Another index formula

For a Jordan curve γ, let us write *int* γ for the bounded interior region of the complement of γ.

Let f be a planar C^1 vector field with an isolated critical point x_0. and let γ be a positively oriented C^1 Jordan curve so that the only critical point of f in $int\ \gamma \bigcup \gamma$ is x_0. Let $\phi(t,x)$ be the local flow of f. A point $y \in \gamma$ at which f is tangent to γ is called an *exterior tangency* if there is an $\epsilon > 0$ such that $\phi(t,x)$ is in the exterior of γ for $t \in (-\epsilon, \epsilon) \setminus \{0\}$. Similarly, the point y of tangency is an *interior tangency* if there is an $\epsilon > 0$ such that $\phi(t,x) \in int\ \gamma$ for $t \in (-\epsilon, \epsilon) \setminus \{0\}$.

Note that a tangency may be neither exterior nor interior.

Theorem. *Suppose that f and γ are as above and there are only finitely many points of tangency of f and γ and all of these tangencies are exterior or interior. Let N_i be the number of interior tangencies and N_e be the number of exterior tangencies. Then,*

$$Ind(x_0, f) = 1 + \frac{1}{2}(N_i - N_e)$$

The main part of the proof involves a combinatorial lemma which we now give.

For a non-negative integer n, let $\mathcal{A}_n$ be the set of all finite sequences $(a_0, a_1, \ldots, a_n)$ for which each $a_j = i$ or e. Given such a finite sequence, $(a_0, \ldots, a_n)$, call n its length.

For $a = (a_0, a_1, \ldots, a_n) \in \mathcal{A}_n$, let $N_i(a)$ be the number of $i's$ occurring in a, and let $N_e(a)$ be the number of $e's$ occurring in a.

Letting $\mathcal{A} = \bigcup_n \mathcal{A}_n$ be the collection of all finite sequences of $i's$ and $e's$, we define a function $\eta : \mathcal{A} \to \mathbf{Z}$ inductively by the following.

1. $\eta(a_0) = 0$ for all a_0.
2. $\eta(i,i) = 1, \eta(e,e) = -1, \eta(i,e) = 0, \eta(e,i) = 0$
3. For $n \geq 2$,

$$\eta(a_0, \ldots, a_n) = \begin{cases} \eta(a_0, \ldots, a_{n-1}) + 1 & \text{if } (a_{n-1}, a_n) = (i,i) \\ \eta(a_0, \ldots, a_{n-1}) - 1 & \text{if } (a_{n-1}, a_n) = (e,e) \\ \eta(a_0, \ldots, a_{n-1}) & \text{otherwise} \end{cases}$$

Lemma. *Let η be as above. Then, for any $(a_0, \ldots, a_n) \in \mathcal{A}_n$ with $a_0 = a_n$*

(a) if $a_0 = a_n = i$, then $\eta(a_0, \ldots, a_n) = N_i - N_e - 1$.

(b) if $a_0 = a_n = e$, then $\eta(a_0, \ldots, a_n) = N_i - N_e + 1$.

Proof. We use induction on the length n. By definition, the Lemma holds for $n = 0, 1$.

Let $n \geq 2$, and assume that lemma holds for every sequence of length less than n.

Let $(a_0, \ldots, a_n)$ have length n.

Suppose $a_0 = a_n = i$

Case 1: There is no integer $j < n$ for which $a_j = e$. Then, $\eta((a_0, \ldots, a_k)) = k-1$ for all $k \leq n$, so $\eta(a_0, \ldots, a_n) = n = N_i - N_e - 1$

Case 2: There is a least integer $j > 0$ for which $a_j = e$. Let $b_k = a_k$ for $k < j$, and $b_k = a_{k+1}$ for $j \leq k < n$. That is, we remove the e at a_j.

Then, the length of b is $n - 1$.

