Limit Sets

Consider an autonomous system of the form $\dot{x} = f(x)$ for which solutions are defined for all time in an open set D. For $x \in D$, the $\omega-$limit set of x, denoted $\omega(x)$ is the set of points y such that there is a sequence $t_1 < t_2 < \ldots$ with $t_i \to +\infty$ as $i \to \infty$, and $\phi(t_i, x) \to y$. Similarly, the $\alpha-$limit set of x is the set of points y for which there is a sequence $t_1 > t_2 > \ldots$ with $t_i \to -\infty$ as $i \to \infty$ and $\phi(t_i) \to y$ as $i \to \infty$.

The $\omega-$limit set of x is denoted $\omega(x)$ and the $\alpha-$limit set of x is denoted $\alpha(x)$. It is easy to show that these are closed subsets of D. They may be empty. One can define $\omega(x)$ in the case that $\phi(t, x)$ exists for all $t > t_0$ for some t_0. A similar statement holds for $\alpha(x)$, if $\phi(t)$ exists for $t < t_0$.

A point x_0 for which $f(x_0) = 0$ is called an *equilibrium* or *stationary point* or *critical point* of $\dot{x} = f(x)$.

We also say that a function $f : D \to \mathbf{R}^n$ is a *vector field* in D. Thus, being given a vector field in D is the same as being given an autonomous differential equation in D.

An *invariant set* for the differential equation $\dot{x} = f(x)$ or for the vector field f in D is a subset Ω of D such that if $x \in \Omega$ and $\phi(t, x)$ is a solution to $\dot{x} = f(x)$ with $\phi(0) = x$, then $\phi(t, x) \in \Omega$ for all t.

Facts:

1. Suppose the f is a C^1 vector field in $D \subset \mathbf{R}^n$ and $x \in D$ has the property that the orbit $\phi(t, x)$ of x remains in a compact subset F of D for $t \geq 0$. Then, $\omega(x)$ is a compact invariant connected subset of F.

2. Any orbit is an invariant set.

Vector Fields as Differential operators

Recall that an autonomous differential equation $\dot{x} = f(x)$ is given by simply giving a function $f : D \to \mathbf{R}^n$ from a domain D in $\mathbf{R}^n$. Suppose that f is C^k for $k \geq 1$. Let $C^k[D, \mathbf{R}]$ be the space of C^k real-valued functions defined on D. We can use f to define an operator $\mathcal{L}_f$ from $C^{k+1}[D, \mathbf{R}]$ to $C^k[D, \mathbf{R}]$ in the following way.

For $x \in D$, let $\phi(t, x)$ be the solution to $\dot{x} = f(x), \phi(0, x) = x$. For, $\psi \in C^{k+1}[D, \mathbf{R}]$, let $(\mathcal{L}_f \psi)(x) = \frac{d}{dt}\psi(\phi(t, x)) \mid_{t=0}$. This defines a mapping from $C^{k+1}[D, \mathbf{R}]$ to $C^k[D, \mathbf{R}]$ which satisfies the following two properties.

1. (linearity). $\mathcal{L}$ is a linear mapping; i.e., $\mathcal{L}_f(\alpha\psi+\beta\eta) = \alpha\mathcal{L}_f(\psi) = \beta\mathcal{L}_f(\eta)$ for any two functions ψ, η and scalars α, β.

2. (derivation). For $\psi, \eta \in C^{k+1}[D, \mathbf{R}]$,

$$\mathcal{L}_f(\psi \cdot \eta) = \mathcal{L}_f(\psi) \cdot \eta + \psi\mathcal{L}_f(\eta).$$

The operator $\mathcal{L}_f$ is called the Lie derivative operator. It maps C^{k+1} functions to C^k functions.

Let $\pi_i : x \to x_i$ be the projection of a vector onto its $i-$th coordinate as a function on $\mathbf{R}^n$.

Facts.

