordinates on \( B \), \( \eta(u) \) is obviously a continuous function of \( u \) since \( L \cap B \) is the graph of a continuous function from the vertical direction to the horizontal direction. This implies that, in the standard coordinates on \( B \), \( \eta(u) \) is still a continuous function of \( u \).

Now, let \( s_w = t_h(w) \). Then, for \( z \) near \( w \) in \( W \), \( \phi(s_w, h(z)) \) is near \( \phi(s_w, h(w)) \) so it is in \( B \) (since \( \phi(t, x) \) is a continuous function of \( (t, x) \)). Then,

\[
\phi(\eta(\phi(s_w, h(z)), \phi(s_w, h(z))) = \phi(\eta(\phi(s_w, h(z))) + s_w, h(z)) \in L.
\]

This gives that \( h(z) \in W \), so \( W \) is open. Also, since \( g(z) = h^{-1} \phi(\eta(\phi(s_w, h(z))) + s_w, h(z)) \),

we get that \( g \) is continuous.

We now prove the monotonicity statement.

Consider a point \( z \in W \). If \( g(z) = z \) for all \( z \), there is nothing to prove, so assume \( z \) is such that \( g(z) > z \). In the opposite case in which \( g(z) < z \) one proceeds similarly.

The solution curve from \( h(z) \) to \( h(g(z)) \) together with the piece, say \( L_1 \) of \( L \) from \( h(g(z)) \) to \( h(z) \) is a Jordan curve \( \gamma \). Since solutions always cross \( L \) moving in the same direction, the forward orbit of a point in the interval \( L_1 \setminus \{h(z)\} \) always lies in the same component of the complement of \( \gamma \). If \( g(g(z)) \) is defined, then \( g(g(z)) \) must be greater than \( g(z) \) since otherwise, the forward orbit of \( h(g(z)) \) would have to pass from one component of the complement of \( \gamma \) to the other one. Now the argument continues replacing \( z \) by \( g(z) \). QED.

**Corollary.** The \( \omega \)-limit set \( \omega(\gamma) \) of an orbit \( \gamma \) can intersect the interior \( L_0 \) of a transversal \( L \) in at most one point. If \( \gamma = \gamma(p) \) is the orbit through \( p \), and \( \omega(\gamma) \) meets the interior of a transversal at \( p_0 \), then, either \( \omega(\gamma) = \gamma \) in which case \( \gamma \) is periodic or the points in the forward orbit \( O_+(p) \) in \( L_0 \) approach \( p_0 \) monotonically in \( L_0 \).

**Proof.**

We assume that \( L_0 \) is small enough that it fits inside a single flow box \( B \).

Since \( \omega(p) \cap L_0 \supset \{p_0\} \), there is a sequence \( t_1 < t_2 < \ldots \) with \( t_k \to \infty \) such that \( \phi(t_k, p) \to p_0 \) in \( L_0 \).

Case 1:

For some \( j < k \), \( \phi(t_j, p) = \phi(t_k, p) \). Then, \( \gamma \) is periodic. It must be equal to its own \( \omega \)-limit set.
Case 2:
For all $k < j$, $\phi(t_j, p) \neq \phi(t_k, p)$.

Constructing a Jordan curve using pieces of orbits $\phi(t_i, p), \phi(t_{i+1}, p)$ and pieces of $L_0$ as before shows that the forward orbit $O_+(p)$ in $L_0$ approach $p_0$ monotonically in $L_0$ as required. QED.

**Corollary.** If some regular point of $O_+(p)$ is also in $\omega(p)$, then $O(p)$ is periodic.

**Proof.** This is another corollary of the Jordan curve theorem and the flow box theorem.

**Theorem.** A bounded minimal set of a $C^1$ autonomous planar vector field is a critical point or a periodic orbit.

**Proof.**

If the minimal set is not a critical point, then it contains no critical points. But each of its orbits must be dense in the set. Thus, each of its points is in its own $\omega$–limit sets. Hence, by the previous corollary, each of its orbits is periodic. Since it is minimal, it must be a single orbit. QED.

