Jordan Canonical Form

Suppose that A is an $n \times n$ matrix with characteristic polynomial.

$$p(\lambda) = (\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_s)^{m_s}$$

and generalized eigenspaces $V_j = ker(A - \lambda_j I)^{m_j}$.

Let L be the linear map defined by $Lx = Ax$ for all x.

The space V_j is mapped into itself by the linear map defined by L. There is a basis for the space V_j such that in that basis the map L has the matrix representation

$$B_j = \lambda_j I + N_j$$

and

N_j is a block matrix $diag(N_{j,1}, \dots, N_{j,d_j})$ where each $N_{j,k}$ is a square matrix whose only non-zero entries are $1's$ on the super-diagonal (i.e., just above the diagonal).

The matrix $B = diag(B_1, \dots, B_s)$ is unique up to a permutation of the $B_k's$ or $N_{j,k}'s$.

This matrix B is called the *Jordan canonical form* of the matrix A.

If the eigenvalues of A are real, the matrix B can be chosen to be real. If some eigenvalues are complex, then the matrix B will have complex entries.

However, if A is real, then the complex eigenvalues come in complex conjugate pairs, and this can be used to give a *real Jordan canonical form.* In this form, if $\lambda_j = a_j + ib_j$ is a complex eigenvalue of A, then the matrix B_j will have the form

$$B_j = D_j + N_j$$

where $D_j = diag(E_j, E_j, \dots, E_j)$ and E_j is the 2×2 diagonal matrix

$$\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

and N_j is a block matrix of the form $diag(N_{j,1}, \dots, N_{j,d_j})$ in which each $N_{j,k}$ is a square matrix whose only non-zero terms lie in blocks of 2×2 identity matrices in the super 2-block diagonal.

For instance, in case $\lambda_j = 2 + i$ and $\bar{\lambda}_j = 2 - i$ has multiplicity 4, one can have the following form for B_j.

$$\begin{bmatrix} 2 & -1 & 1 & 0 & & & & \\ 1 & 2 & 0 & 1 & & & & \\ & & 2 & -1 & 1 & 0 & & \\ & & 1 & 2 & 0 & 1 & & \\ & & & & 2 & -1 & 1 & 0 \\ & & & & 1 & 2 & 0 & 1 \\ & & & & & & 2 & -1 \\ & & & & & & 1 & 2 \end{bmatrix}$$

Linear Periodic Systems

Consider the homogeneous linear periodic system

$$\dot{x} = A(t)x, \quad A(t+T) = A(t), \quad T > 0 \tag{1}$$

where $A(t)$ is a continuous $n \times n$ real or complex matrix which is periodic of period T in t.

Lemma 1. *If C is an $n \times n$ complex matrix with det $C \neq 0$, then there is a complex matrix B such that $C = e^B$.*

Proof.

Since *det* $C \neq 0$, the eigenvalues of C must all be non-zero.

Using the decomposition into generalized eigenspaces, it suffices to deal with a single generalized eigenspace and a single non-zero eigenvalue, say λ.

Thus, we may assume there is a non-singular matrix P such that

$$PCP^{-1} = \lambda I + N \equiv C_1$$

where $N^r = 0$ for some $r > 0$.

Let us first work with $C_1 = \lambda I + N = \lambda(I + \frac{1}{\lambda}N)$.

Let $b = \log \lambda$ (complex if λ is not real and positive).

We claim: there is a matrix S such that $e^S = I + \frac{1}{\lambda}N$.

Assuming the claim, we get

$$\begin{aligned} C_1 &= e^b e^S \\ &= e^{bI} e^S \\ &= e^{bI+S} \\ &= e^{B_1} \end{aligned}$$

with $B_1 = bI + S$.
Then,

$$PCP^{-1} = C_1 = e^{B_1}$$

and,

$$C = P^{-1}C_1P = P^{-1}e^{B_1}P = e^{P^{-1}B_1P}$$

so, it suffices to prove the claim.
We want S such that $e^S = I + \frac{1}{\lambda}N$.
The real power series for $\log(1+x)$ is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

So, we might try

$$S = \log(I + \frac{1}{\lambda}N) = \frac{N}{\lambda} - \frac{N^2}{2\lambda^2} + \frac{N^3}{3\lambda^3} - \dots$$

Since N is nilpotent, this is a finite sum, and we leave it as an exercise to show that $e^S = C_1$ as required. QED.

