Hyperbolic Periodic Orbits

Definition. A C^k *hypersurface* in $\mathbf{R}^n$ is a pair (ψ, U) in which

1. U is an open set in $\mathbf{R}^{n-1}$.
2. ψ is a 1-1 C^k map from U into $\mathbf{R}^n$.
3. $D\psi(x)$ has rank $n-1$ at each $x \in U$.

Given a C^k hypersurface (ψ, U) in $\mathbf{R}^n$ and a point $p \in \psi(U)$, we let $T_p\psi(U)$ denote the tangent space to $\psi(U)$ at p. This is an affine hyperplane in $\mathbf{R}^n$ passing through p.

Sometimes we call the image set $\psi(U)$ the hypersurface, or we just say Σ is a hypersurface in $\mathbf{R}^n$ when we mean that Σ is the image of a hypersurface, and we call the defining map ψ a *parametrization* of Σ.

Suppose that X is a C^k vector field, $k \geq 1$, in an open set in $\mathbf{R}^n$. Let γ be a non-critical orbit of X, and let $p \in \gamma$. A C^k hypersurface transverse to γ at p is a C^k hypersurface (ψ, U) such that $\psi(U) \bigcap \gamma = \{p\}$ and $X(p) \notin T_p\psi(U)$.

Now suppose that γ is a non-critical periodic orbit of X and $p \in \gamma$. Let (ψ, U) be a C^k transverse hypersurface to γ at p. Let $\phi(t, x)$ denote the local flow of X. There is a neighborhood V of p in $\psi(U)$ and a C^k function $\tau : V \to \mathbf{R}$ such that

1. τ is bounded below; i.e., there is a constant $c > 0$ such that $\tau(x) > c$ for all $x \in V$.
2. $\phi(\tau(x), x) \in \psi(U)$ and $\phi(s, x) \in \psi(U)$ for $0 < s < \tau(x)$.

The function τ is called the *first-return time* to $\psi(U)$. It depends on the choice of V.

Given such a V and τ as above there is a map $\Phi : V \to \psi(U)$ defined by $\Phi(x) = \phi(\tau(x), x)$. This is called the Poincare map (or first return map) of X on V.

Facts:

1. There is a neighborhood V_1 of p in V such that the map $\psi^{-1} \circ \Phi \circ \psi$ is a C^k diffeomorphism from $\psi^{-1}(V_1)$ onto its image in $\mathbf{R}^{n-1}$.
2. The eigenvalues of the derivative of $\psi^{-1} \circ \Phi \circ \psi$ at $\psi^{-1}(p)$ are independent of the choice of $p \in \gamma$, ψ and V_1.

These eigenvalues are called the *characteristic multipliers* of γ. If γ is such that its characteristic multipliers have norm different from 1, then we call γ a *hyperbolic* periodic orbit for X.

Remark. A sufficient condition that a periodic orbit be asymptotically stable is that it characteristic multipliers all have norm less than 1.

We will leave the verification of the above remarks and facts to the exercises.

Lyapunov Stability

It is frequently of interest to determine stability and asymptotic stability of orbits of a vector field X. The method of Lyapunov (usually called Lyapunov's Direct Method) provides a tool for doing this without the need for finding actual solutions (integral curves) of X. It is mainly of use in analyzing stability of critical points and periodic solutions.

We proceed to discuss first the case of critical points.

Let $\Omega \subset \mathbf{R}^n$ be an open set in $\mathbf{R}^n$, and let $V : \mathbf{R}^n \to \mathbf{R}$ be a function. We say that $x_0 \in \Omega$ is a *strict relative minimum of* V if there is a neighborhood N of x_0 in Ω such that, for $x \in N \setminus \{x_0\}$, $V(x) > V(x_0)$.

If X is a C^1 vector field in Ω, a function $V : \Omega \to \mathbf{R}$ is called a *Lyapunov function* for X on Ω if

1. V is continuous on Ω.

2. V is non-increasing on orbits of X.

If in addition, V is strictly decreasing along non-critical orbits of X in Ω, then V is called a *strict Lyapunov function* for X.

If x_0 is a critical point of X, and V is a Lyapunov function defined in a neighborhood of x_0, we say that V is a Lyapunov function for X *near* x_0. Similarly, we define a strict Lyapunov function *near* x_0.

Throughout the following, we assume that the vector field X is defined and C^1 in the open set $\Omega \subseteq \mathbf{R}^n$.

