1. Find the general real solution to the differential equation $\dot{x} = Ax$ for each of the following matrices A. Also, draw a rough sketch of the orbits in each case.

   (a)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   (b)

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}$$

   (c)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   (d)

$$A = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the following exercises, we let $\mathcal{X}^1(\mathbf{R}^n)$ denote the set of complete (autonomous) C^1 vector fields on $\mathbf{R}^n$. Thus, an element $X \in \mathcal{X}^1(\mathbf{R}^n)$ is a C^1 map from $\mathbf{R}^n$ into itself such that the domain of each integral curve of X is the entire real line $\mathbf{R}$.

For $X \in \mathcal{X}^1(\mathbf{R}^n)$ with flow $\{\phi_t\}$ and $x \in \mathbf{R}^n$, we define the $\omega-$limit set of x to be the set of points $y \in \mathbf{R}^n$ such that there is a sequence $t_1 < t_2 < \ldots$ with $t_i \to \infty$ as $i \to \infty$ such that

$$\lim_{t_i \to \infty} \phi_{t_i}(x) = y$$

Similarly, we define the $\alpha-$limit set of x to be the set of points $y \in \mathbf{R}^n$ such that there is a sequence $t_1 > t_2 > \ldots$ with $t_i \to -\infty$ as $i \to \infty$ such that

$$\lim_{t_i \to -\infty} \phi_{t_i}(x) = y$$

We denote these sets by $\omega(x) = \omega_X(x)$ and $\alpha(x) = \alpha_X(x)$, respectively.

For $X \in \mathbf{R}^n$, we define the *positive limit set* of X to be the closure of the set of all $\omega-$limit points of X. This set is denoted by $L^+(X)$. Similarly, we define the *negative limit set* of X to be the closure of the set of all $\alpha-$limit points of X. This is denoted by $L^-(X)$.

The limit set $L(X)$ is defined to be $L^+(X) \bigcup L^-(X)$.

By definition, this is a closed subset of $\mathbf{R}^n$ (which may be empty).

2. Let $X \in \mathcal{X}^1(\mathbf{R}^n)$ and let $x \in \mathbf{R}^n$ be a point such that the forward orbit $O_+(x)$ is bounded. Prove that $\omega(x)$ is a non-empty, compact, invariant, connected subset of $\mathbf{R}^n$.

3. Let $X \in \mathcal{X}^1(\mathbf{R})$, and let $x \in \mathbf{R}$. Prove that if $\omega(x)$ is non-empty, then it consists of a critical point.

4. Give an example of and element X in $\mathcal{X}^1(\mathbf{R}^2)$ plane such that $L(X)$ is the union of the unit circle and the origin. (Hint: Use polar coordinates)

5. Let X and Y be elements of $\mathcal{X}^1(\mathbf{R}^n)$ with flows $\{\phi_t\}$, $\{\psi_t\}$, respectively. A *topological conjugacy from* X to Y is a homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$h(\phi_t(x)) = \psi_t(h(x)) \tag{1}$$

for every $t \in \mathbf{R}$ and $x \in \mathbf{R}^n$.

When such an h exists, we say that X and Y are *topologically conjugate.*

(a) Prove that the relation of topological conjugacy is an equivalence relation in $\mathcal{X}^1(\mathbf{R}^n)$.

(b) Say that $X \in \mathcal{X}^1(R^n)$ is *globally stable* if $L^+(X)$ is a single point. Prove that if X and Y are globally stable vector fields on $\mathbf{R}$, then they are topologically conjugate. (This is also true in $\mathbf{R}^n$)