Index for planar C^2 vector fields

Let Ω be an open set in $\mathbf{R}^n$. We say that Ω is *connected* if it cannot be written as the disjoint union of two non-empty open subsets.

A *curve* or *path* in Ω is a continuous map $\gamma : [a, b] \to \Omega$ where $[a, b]$ is a closed interval in $\mathbf{R}$. Sometimes we abuse the language by referring to the image $Image(\gamma)$ for such a γ as a *curve* or *path*. We can always reparametrize a curve γ so that it is defined on the closed unit real interval $[0, 1]$ and its image is unchanged. We simply use $\gamma_1(t) = \gamma((1 - t) * a + t * b)$.

We say that Ω is *path connected* if for any two points $p_1, p_2 \in \Omega$, there is a curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = p_1$ and $\gamma(1) = p_1$.

It is a simple exercise to verify that an open connected subset of $\mathbf{R}^n$ is path connected.

An *region* in $\mathbf{R}^n$ is defined to be an open connected set.

Let $\gamma_i : [0, 1] \to \Omega, i = 1, 2$ be two curves in the region $\Omega \subset \mathbf{R}^n$.

We say that γ_1 is *homotopic* to γ_2 if there is a continuous map $F : [0, 1] \times [0, 1] \to \Omega$ such that

$$F(t, 0) = \gamma_1(t) \text{ and } F(t, 1) = \gamma_2(t)$$

for all $t \in [0,1]$. This is the precise way of saying the γ_1 can be *continuously deformed* into γ_2.

We call F a *homotopy* from γ_1 to γ_2.

When this is the case, we write $\gamma_1 \simeq \gamma_2$.

Given a curve $\gamma : [0,1] \to \Omega$, define its *negative* or *inverse* or *reverse* curve $-\gamma(t)$ by

$$\gamma(t) = \gamma(1-t).$$

If γ_1, γ_2 are C^r for $r \geq 1$, then we say that γ_1 is C^r homotopic to γ_2 if the map F above can be taken to be C^r.

Facts.

1. The relation $\simeq$ is an equivalence relation on curves (parametrized by [0,1]).
2. If $\gamma_1 \simeq \gamma_2$, then $-\gamma_1 \simeq -\gamma_2$.

If $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ and a homotopy F can be chosen so that $F(0,s) = \gamma_1(0)$ and $F(1,s) = \gamma_1(0)$ for all s, then we say that γ_1 is homotopic to γ_2 *relative to their boundaries* and we write $\gamma_1 \simeq_\partial \gamma_2$.

A curve $\gamma : [0,1] \to \Omega$ is called a *loop* or *closed curve* in Ω if $\gamma(0) = \gamma(1)$.

A *constant curve* in Ω is a continuous map $\gamma : [0,1] \to \Omega$ such that $\gamma(t) = \gamma(s)$ for all $s \in [0,1]$. Thus, the image is a single point.

A loop γ in Ω is *null-homotopic* or *in-essential* if it is homotopic to a constant curve. Sometimes, we say it is *homotopic to a constant* or *homotopic to a point.*

A region Ω is called *simply connected* if every loop in Ω is null-homotopic.

Examples

1. An open ball is simply connected.
2. The complement of a point in $\mathbf{R}^2$ is not simply connected.
3. The unit circle in $\mathbf{R}^2$ is not simply connected.
4. If $n \geq 3$, then the complement of any finite (non-empty) set in $\mathbf{R}^n$ is simply connected. Also, any sphere S^n (i.e., boundary of a ball) is simply connected.
5. The two dimensional torus $S^1 \times S^1$ is not simply connected.

Let (x, y) represent the coordinates of a point in Ω, and let $P(x, y), Q(x, y)$ be C^1 real-valued functions from Ω to $\mathbf{R}$.

The formal expression

$$\omega(x, y) = P(x, y)dx + Q(x, y)dy$$

is called a (C^1) differential 1-form (or simply a 1-form) Ω. We sometimes write $\omega = Pdx + Qdy$ leaving out the explicit dependence on (x, y).