**Theorem (Poincare-Bendixson).** Suppose that $O_+(x)$ is a bounded positive semi-orbit of an autonomous $C^1$ vector field $f$ in the plane. If $\omega(x)$ does not contain a critical point, then $\omega(x)$ consists of a periodic orbit $O(p)$. Either $O(p) = O(x)$ or $O(p) = \text{Closure}(O_+(x)) \setminus O_+(x)$.

**Proof.**

Since $O_+(x)$ is bounded, $\text{Closure}(O_+(x)) \neq \emptyset$ and $\omega(x)$ is a non-empty, invariant set. By hypothesis, it contains only regular points. It also contains a minimal set $\Sigma$ which must be a periodic orbit, $O(p)$. Let $L_0$ be a small open transversal arc to the periodic orbit $O(p)$ at $p$. Since, $p$ is in $\omega(x)$, there is a sequence $t_1 < t_2 \to \infty$ such that $\phi(t_i, x) \in L_0$ and $\phi(t_i, x) \to p$ as $i \to \infty$. Let $z_i = \phi(t_i, x)$. If, for some $i_0$, $z_i = z_{i+1}$, then $O(x)$ is periodic and must equal $O(p)$. If not, then the sequence of points $z_i$ with different $i$ is a sequence of distinct points in $L_0$. By the flow-box theorem and the Jordan curve theorem, the sequence $z_i$ converges monotonically to $p$. It follows that $O(p) = \text{Closure}(O_+(x)) \setminus O_+(x)$. QED.

**Lemma.** Suppose that $\omega(p_0)$ contains a regular point, $p_1$ which is not in the orbit of $p_0$. Then, $p_0$ cannot be in $\omega(x)$ for any $x$.

**Proof.**

Take a small open transversal $L_0$ to $p_1$. The positive orbit of $p_0$ must cross $L_0$ and monotonically converge to $p_1$. Then, pieces of this orbit together with pieces of $L_0$ form Jordan curves which trap the positive orbit of any point near $p_0$. Hence, $p_0$ cannot be in $\omega(x)$ for any $x$. QED.
Theorem. Suppose $O_+(x)$ is a positive semi-orbit in a closed bounded subset $K$ of the plane for a $C^1$ vector field $f$. Assume that $K$ contains only a finite number of critical points. Then, one of the following holds.

(i) $\omega(x)$ is a critical point.

(ii) $\omega(x)$ is a periodic orbit.

(iii) $\omega(x)$ consists of a finite number of critical points and a set of orbits $\gamma_i$ such that each $\gamma_i$ has its $\omega$–limit set $\omega(\gamma_i)$ and $\alpha$–limit set $\alpha(\gamma_i)$ consisting of a critical point.

Definition. A cycle of critical points is a finite sequence $p_1, p_2, \ldots, p_n$ of critical points such that $p_1 = p_n$ and, for each $1 \leq i < n$, there is a point $x_i$ such that $\alpha(x_i) = p_i$ and $\omega(x_i) = p_{i+1}$. A solution $\gamma$ whose $\alpha$ and $\omega$ limit sets are critical points is called a separatrice.

Remark. One way in which condition (iii) occurs is that $\omega(x)$ consists of separatrices of a cycle of critical points. There can also be several regular orbits whose $\alpha$ and $\omega$ limits are the same critical point.

Proof.

If $\omega(x)$ contains no regular points, then since it is connected, it must consist of a single critical point. This is case (i).

Thus, we may assume $\omega(x)$ contains at least one regular point, say $p_0$. If $O(p_0)$ is periodic, and $L_0$ is an open transversal at $p_0$, then $O_+(x) \cap L_0$ converges monotonically to $p_0$, so $\omega(x) = O(p_0)$ which is case (ii).

If the orbit $O(p_0)$ is not periodic, then its $\omega$–limit set must be disjoint from $O(p_0)$. If $\omega(p_0)$ contained a regular point, then the previous lemma would contradict the assumption that $p_0$ is itself an $\omega$–limit point (of $x$). Therefore, $\omega(p_0)$ consists only of critical points. Since it is connected, it must be a single critical point. A similar argument works for $\alpha(p_0)$.

We have therefore proved that each regular, non-periodic, $\omega$–limit point or $\alpha$–limit point of $x$ must have each of its own $\omega$–limit and $\alpha$–limit sets reducing to single critical points. QED.