General Properties of Autonomous Systems

Consider an autonomous system

$$\dot{x} = f(x) \tag{1}$$

with $f : D \to \mathbf{R}^n$ a locally Lipschitz map from an open set D in $\mathbf{R}^n$ into $\mathbf{R}^n$.

According to the E-U theorem, for each point $x \in D$, there is a unique solution $\phi(t)$ with $\phi(0) = x$ defined in an interval I about 0 in $\mathbf{R}$. Let us denote this function by $\phi(t, x)$. We claim

$$\phi(t + s, x) = \phi(t, \phi(s, x))$$

for $t, s, t + s \in I$.

Indeed, the function $t \to \phi(t+s, x)$ is a solution to the differential equation $\dot{x} = f(x)$ and $\phi(0 + s, x) = \phi(s, x)$. By uniqueness of solutions, $\phi(t + s, x) = \phi(t, \phi(s, x))$.

This property is called the local flow property of autonomous systems. The map $t \to \phi(t, x), t \in I$ then defines a C^1 curve in D

If, in addition, for each $x \in D$, the solution $\phi(t, x)$ is defined for all time, it is called a *flow* in D. We then have $\phi(t + s, x) = \phi(t, \phi(s, x))$ for all x, s, t. Writing ϕ_t for the mapping $\phi(t, \cdot)$, we can write this last property as

$$\phi_{t+s} = \phi_t \circ \phi_s.$$

Definition. Let D_1, D_2 be two non-empty open sets in $\mathbf{R}^n$. Let r be a positive integer. A C^r map $g : D_1 \to D_2$ is called a C^r *diffeomorphism* if g is one-to-one, onto, and its inverse mapping $g^{-1} : D_2 \to D_1$, is also C^r.

By the inverse function theorem, a C^r map $g : D_1 \to D_2$ is a C^r diffeomorphism if and only if it is 1-1, onto, and, for each $x \in D_1$, the derivative of g at x is a linear isomorphism of $\mathbf{R}^n$.

The set $\mathcal{D}^r(D)$ of C^r diffeomorphisms from D to itself is a non-commutative group with the composition operation as group product

$$g_1 \cdot g_2 = g_1 \circ g_2.$$

The identity of this group is simply the identity transformation.

Exercise. Recall that a *topological group* is a triple $(G, \mathcal{T}, \cdot)$ in which $(G, \mathcal{T})$ is a topological space, $(G, \cdot)$ is a group (i.e., $\cdot$ is an associative operation with an identity and each element h has an inverse h^{-1}), and the mapping $(g, h) \to g \cdot h^{-1}$ from the product space $G \times G$ to G is continuous. Sometimes we say that the group G is a topological group and we suppress writing the explicit topology and group operation. Give the set $\mathcal{D}^r(D)$ the topology of uniform C^r convergence on compact subsets of D. Then this topology makes the group $(\mathcal{D}^r(D), \circ)$ into a topological group.

Given a system (1), and a point $x \in D$, we call the maximal soltution curve $t \to \phi(t, x)$ the *orbit* or *path* through x. Sometimes we also call the image set $\{\phi(t, x)\}$ of a maximal solution through x, the orbit through x.

Remarks.

1. Any two orbits are distinct or identical
2. If the open set D has the property that it contains all maximal solutions through its points, then D can be represented as the disjoint union of distinct orbits.
3. Any orbit is a point, a 1-1 continuous image of an open interval (including the whole line), or a circle. In the second case the orbit can accumulate on itself infinitely often.
4. The solution whose orbit is a topological circle is called a periodic solution. Alternatively, a periodic solution $\phi(t, x)$ is one for which there is a real $\tau > 0$ such that $\phi(\tau, x) = x$ and $\phi(s, x) \neq x$ for $0 < s < \tau$. The number τ is called the *period* of the periodic solution. Sometimes periodic solutions are called *periodic orbits* or *closed orbits*.

Since any solution may be extended to a maximal solution, from now on, unless explicitly stated otherwise, every solution will be assumed to be maximal.