Exercises–Set 2, Math 848, Fall 2011

1. Consider the system

$$\begin{aligned} \dot{x} = & \ 2x - y + z + t^2 * cos(t) \\ \dot{y} = & \ x + y + z \\ \dot{z} = & \ x - 2y + 3z + 2 - t^3 * exp(t) \end{aligned}$$

   Prove that any maximal solution is defined on the whole real line.

2. Let $g : \mathbf{R} \to \mathbf{R}$, be a C^1 function with derivative $g'(t)$ for $t \in \mathbf{R}$. Assume that $g(0) = 0$ and $g'(0) > 0$.

   Consider the system

$$(*) \qquad \begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x). \end{aligned}$$

   Let $G(x) = \int_0^x g(s)ds$, and let $V(x, y) = \frac{y^2}{2} + G(x)$.

   (a) Prove that V is constant on solutions of (*).

   (b) Prove that there is a $\delta > 0$ such that if $| (x, y) | < \delta$, then the solution which passes through the point (x, y) is periodic.

3. For this question, we need to discuss the concepts of *stability* and *asymptotic stability.* We will do it only for autonomous equations here.

   Let $f(x)$ be a C^1 vector field defined in an open subset $D \subset \mathbf{R}^n$, and, for $x_0 \in D$, let $\phi(t, x_0)$ denote the unique solution to $\dot{x} = f(x)$ such that $\phi(0, x_0) = x_0$. We say the solution $t \to \phi(t, x_0)$ is *stable* if, given $\epsilon > 0$, there is a $\delta > 0$ such that if $| x - x_0 | < \delta$, then

   (a) the solution $\phi(t, x)$ is defined for all $t \geq 0$, and

   (b) $| \phi(t, x) - \phi(t, x_0) | < \epsilon$ for all $t \geq 0$.

   We say that $\phi(t, x_0)$ is *asymptotically stable* if it is stable and, there is a $\delta > 0$ such that for $| x - x_0 | < \delta$, we have

$$\lim_{t \to \infty} | \phi(t, x) - \phi(t, x_0) | = 0.$$

For $n > 1$, an $n-$th order ODE is *stable* iff its associated first order system is stable. A similar definition holds for asymptotically stable $n-$th order equations.

Discuss the stability and asymptotic stability of every solution of the equations $\dot{x} = x^3 - x,\ \ddot{x} + 3x = 0$.

4. Suppose X is a C^1 vector field in $\mathbf{R}^n$ and all solutions $\phi(t, x)$ exist for all time t. A point x is called *non-wandering* for X if, for every $\epsilon > 0$ and every $T > 0$ there are a point y and a $t > T$ such that

$$\mid y - x \mid < \epsilon$$

and

$$\mid \phi(t, y) - x \mid < \epsilon$$

The set of non-wandering points for X is denoted $NW(X)$.

   (a) Give an example of an X on $\mathbf{R}^2$ for which $NW(X)$ is empty.

   (b) Give examples of vector fields X_1, X_2 on $\mathbf{R}^2$ for which

      i. $NW(X_1)$ is a single point

      ii. $NW(X_2)$ consists of exactly two points

   (c) Prove that if $x \in NW(X)$, then $\phi(t, x) \in NW(X)$ for all $t \in \mathbf{R}$. (This property of $NW(X)$ is called *invariance*)

   (d) Prove that $\omega(x) \subset NW(X)$ for every $x \in \mathbf{R}^n$.

   (e) Prove that if X is a vector field on the line $\mathbf{R}^1$, then each point in $NW(X)$ is a critical point.

   (f) Give an example of a vector field X on $\mathbf{R}^2$ for which there is a point $y \in NW(X)$ such that y is not in the $\omega-$limit set of any point in $\mathbf{R}^n$.

5. Let $f : D \to \mathbf{R}$ be a C^2 real-valued function defined in the open set $D \subset \mathbf{R}^n$. Let $grad(f)(x)$ be the gradient of x at the point $x \in D$; i.e., $grad(f)(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x))$. The system

$$\dot{x} = grad(f)(x) \tag{1}$$

is called a *gradient system* with potential function f. Let $\phi(t, x)$ be the local flow of the system (1).

(a) Show that if x is not a critical point of (1), then the function $f(\phi(t, x))$ is strictly increasing for t near 0.

(b) Suppose f is defined and C^2 on all of $\mathbf{R}^n$. Show that if $\phi(t, x)$ is a bounded solution of $grad(f)$, then $\omega(x)$ consists of critical points of f.

6. Determine, with justification, which of the following systems in $\mathbf{R}^2$ is a gradient system. If the system is a gradient system, determine the potential function f.

(a) $$\begin{aligned} \dot{x} &= y^2 - sin(x) \\ \dot{y} &= -y^2 + 2xy \end{aligned}$$ (b) $$\begin{aligned} \dot{x} &= y + exp(x) \\ \dot{y} &= x \end{aligned}$$ (c) $$\begin{aligned} \dot{x} &= y^3 - sin(x) \\ \dot{y} &= -x^2 + y \end{aligned}$$