Theorem.(Stability Criterion) *Suppose x_0 is a critical point of X and there is a Lyapunov function near x_0 which has x_0 as a strict relative minumum. Then, x_0 is a stable critical point of X. If x_0 is an isolated critical point of X and there is a strict Lyapunov function V for X near x_0 having x_0 as a strict relative minimum, then x_0 is asymptotically stable.*

Proof.

Let $\phi(t,x)$ denote the solution of $\dot{x} = X(x)$, with $\phi(0,x) = x$.

Let $B_\delta(x), \bar{B}_\delta(x)$ denote, respectively, the open and closed balls of radius δ centered at x.

Assume that V is a Lyapunov function for X near x_0 having x_0 as a strict relative minimum, and let $\epsilon_1 > 0$ be small enough so that

$$\bar{B}_{\epsilon_1}(x_0) \subset \Omega \tag{1}$$

$$V(x) \text{ is defined and continuous in } \bar{B}_{\epsilon_1}(x_0) \tag{2}$$

and

$$V(x) > V(x_0) \text{ for } x \in \bar{B}_{\epsilon_1}(x_0) \setminus \{x_0\} \tag{3}$$

Let $\epsilon \in (0, \epsilon_1)$ We wish to find $\delta > 0$ so that if $x \in B_\delta(x_0)$, and $t \geq 0$, then

$$\phi(t,x) \text{ is defined for } t \geq 0 \tag{4}$$

$$\phi(t,x) \in B_\epsilon(x_0) \text{ for } t \geq 0. \tag{5}$$

Note that if I is a maximal interval on which the solution $\phi(t,x)$ is defined, and, for $t \in I \bigcap [0,\infty)$, we have $\phi(t,x) \in B_\epsilon(x_0)$, then condition (4) above will be satisfied since $X(\phi(t,x))$ cannot become unbounded and $\phi(t,x)$ cannot approach the boundary of Ω.

So, it suffices to obtain condition (5).

Since, $V(u) > V(x_0)$ for $u \in \partial B_\epsilon(x_0)$ and V is continuous, there is a $C > 0$ such that

$$V(u) > V(x_0) + C$$

for $u \in \partial B_\epsilon(x_0)$ where ∂E denotes the boundary of a set E.

Since V is continuous in Ω, there is a $\delta \in (0, \epsilon)$ such that if $x \in B_\delta(x_0)$, then $V(x) < V(x_0) + C$.

Now, since V is non-increasing along solutions, it follows that if $t \geq 0$, then $V(\phi(t,x)) \leq V(x) < V(x_0) + C$, so $\phi(t,x)$ can never meet the boundary $\partial B_\epsilon(x_0)$.

This implies (5) and proves that x_0 is stable.

Now, suppose that V is a strict Lyapunov function for X near x_0, x_0 is an isolated critical point of X, and a strict relative minimum of V.

Let ϵ, δ be as above. We wish to show that

(*) if $x \in B_{\delta}(x_0)$ then $\phi(t,x) \to x_0$ as $t \to \infty$.

If (*) fails, then there is an $\omega-$limit point, say y, in $B_\epsilon(x_0) \setminus \{x_0\}$.

Thus, there is a sequence $0 < t_1 < t_2 < \ldots$ with $t_{i+1} - t_i \to \infty$ and $\phi(t_i, x) \to y$ as $i \to \infty$.

Since x_0 is an isolated critical point of X, we may assume that X has no critical points in $B_\epsilon(x_0)$ except x_0. Thus, $X(y) \neq 0$.

Since $t \to V(\phi(t,y))$ is strictly decreasing, there is a constant $C_1 > 0$ and a time $\bar{t} > 0$ such that

$$V(\phi(\bar{t}, y)) < V(y) - C_1 \tag{6}$$

By continuity of solutions on initial conditions, and continuity of V, there is an $i_0 > 0$ such that if $i \geq i_0$, then

$$V(\phi(\bar{t} + t_i, x)) < V(y) - \frac{C_1}{2} \tag{7}$$

and,

$$V(\phi(t_i, x)) > V(y) - \frac{C_1}{2} \tag{8}$$

Now, take $s > i_0$ such that $t_{s+1} - t_s > \bar{t}$.

Since $t \to V(\phi(t,x))$ is strictly decreasing, we have

$$V(y) - \frac{C_1}{2} < V(\phi(t_{s+1}, x)) < V(\phi(\bar{t} + t_s, x)) < V(y) - \frac{C_1}{2}$$

which is a contradiction.