Given a C^1 curve, $\gamma : [a, b] \to \Omega$, $\gamma(t) = (x(t), y(t))$ and a 1-form ω in Ω, one defines the line integral

$$\int_\gamma \omega = \int_a^b P(x(t), y(t))x'(t)dt + P(x(t), y(t))y'(t)dt.$$

The 1-form w is called *closed* (in Ω) if $P_y(x, y) = Q_x(x, y)$ for all $(x, y) \in \Omega$. It is called *exact* if there is a C^2 real-valued function $\psi(x, y)$ defined in Ω such that $\psi_x = P$, and $\psi_y = Q$. That is, we can formally write

$$d\psi = \psi_x dx + \psi_y dy = Pdx + Qdy$$

It follows from the equality of mixed partial derivatives of ψ that every exact 1-form is closed. The converse is true if the region is simply connected.

It is proved in the calculus of several variables that, in a simply connected region, the line integral $\int_\gamma \omega$ of a closed 1-form is *path-independent*. Indeed, letting ψ be the function such that $d\psi = \omega$, if γ_1 and γ_2 are two paths starting at the point p_1 and ending at the point p_2, then

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega = \psi(p_2) - \psi(p_1).$$

Now, let Ω be an open connected subset of the plane $\mathbf{R}^2$, and let $\eta = (\eta_1, \eta_2)$ be a C^2 non-vanishing vector field defined in Ω.

Consider the 1-form

$$\begin{aligned} \alpha_\eta &= \frac{\eta_1 d\eta_2 - \eta_2 d\eta_1}{\eta_1^2 + \eta_2^2} \\ &= \frac{\eta_1(\eta_{2x}dx + \eta_{2y}dy) - \eta_2(\eta_{1x}dx + \eta_{1y}dy)}{\eta_1^2 + \eta_2^2} \end{aligned}$$

In a region where $\eta_1 \neq 0$, this is

$$d\left(arctan(\frac{\eta_2}{\eta_1})\right)$$

while in a region where $\eta_2 \neq 0$, this is

$$-d\left(arctan(\frac{\eta_1}{\eta_2})\right) = d\left(arccot(\frac{\eta_2}{\eta_1})\right).$$

If $\gamma : [0, 1] \to \Omega$ is a curve and η does not vanish on the image of γ, then

$$\int_\gamma \alpha_\eta$$

gives the *total change in the angle that* η *makes with the positive horizontal direction* as one moves along the curve γ. Since η points in the same direction at $\gamma(0)$ and $\gamma(1)$, this must be an integer multiple of 2π.

We define the *index* of η over γ to be the integer

$$Ind(\gamma, \eta) = \frac{1}{2\pi}\int_\gamma \alpha_\eta$$

Definition. The standard parametrization of the unit circle S^1 is the map $\gamma : [0,1] \to S^1$ defined by

$$\gamma(t) = (cos(2\pi t), sin(2\pi t)$$

Exercises:

1. Using the standard parametrization γ of the unit circle S^1, compute the index $Ind(\gamma, \eta)$ of each of the following vector fields.

   (a) $\eta(x, y) = x\ \partial_x + y\ \partial_y$

   (b) $\eta(x, y) = x\ \partial_x - y\ \partial_y$

   (c) $\eta(x, y) = -x\ \partial_x - y\ \partial_y$

   (d) Given a complex number $z = x + iy$, with $i = \sqrt{-1}$, we let $Re(z) = x, Im(z) = y$ denote its real and imaginary parts. Let n be an integer, and consider the vector field

   $$\eta(z) = z^n$$

   (i.e., $\eta(x, y) = Re(z^n)\ \partial_x + Im(z^n)\ \partial_y$).
   Compute $Ind(\gamma, \eta)$.

2. Let Ω be a region in the plane, and let $[a, b]$ and $[a_1, b_1]$ be two closed intervals in $\mathbf{R}$. Let $\gamma : [a, b] \to \Omega$ and $\gamma_1 : [a_1, b_1] \to \Omega$ be two C^2 curves. We say that γ_1 is a reparametrization of γ if there is C^2 map $\rho : [a_1, b_1] \to [a, b]$ such that

(a) $\rho(a_1) = a$ and $\rho(b_1) = b$,

(b) ρ is strictly increasing (i.e., $a_1 \le s < t \le b_1$ implies that $\rho(s) < \rho(t)$),

(c) $\rho'(t) = \frac{D\rho}{dt}(t) \neq 0$ for $t \in [a_1, b_1]$, and

(d) $\gamma_1 = \gamma \circ \rho$.