Remark. Note that even if C were a real matrix with $det(C) \neq 0$, the above lemma may only yield a complex matrix S such that $e^S = C$. It is interesting to ask for the conditions under which the matrix S can also be chose to be real. The next lemma provides the answer.

Lemma 2. *If C is an $n \times n$ real matrix, then there is a real matrix S such that $C = e^S$ if and only if $det(C) \neq 0$ and C is a square (i.e., there is a real matrix A such that $C = A^2$.*

Proof. The necessity is easy since we simply take $A = e^{\frac{1}{2}S}$, and observe that $det(e^S) \neq 0$ for any S.

For sufficiency, again we have that all the eigenvalues of C are non-zero. Since C is real, each complex eigenvalue λ its complex conjugate $\bar{\lambda}$ occurs with the same multiplicity. Also, since $C = A^2$, the real negative eigenvalues must have even multiplicity. Identifying C with its associated linear operator as usual, and taking the direct sum decomposition of $\mathbf{R}^n = V_1 \oplus V_2 \oplus \dots \oplus V_k$ into generalized eigenspaces of C, we can express C as the sum

$$C = A_1 + A_2 + \dots + A_k$$

where

1. $A_i A_j$ is the zero matrix for $i \neq j$,
2. for each $1 \leq i \leq k$, $dim(V_i)$ is even, and
3. $A_i \mid V_i$ is non-singular and has either a single non-zero real eigenvalue or a single pair of non-zero complex conjugate eigenvalues; the negative real eigenvalues have even multiplicity.

It then suffices to show that there is a real matrix S_i such that $A_i = e^{S_i}$ for each i. We leave this last part as an exercise. QED.

Theorem.(Floquet) *Every fundamental matrix* $\Phi(t)$ *for (1) has the form*

$$\Phi(t) = P(t)e^{Bt} \tag{2}$$

where $P(t)$ *is a periodic matrix of period* T *and* B *is a constant matrix (which may by complex). We may always obtain (1) with a real matrix* B *where* $P(t)$ *has period* $2T$*.*

Proof.

Let $\Phi(t)$ be a fundamental matrix for (1).

Then, letting $u = u(t) = t + T$, and using $A(t+T) = A(t)$, we get

$$\begin{aligned} \frac{d}{dt}\Phi(t+T) &= \frac{d}{du}\Phi(u) \\ &= A(u)\Phi(u) \\ &= A(t+T)\Phi(t+T) \\ &= A(t)\Phi(t+T) \end{aligned}$$

so, $\Phi(t+T)$ is also a solution matrix. Since it is non-singular, it is a fundamental matrix. Thus, there is a non-singular matrix C (possibly comples) such that

$$\Phi(t+T) = \Phi(t)C \tag{3}$$

By Lemma 1, there is a (possibly complex) matrix B such that $e^{BT} = C$. Now, letting $P(t) = \Phi(t)e^{-Bt}$ we get $\Phi(t) = P(t)e^{Bt}$ and

$$P(t+T) = \Phi(t+T)e^{-B(t+T)} = \Phi(t)e^{-Bt} = P(t).$$

In order to choose B to be real, we simply need the matrix C to be a square of some real matrix. But by (3), we have

$$\Phi(t+2T) = \Phi(t+T+T) = \Phi(t+T)C = \Phi(t)C^2.$$

Thus, replacing T by $2T$ in (3), we may obtain a real matrix B such that $e^{B2T} = C^2$. Repeating the above argument then gives the result. QED

Corollary. *There is a nonsingular periodic transformation of variables (of period T or $2T$) taking (1) into a linear differential equation with constant coefficients.*

Proof.

Let $P(t), B$ be as above, and set $x = P(t)y$.

We may choose $P(t)$ to be of period T or $2T$ as above.

Then,

$$\begin{aligned} \dot{x} &= \dot{P}y + P\dot{y} \\ &= Ax \\ &= APy \end{aligned}$$

So,

$$APy = \dot{P}y + P\dot{y}$$

But, $P = \Phi e^{-Bt}$, or $Pe^{Bt} = \Phi$, so

$$\dot{P}e^{Bt} + PBe^{Bt} = APe^{Bt}$$

or

$$\dot{P} + PB = AP$$

or

$$APy = (AP - PB)y + P\dot{y}$$

or

$$PBy = P\dot{y}$$

or $By = \dot{y}$ since P is non-singular. QED