1. The value of the function $\mathcal{L}_f(\psi)$ can be computed from knowledge of the vector field and the partial derivative functions of ψ by the formula

$$\mathcal{L}_f(\psi)(x) = \sum_{i=1}^{n} \frac{\partial\psi}{\partial x_i}(x)(\pi_i \circ f)(x). \tag{1}$$

2. It follows from this formula that the component functions of the vector field f are equal to the functions $\mathcal{L}_f(\pi_i)$. Indeed, if $f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) = (\pi_1 \circ f(x), \ldots, \pi_n \circ f(x))$, then

$$\mathcal{L}_f(\pi_i) = f_i.$$

   Thus, the operator $\mathcal{L}_f$ and the vector field f completely determine each other, and we can think of vector fields as differential operators on real-valued functions or as assignments of vectors at each point in a domain D.

3. The function ψ is constant along solution curves of $\dot{x} = f(x)$ if and only if $\mathcal{L}_f(\psi)$ is the zero function in D.

Let e_i be the unit vector in $\mathbf{R}^n$ whose $i-$th coordinate is 1 and whose other coordinates are 0. It is common to write $\frac{\partial}{\partial x_i}$ for the operator $\mathcal{L}_f$ where $f(x) = e_i$ is the constant vector field whose value at each x is e_i.

We will often identify an autonomous differential equation $\dot{x} = f(x)$ with the vector field f and with the operator $\mathcal{L}_f$.

In this sense, we can write

$$f(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}.$$

This means that given a function $f : D \to \mathbf{R}^n$, with $f(x) = (f_1(x), \ldots, f_n(x))$, we get any one of three objects: the system of differential equations

$$\dot{x}_i = f_i(x), \; i = 1, \ldots, n$$

the vector field

$$x \to f(x), x \in D$$

the operator

$$\psi \to \mathcal{L}_f(\psi)$$

Structure of autonomous differential equations near a non-critical point

Definition. Suppose f is a vector field in the domain $D \subset \mathbf{R}^n$. Let $\rho : D \to D'$ be a smooth change of coordinates from D to the domain D'. Then, ρ maps the vector field f to the new vector field $\rho_\star f$ defined by

$$\rho_\star(f)(y) = D\rho_{\rho^{-1}y}(f(\rho^{-1}y))$$

Thus, we can write $\rho_\star = D\rho \circ f \circ \rho^{-1}$ as vector valued funtions.

Theorem(Flow-box theorem, path-cylinder theorem). *Let $k \geq 1$. Suppose f is a C^k vector field in a domain D and x_0 is a point in D such that $f(x_0) \neq 0$. Then, there is a C^k change of coordinates ρ from a neighborhood U of 0 in $\mathbf{R}^n$ to a neighborhood V of x_0 such that ρ carries solutions of the constant vector field $\frac{\partial}{\partial x_1}$ onto those of $\dot{x} = f(x)$.*

Proof. Since $f(x_0) \neq 0$, we may consider $f(x_0)$ as a vector attached to the origin 0 in $\mathbf{R}^n$ and pick non-zero unit vectors $v_2, v_3, \ldots v_n$ so that the vectors $f(x_0), v_2, v_3, \ldots, v_n$ are linearly independent. Let $\tilde{H}$ be the subspace of $\mathbf{R}^n$ spanned by the vectors $v_i, i \geq 2$. The affine subspace $H = x_0 + \tilde{H}$

is then transverse to the vector field $f(x_0)$ at x_0. By the local continuity of solutions to $\dot{x} = f(x)$ on initial conditions and the continuity of f, there are a neighborhood V_1 of x_0 in H and an interval I about 0 in $\mathbf{R}$ such that if $x \in V_1$, then $\phi(t, x)$ is defined on all of I and meets H only for $t = 0$. For $(y_2, \ldots, y_n) = y$ near 0 in $\mathbf{R}^{n-1}$, we have an associated point $\eta(y) = x_0 + \sum_j y_j v_j \in H$. Write (y_1, y) for the point $(y_1, y_2, \ldots, y_n)$ in $\mathbf{R}^n$ with $y \in \mathbf{R}^{n-1}$.