Let $\omega = Pdx + Qdy$ be a C^2 1-form in Ω. Show that if γ_1 is a reparametrization of γ then

$$\int_{\gamma_1} \omega = \int_{\gamma} \omega$$

Proposition 1. *Let Ω be a region in the plane and let η be a non-vanishing vector field in Ω. Let γ_0 and γ_1 be two loops in Ω which are C^2 homotopic through loops in Ω. Then,*

$$Ind(\gamma_0, \eta) = Ind(\gamma_1, \eta)$$

Proof. Let $F(t, s)$ be a C^2 homotopy from γ_0 to γ_1 so that $F(t, 0) = \gamma_0$ and $F(t, 1) = \gamma_1(t)$ and $F(0, s) = F(1, s)$ for all $s \in [0, 1]$.

Let $\gamma_s(t) = F(t, s)$ for each t, s. Then, each γ_s is a loop in Ω so $Ind(\gamma_s, \eta)$ is an integer for each s. But, $Ind(\gamma_s, \eta)$ is an integral $\int_0^1 \psi(s, t)dt$ in which the function $\psi(s, t)$ is continuous. So, $Ind(\gamma_s, \eta)$ depends continuously on s. Since it is integer valued, it must be constant. QED.

Proposition 2. *Let Ω be a region in the plane and let γ be a loop in Γ. Let η_1, η_2 be two C^2 non-vanishing vector fields in Ω so that η_1 is C^2 homotopic to η_2 through non-vanishing vector fields in Ω. Then,*

$$Ind(\gamma, \eta_1) = Ind(\gamma, \eta_2)$$

Proof. This is similar to the preceding proof. Let $F(x,t)$ be the homotopy from η_1 to η_2 through non-vanishing vector fields. That is, $F(x,t)$ is a C^2 map from $\Omega \times [0,1]$ to Ω such that $F(x,0) = \eta_1(x)$ and $F(x,1) = \eta_2(x)$ for each $x \in \Omega$. Letting $F_s(x) = F(x,s)$, the map $s \to F_s$ is a C^2 map from $[0,1]$ to $\mathbf{R}^2 \setminus \{0\}$.

Hence, the function $s \to Ind(\gamma, F_s)$ is continuous. Again, since it is integer valued, it must be constant. QED.

Theorem 3. *Let Ω be a simply connected region in the plane. Let η be a vector field in Ω and suppose that there as a loop γ in Ω such that*

$$Ind(\gamma, \eta) \neq 0.$$

Then, η has a critical point in Ω.

Proof.

Assume that η has no critical point in Ω, and let $p_0 \in \Omega$. Then, there is an $\epsilon > 0$ such that $B = B_\epsilon(p_0) \subset \Omega$. If

ϵ is small enough then η is nearly constant in B, so, any loop γ_1 whose image is in B has $Ind(\gamma_1, \eta) = 0$. But, simple connectivity gives that $\gamma \simeq \gamma_1$ which implies that $Ind(\gamma, \eta) = 0$. This contradiction proves the theorem. QED.

Remark One can generalize the notion of $Ind(\gamma, \eta)$ where γ and η are only continuous. This can most easily be done through C^2 approximations. Suppose that $\gamma : [0, 1] \to \Omega$ is a continuous loop in Ω and $\eta : \Omega \to \mathbf{R}^2$ is a continuous non-vanishing vector field in Ω. Then, given $\epsilon >$ there exist a C^2 loop $\gamma_1 : [0, 1] \to \Omega$ such that $\mid \gamma(t) - \gamma_1(t) \mid < \epsilon$ for all $t \in [0, 1]$. For ϵ small enough, if γ_2 is another C^2 loop within ϵ of γ, then $\gamma_1 \simeq \gamma_2$ in Ω. The closure of the image G of a homotopy between γ_1 and γ_2 is a compact subset of Ω. There is a C^2 vector field η_1 in Ω such that $\mid \eta(x) - \eta_1(x) \mid < \epsilon$ for $x \in G$. We have defined $Ind(\gamma_1, \eta_1)$. If we choose another C^2 loop γ_2 C^0 near γ and another C^2 vector field η_2 C^0 near η, then our invariance under homotopies shows that $Ind(\gamma_1, \eta_1) = Ind(\gamma_2, \eta_1) = Ind(\gamma_2, \eta_2)$. Thus, the index is independent of the smooth approximations of γ and η. So, we can define $Ind(\gamma, \eta)$ using any two sufficiently close C^2 approximations. The properties in the previous Propositions 1 and 2 and Theorem 3 remain valid.