Case 2a: $a_{j+1} = i$. Then, $\eta(b_0, \ldots, b_j) = \eta(b_0, \ldots, b_{j-1}) + 1$, and adding more $b_\ell' s$ gives $\eta(b) = \eta(a) + 1$.

By induction, $\eta(b) = N_i(b) - N_e(b) - 1$. But, $N_i(a) = N_i(b), N_e(a) = N_e(b) + 1$.

So, $\eta(a) = \eta(b) - 1 = N_i(b) - (N_e(a) - 1) - 1 - 1 = N_i(a) - N_e(a) - 1$.

Case 2b: $a_{j+1} = e$. Then, $\eta(b) = \eta(a) + 1$. Also, $N_i(a) = N_i(b), N_e(a) = N_e(b) + 1$.

So, $\eta(a) = \eta(b) - 1 = N_i(b) - N_e(b) - 1 - 1 = N_i(a) - N_e(a) - 1$.

This takes care of the case in which $a_0 = a_n = i$.

Now suppose $a_0 = a_n = e$.

Proceding in the same way, we get $\eta(a) = N_i(a) - N_e(a) + 1$. QED

Proof of the theorem.

Let $y_0, y_1, \ldots, y_n$ be the tangencies of f at the curve γ where γ is as in the statement of the theorem.

Let $a_j = i$ if the tangency at y_j is interior, and let $a_j = e$ if the tangency at y_j is exterior.

Let $var(y_0, y_m, \gamma)$ be the angular variation of the tangent vector to γ from y_0 to y_m, and let $var(y_0.y_m, f)$ be the angular variation of f from y_0 to y_m.

Let $\beta : \bigcup_{0\leq m\leq n} \mathcal{A}_m \to \mathbf{Z}$ be defined as follows.

$$\beta(a_0, a_m) = var(y_0.y_m, f) - var(y_0, y_m, \gamma)$$

Then, one can check that $\frac{1}{\pi}\beta$ on the sequences in $\bigcup_{0\leq m\leq n} \mathcal{A}_m$ satisfies the same conditions as the function η of the preceding lemma.

Case 1: $a_0 = a_n = i$

We get, for $a = (a_0, \dots, a_n)$,

$$\beta(a_0, a_n) = \pi(N_i(a) - N_e(a) - 1)$$

This gives

$$var(y_0, y_n, f) = var(y_0, y_n, \gamma) + \pi(N_i(a) - N_e(a) - 1)$$

or,

$$2\pi Ind(f, x_0) = 2\pi + \pi(N_i(a) - N_e(a) - 1)$$

But, $N_i = N_i(a) - 1, N_e = N_e(a)$, so we get

$$Ind(f, x_0) = 1 + \frac{1}{2}(N_i - N_e)$$

as required.

Case 2: $a_0 = a_n = e$.

We get, for $a = (a_0, \dots, a_n)$,

$$\beta(a_0, a_n) = \pi(N_i(a) - N_e(a) + 1)$$

This gives

$$var(y_0, y_n, f) = var(y_0, y_n, \gamma) + \pi(N_i(a) - N_e(a) + 1)$$

or,

$$2\pi Ind(f, x_0) = 2\pi + \pi(N_i(a) - N_e(a) + 1)$$

But, $N_i = N_i(a), N_e = N_e(a) - 1$, so we get

$$Ind(f, x_0) = 1 + \frac{1}{2}(N_i - N_e)$$

as required. QED

Let us continue with our assumptions that f is a C^1 planar vector field with an isolated critical point at x_0, and suppose γ is a small positively oriented C^1 Jordan curve γ with $x_0 \in int\ \gamma$ such that f does not vanish on $int\ \gamma \bigcup \gamma$ except at x_0 .

Let $\phi(t, x)$ be the local flow of f. Recall we are assuming that $\phi(t, x)$ is defined for all t.

A solution $\phi(t, x)$ through a point $x \in \gamma$ is called a *positive null solution relative to* γ if it satisfies the following condition.

