2. Linear Transformations

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is *linear* if it satisfies the following two properties:

1. $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathcal{X}$
2. $f(\alpha x) = \alpha f(x)$ for all $x \in \mathcal{X}, \alpha \in \mathbf{R}$.

A linear map $f : \mathcal{X} \to \mathcal{Y}$ is called *bounded* if there is a constant $C > 0$ such that $| f(x) | \leq C | x |$ for all $x \in \mathcal{X}$.

Facts:

1. A linear map f is bounded if and only if it is continuous.
2. The linear map f is bounded if and only if the quantity $\sup_{| x | \leq 1} | f(x) |$ is finite.
3. The quantity $\sup_{| x | \leq 1} | f(x) |$ in the preceding statement is also equal to $\sup_{| x | = 1} | f(x) |$
4. Every linear map whose domain is $\mathbf{R}^n$ or $\mathbf{C}^n$ is bounded (hence continuous).

If f is a bounded linear map (transformation), we set $| f | = \sup_{| x | = 1} | f(x) |$. This defines a norm in the space $L(\mathcal{X}, \mathcal{Y})$ of bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$, making it into a Banach space also.

Fixed Point Theorems

Many existence theorems for differential equations can be reduced to fixed point theorems in appropriate function spaces. Here we will discuss a few relevant results.

Let X be a metric space and let $T : X \to X$ be a mapping. A fixed point of T is a point $x \in X$ such that $T(x) = x$.

A self-map T of a metric space $\mathcal{X}$ is called a contraction (or contraction map or mapping) if there is a constant $0 < \lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in \mathcal{X}$. Thus, $T : \mathcal{X} \to \mathcal{X}$ is a contraction if and only it is Lipschitz with Lipschitz constant less than 1.

Theorem. (Contraction Mapping Theorem) *Suppose $\mathcal{X}$ is a complete metric space and $T : \mathcal{X} \to \mathcal{X}$ is a contraction map. Then, T has a unique fixed point $\bar{x}$ in $\mathcal{X}$. Moreover, if x is any point in $\mathcal{F}$, then the sequence of iterates $x, Tx, T^2x, \ldots$ converges to $\bar{x}$ exponentially fast.*

Proof.

Uniqueness:

If $0 < \lambda < 1$ is the contraction constant for T and $Tx = x, Ty = y$, then

$$d(x, y) = d(Tx, Ty) \leq \lambda d(x, y)$$

which implies that $d(x, y) = 0$. This in turn implies that $x = y$. QED.

Existence:

Let $x_0 = x, x_1 = Tx, x_i = T^i x, \ldots$.

Then,

$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq \ldots \leq \lambda^n d(x_1, x_0)$ for $1 \leq n$. Thus, for $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ &\leq (\lambda^{m-1} + \lambda^{m-2} + \ldots + \lambda^n) d(x_1, x_0) \\ &= \frac{\lambda^n (1 - \lambda^{m-n})}{1 - \lambda} d(x_1, x_0) \\ &\leq C\lambda^n d(x_1, x_0) \end{aligned}$$

This implies that the sequence $x_1, x_2, \ldots$ is a Cauchy sequence. By completeness of X, it converges, say to an element $\bar{x}$ of $\mathcal{X}$. But, since T is continuous,

$$T(\bar{x}) = T(lim_{n\to\infty} x_n) = lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = \bar{x},$$

so, $T(\bar{x}) = \bar{x}$.

This proves the existence and the exponential convergence. QED.

The preceding theorem gives a useful sufficient condition for the existence of fixed points in a wide variety of situations. It is frequently useful to know when such fixed points depend continuously on parameters. This leads us to the next result.

Definition. Let Λ be a topological space (e.g. a metric space), and let $\mathcal{X}$ be a complete metric space. A map T from Λ into the space of maps $\mathcal{M}(X, X)$ is called a *continuous family of self-maps of* $\mathcal{X}$ if the map $\bar{T}(\lambda, x) = T(\lambda)(x)$ is continuous as a map from the product space $\Lambda \times \mathcal{X}$ to $\mathcal{X}$. The map T is called a *uniform family of contractions* on $\mathcal{X}$ if it is a continuous family of self-maps of $\mathcal{X}$ and there is a constant $0 < \alpha < 1$ such that

$$d(\bar{T}(\lambda, x), \bar{T}(\lambda, y)) \leq \alpha d(x, y)$$

for all $x, y \in \mathcal{X}, \lambda \in \Lambda$.

Thus, the continuous family is a uniform family of contractions if and only if all the maps in the family have the same upper bound $\alpha < 1$ for their Lipschitz constants.

Given the family T as above, we define the map $T_\lambda : \mathcal{X} \to \mathcal{X}$ by

$$T_\lambda(x) = T(\lambda)(x) = \bar{T}(\lambda, x)$$

Theorem. *If $T : \Lambda \to \mathcal{M}(X, X)$ is a uniform family of contractions on $\mathcal{X}$, then each map T_λ has a unique fixed point x_λ which depends continuously on λ. That is, the map $\lambda \to x_\lambda$ is a continuous map from Λ into $\mathcal{X}$.*

Proof. Let $g(\lambda)$ be the fixed point of the map T_λ which exists since the map T_λ is a contraction.

For $\lambda_1, \lambda_2 \in \Lambda$, we have

$$\begin{aligned} d(g(\lambda_1), g(\lambda_2)) &= d(T_{\lambda_1} g(\lambda_1), T_{\lambda_2} g(\lambda_2)) \\ &\leq d(T_{\lambda_1} g(\lambda_1), T_{\lambda_1} g(\lambda_2)) + d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2)) \\ &\leq \alpha d(g(\lambda_1), g(\lambda_2)) + d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2)) \end{aligned}$$

This implies that

$$d(g(\lambda_1), g(\lambda_2)) \leq (1 - \alpha)^{-1} d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2))$$

Since the map $\lambda \to T_\lambda g(\lambda_2)$ is continuous for fixed λ_2, we see that $\lambda \to g(\lambda)$ is continuous. QED.

There is another useful criterion for the existence of fixed points of transformations in Banach spaces.

Let X be a Banach space. Let $x, y \in X$. The line segment in X from x to y is the set of points $\{(1-t)x + ty : 0 \leq t \leq 1\}$. A subset F of X is called *convex* if for any two points $x, y \in F$, each point in the line segment from x to y is contained in F.

Examples:

1. Linear subspaces are convex.

2. open and closed balls are convex

The following are three remarkable theorems.

Theorem.(Brouwer Fixed Point Theorem). *Every continuous map T of the closed unit ball in $\mathbf{R}^n$ to itself has a fixed point.*

Theorem.(Schauder Fixed Point Theorem). *Every continuous self-map of a compact convex subset of a Banach space has a fixed point.*

Theorem.(Schauder-Tychonov Fixed Point Theorem).*Every continuous self-map of a compact convex subset of a locally convex linear topological space to itself has a fixed point.*