We define a mapping $\rho(y_1, y)$ by

$$\rho(y_1, y) = \phi(y_1, \eta(y)).$$

We claim that this transformation ρ is the required change of coordinates.

First, note that ρ is a C^k mapping of the variables (y_1, y).

To prove that ρ is a change of coordinates, it suffices to show that its jacobian determinant at 0 is not zero and use the implicit function theorem.

Now, at $(y_1, y) = 0$, the first column of the jacobian matrix of ρ, $\frac{\partial \rho}{\partial y_1}$ is just $f(x_0)$, while the $j-th$ column is just $\frac{\partial \rho}{\partial y_j}$ is v_j (exercise). By the choice of the $v_j's$, these vectors are linearly independent. Thus, the required jacobian determinant is not zero.

Finally, we have to show that the mapping ρ carries solutions to $\frac{\partial}{\partial x_1}$ to those of f.

A solution to the constant vector field $\frac{\partial}{\partial x_1}$ is simply a function $(t, (y_1, y)) \rightarrow (t + y_1, y)$. Transforming this by ρ gives the function $(t, (y_1, y)) \rightarrow \rho(t + y_1, y) = \phi(t + y_1, \eta(y))$. But, as we saw in the proof of the local flow property of autonomous systems, if $\phi(t, z)$ is a solution, then so is $\phi(t + s, z)$. Thus, the function $t \rightarrow \phi(t + y_1, \eta(y))$ is a solution to the equation $\dot{x} = f(x)$.

Suppose the f is a C^1 vector field defined in an open set $D \subset \mathbf{R}^n$.

Definition. An invariant set K is called a *minimal set* if it is compact, non-empty, and does not properly contain another compact, non-empty, invariant set.

Proposition. *Any compact invariant set contains a minimal set.*

Proof. Let K be a compact invariant set. The set $\mathcal{C}$ of non-empty compact invariant subsets of K is partially ordered by inclusion $A \prec B$ if and only if $A \supseteq B$. Each totally ordered subset has an upper bound, so by Zorn's lemma, $\mathcal{C}$ contains a maximal element, say Σ. Then, Σ is a minimal set. QED.

Example and Remark.

1. A critical point or periodic orbit is a minimal set.
2. It is remarkable fact that in the plane for a C^1 autonomous vector field, there are no other minimal sets.
3. In $\mathbf{R}^n, n > 2$, there are many examples of non-trivial minimal sets. We will see this later.

Proposition. *Suppose f is a C^1 vector field in an open set $D \subset \mathbf{R}^n$ and there is a closed non-empty ball $B \subset D$ such that f is non-zero and nowhere tangent on the boundary of B. Then, f possesses a critical point in B.*

Proof. Let $\phi(t, x)$ be the local flow of f. Since, f is non-zero and not tangent to the boundary of B, orbits at the boundary either flow into or outof B. We suppose they flow into B. In the other case, replace f by $-f$.

For $x \in B$, the solution $\phi(t, x)$ is defined and remains in B for all $t > 0$. Let $m > 0$ be a positive integer, and consider the mapping $x \to \phi_{\frac{1}{m}}(x)$. This is a continuous self-map of the closed ball B to itself. By the Brouwer fixed point theorem, it has a fixed point, say x_m. Since B is compact, the sequence x_m has a subsequence x_{m_k} which converges, say to the point y as $k \to \infty$.

Let us show that $f(y) = 0$. If not, then by the flow box theorem, there are a neighborhood U of y in D and an interval $I_\epsilon = [-\epsilon, \epsilon]$ about 0 in $\mathbf{R}$ such that,

(*) for $z \in U$, the solution $\phi(t, z)$ is defined for all $t \in [-\epsilon, \epsilon]$

(**) $\phi(t_1, z) \neq \phi(t_2, z)$ for $t_1 \neq t_2 \in I_\epsilon$

But, if k is large enough, then $x_{m_k} \in U$, and $\frac{1}{m_k} < \epsilon$. But then, $\phi_{\frac{1}{m}}(x_{m_k}) \neq x_{m_k}$ by (**) which contradicts the definition of x_{m_k}. QED.