- $\phi(t, x)$ is defined for all $t \geq 0$, there is a $t_1 > 0$ such that $\phi(t, x) \in int\ \gamma$ for all $t > t_1$ and $\phi(t, x) \to x_0$ as $t \to \infty$.

The solution $\phi(t, x)$ is called a *negative null solution relative to* γ if $\phi(-t, x)$ is a positive null solution for $-f$.

A *null solution relative to* γ is either a positive or negative null solution relative to γ. When γ is understood, we refer simply to null solutions, positive null solutions, or negative null solutions. We also speak of null, positive null, and negative null orbits for the set of points along the images of such solutions.

A solution $\phi(t, x)$ is called *elliptic* if it is both a positive and negative null solution.

A *base interval* in γ is an open interval U in γ whose boundary points belong to null orbits.

The base interval U is a *parabolic* interval if all of its points belong to positive null orbits or all of its points belong to negative null orbits. The base interval is an *elliptic* interval if all of its points belong to elliptic orbits.

A base interval which is called *hyperbolic* if none of its points belongs to a null orbit. Suppose that there is at least one null solution. Then, it can be shown that, for any point x in a hyperbolic interval there are times $t_2(x) > 0, t_1(x) < 0$ such that $\phi(t, x)$ is exterior to γ for all $t \notin (t_1(x), t_2(x))$.

The union of the set of orbits of points belonging to a parabolic, elliptic, or hyperbolic base interval will be called, respectively, a parabolic, elliptic, or hyperbolic sector.

The following theorem can be proved more or less like the previous one.

Theorem. *Suppose that f, x_0, and γ are as above and that there are finitely many points $y_0, y_1, \ldots, y_n$ on γ whose orbits are base solutions. Assume that the points in $\gamma \setminus \{y_0, \ldots, y_n\}$ belong only to elliptic, hyperbolic, and parabolic intervals. Let N_{ell}, N_{hyp} denote, respectively the number of elliptic, hyperbolic intervals. Then,*

$$Ind(f, x_0) = 1 + \frac{1}{2}(N_{ell} - N_{hyp})$$

For more general information about these topics, see (P. Hartman, *Ordinary Differential Equations*, 1973, Chapter 7).

Note that Hartman's definitions of sectors are slightly different than the ones given here.

We now briefly discuss the concept of index for an isolated critical point of an autonomous vector field in $\mathbf{R}^{n+1}$ for arbitrary $n \geq 1$. This involves the notion of degree of a continuous self-mapping of the $n-$sphere. There are many equivalent ways to define this notion. The simplest involves homology theory. Since we do not assume knowledge of this theory, we will give a definition in terms of integration on the $n-$sphere S^n.

We refer to standard texts; e.g. for the basic concepts of differential forms and integration of manifolds, see (F. Warner, Foundations of differentiable manifolds and Lie groups, Springer, 1983.)

Let D^n be the open unit ball in $\mathbf{R}^n$, and let S^n be the (unit) $n-$sphere; i.e., the set of vectors in $\mathbf{R}^{n+1}$ whose distance from the origin is exactly equal to 1. Write $x = (x_1, x_2, \ldots, x_n)$ for coordinates in $\mathbf{R}^n$. Let $r \geq 1$. A C^r coordinate parametrization in S^n is a pair (U, ϕ) where ϕ is a 1-1 C^r map from D^n into $\mathbf{R}^{n+1}$ such that

1. $\phi(x) \in S^n$ for all $x \in D^n$

2. The columns of the Jacobian matrix of ϕ, $\frac{\partial \phi}{\partial x_i}$ are linearly independent in $\mathbf{R}^{n+1}$.

3. $U = image(\phi)$ and U is an open subset of S^n.

If (U, ϕ) and (V, ψ) are two C^r coordinate parametrizations in S^n, and $U \bigcap V \neq \emptyset$, then the map $\psi^{-1} \circ \phi$ is a C^r diffeomorphism from the open set $\phi^{-1}(U \bigcap V)$ to $\psi(U \bigcap V)$.