Hence, (*) holds. QED

Suppose now that X is C^1 and V is a C^1 function, both on the open set Ω.

If $\phi(t,x)$ is the solution starting at x as above, then

$$\frac{d}{dt} V(\phi(t,x)) = (grad\ V)(\phi(t,x)) \cdot X(\phi(t,x))$$

where $(grad\ V)(x)$ is the vector $(\frac{\partial V}{\partial x_1}(x), \ldots, \frac{\partial V}{\partial x_n}(x))$.

So, a sufficient condition for V to be a Lyapunov function is

$$(grad\ V)(x) \cdot X(x) \leq 0$$

at each $x \in \Omega$.

Similarly,

$$(grad\ V)(x) \cdot X(x) < 0$$

at each non-critical x is sufficient for V to be a strict Lyapunov function.

The next result shows that a Lyapunov function is constant on $\omega-$limit sets of bounded orbits.

Theorem. *Suppose that V is a Lyapunov function for a C^1 vector field X and V is continuous on a closed set $G \subseteq \mathbf{R}^n$ which is invariant for X (i.e., $X-$orbits starting in G stay there for all time). Suppose that $x \in G$ and the forward orbit of x is bounded. Then, V is constant on the $\omega-$limit set of x.*

Proof.

Let $\phi(t, x)$ denote the orbit of X with $\phi(0, x) = x$ as usual.

The function $V(\phi(t, x))$ is non-increasing in t. Since, $o_+(x)$ is bounded and in G, , $\{V(\phi(t, x)), t \geq 0\}$ is bounded, so

$$\lim_{t \to \infty} V(\phi(t, x)) \equiv c$$

exists.

Let $y \in \omega(x)$. Since G is closed, $y \in G$ and V is defined at y. By continuity of $t \to V(\phi(t, x))$, $V(y) = c$. Thus, $\omega(x) \subseteq \{y : V(y) = c\}$ QED.

We give a final result which guarantees global stability of a critical point.

Theorem 0.1 *Suppose X is a C^1 vector field on all of $\mathbf{R}^n$ with a critical point at x_0, and that V is a C^1 Lyapunov function for X having x_0 as a strict absolute minimum and such that*

$$V(x) \to \infty \ \textit{as} \ |\ x\ | \to \infty \tag{9}$$

and

$$(grad\ V)(x) \cdot X(x) < 0 \tag{10}$$

for each $x \neq x_0$. Then, x_0 is globally asymptotically stable. That is, x_0 is asymptotically stable, and, for any $x \in \mathbf{R}^n$, $\phi(t,x) \to x_0$ as $t \to \infty$.

Proof.

Since V has x_0 as a strict minimum and (10) holds, we see that V is a strict Lyapunov function and x_0 is the only critical point of X in $\mathbf{R}^n$. It follows that x_0 is asymptotically stable.

It remains to show that, for $x \in \mathbf{R}^n$,

$$\phi(t,x) \to x_0 \text{ as } t \to \infty. \tag{11}$$

Fix $x \in \mathbf{R}^n$.

By (9), and the fact that V decreases along the curve $\phi(t,x)$, there is a constant $C > 0$ such that $\phi(t,x) \in B_C(x_0)$ for all $t \geq 0$.

Then, the positive orbit of x, $o_+(x)$ is bounded.

Let $y \in \omega(x)$. Then, V is constant on the orbit of y by the previous theorem.

It follows from condition (10) that $y = x_0$. This proves (11). QED.

Some special Lyapunov functions

For critical points of certain vector fields, one can explicity construct Lyapunov functions. Although this is fairly rare, it is worthwhile to descbibe some conditions when it is possible.

Let $m, n > 0$ be positive even integers, and let $a, b > 0$ be positive real numbers. Consider the function $V : \mathbf{R}^2 \to \mathbf{R}$ defined by $V(x,y) = ax^m + by^n$. This is clearly a smooth function with $(0,0)$ as a strict absolute minimum.

Let

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

be a C^1 planar vector field having $(0,0)$ as a critical point.

The function V, the vector field X, and the critical point $x_0 = (0,0)$ will satisfy the conditions of Theorem 0.1 provided that

$$V_x P + V_y Q < 0 \tag{12}$$

for each (x,y) near $(0,0)$ and different from $(0,0)$.

Exercise. Use this method (i.e., find appropriate a, b, m, n) to prove that $(0,0)$ is globally asymptotically stable for

$$P(x,y) = -x^3 + 3xy^3, \ Q(x,y) = -x^2 - y^3.$$