If (U, ϕ) is a coordinate parametrization in S^n, we sometimes call the pair (U, ϕ^{-1}) a coordinate chart. Thus, the maps in coordinate parametrizations go from D^n into S^n, and the maps in coordinate charts go from open subsets of S^n to D^n. Coordinate charts (U, η), (V, γ) have the property of differentiability on overlaps in the sense that $\eta \circ \gamma^{-1}$ is C^r where it is defined.

A C^k map $f : S^n \to \mathbf{R}^{n+1}$ is a continuous map from S^n to $\mathbf{R}^{n+1}$ such that, for each C^k coordinate chart (U, ϕ) in S^n, the composition $f \circ \phi^{-1}$ is a C^k map from D^n into $\mathbf{R}^{n+1}$.

A C^k tangent vector field X on S^n is a C^k map $X : S^n \to \mathbf{R}^{n+1}$ such that, for each $x \in S^n$, $X(x)$ is tangent to S^n at x. A positively oriented C^k orthonormal $n-$frame field on S^n is an $n-$tuple $(v_1, \ldots, v_n)$ of C^k tangent vector fields on S^n with the property that, for each $x \in S^n$, the set of vectors $\{v_1(x), \ldots, v_n(x)\}$ is an orthonormal basis for the tangent space to S^n at x, and the determinant $Det(v_1(x), v_2(x), \ldots, v_n(x), x)$ is positive.

There is a C^∞ $n-$form on S^n which gives the value 1 to each positively oriented orthonormal $n-$frame field on S^n. We call this form the *unit volume form* and write it dv.

Using this form, one can define integration for continuous functions $g : S^n \to \mathbf{R}$ by the formula

$$\int_{S^n} g(x) = \int_{S^n} g(x) dv(x)$$

The volume of the whole sphere is then the constant $vol(S^n)$ which is the integral of the constant function whose value is 1 for each point of S^n.

Given a C^1 map $f : S^n \to S^n$, one can pull back the form dv to a form $f^\star dv$, and integrate this form over S^n. One gets an integer multiple of $vol(S^n)$ and this integer is called the *degree* of the map f. We write $deg(f)$ for this integer.

Thus, we have

$$\int_{S^n} f^\star dv = deg(f) vol(S^n)$$

Now, let f be a C^1 vector field in $\mathbf{R}^n$ with an isolated critical point x_0. One defines the index of f at x_0, $Ind(f, x_0)$ in the following way.

Let S_ϵ be a small $(n-1)-$sphere of radius ϵ centered at x_0, and assume that $f(x) \neq 0$ for all x with $0 < \mid x - x_0 \mid \leq \epsilon$. Use f on S_ϵ to define a map $\bar{f} : S^{n-1} \to S^{n-1}$ by the formula

$$\bar{f}(y) = \frac{f(x_0 + \epsilon y)}{\mid f(x_0 + \epsilon y) \mid}$$

Then, we define

$$Ind(f, x_0) = deg(\bar{f})$$

One can show that the definition is independent of the choice of small $(n-1)$-sphere S_ϵ containing x_0 in its interior. It actually can be defined for any sphere S on which f does not vanish. If f and g are two vector fields which can be continuously deformed into one another without vanishing on a sphere S, then they have the same index on S. This index satisfies many nice properties. For instance, there is an analogous formula to that in the theorem earlier in this section which says that, given a smooth vector field X with only isolated critical points on a compact smooth manifold M, the sum of the indices of the X equals the Euler Characteristic of M.

For more information on these topics, we refer to the following books.

1. J. Milnor, *Topology from the differentiable viewpoint*, University Press of Virginia, Charlottesville, Va., 1965

2. M. Hirsch, *Differential Topology*, Springer-Verlag, 1976.