Cantor sets and bounded orbits in certain "tent" maps

For a subset E in a metric space, let ∂E denote the boundary of E. This is defined to by

$$\partial E = Closure(E) \setminus Interior(E).$$

A subset E in a metric space X is called *totally disconnected* if for each $x \in E$, and each neighborhood U of x there is an open set V with $x \in V \subset U$ such that $\partial V = \emptyset$. A subset E is *perfect* if each point $x \in E$ is an accumulation point of E. That is, for $x \in E$ and any neighborhood V of x there is a point $y \in V \setminus \{x\} \bigcap E$.

A *Cantor set* $E \subset X$ is a compact, perfect, totally disconnected set.

Fact. Any two Cantor sets are homeomorphic. For instance, if C is a Cantor set, then so is $C \times C$, so C is, in fact, homeomorphic to $C \times C$.

Example. The set of numbers $x \in [0, 1]$ such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

where $a_i = 0$ or $a_i = 2$ for all i is a Cantor set. (It is the middle third set).

Let $0 < \alpha < 1$. One can define the middle $\alpha-$set $F(\alpha)$ in the interval $I = [0, 1]$ as follows.

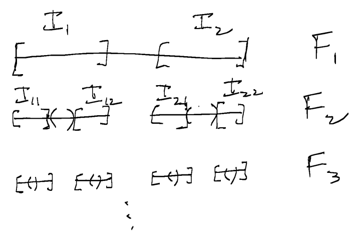
Let

$$\begin{aligned} F_1 &= I \setminus (\frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha}{2}) \\ &= I \setminus \text{ (interval of length } \alpha \text{ centered at } \frac{1}{2}) \\ &= I_1 \bigcup I_2 = \text{ two disjoint closed intervals} \end{aligned}$$

Next, remove the center interval in I_1 of length $\alpha \mid I_1 \mid$ to get two remaining closed subintervals I_{11}, I_{12}. Proceed similarly in I_2 to get two closed subintervals I_{21}, I_{22} in I_2.

Let

$$F_2 = \bigcup_{\substack{a_0 = 1,2 \\ a_1 = 1,2}} I_{a_0 a_1}$$



This is a union of four closed intervals in I, and $F_2 \subset F_1$.

Continue in this way defining $I_{a_0 a_1 \ldots a_k}$ such that, for all k,

$$a_i = 1 \text{ or } 2$$

$$I_{a_0 a_1 \ldots a_{k-1},1} \text{ and } I_{a_0 a_1 \ldots a_{k-1},2}$$

are disjoint closed intervals in $I_{a_0 \ldots a_{k-1}}$ obtained by removing the central interval in $I_{a_0 \ldots a_{k-1}}$ whose length is $\alpha | I_{a_0 \ldots a_{k-1}} |$.

Let

$$F_k = \bigcup_{a_0 \ldots a_k} I_{a_0 \ldots a_k},$$

and let $F(\alpha) = \bigcap_{k \geq 1} F_k$.

Observe that $F(\alpha)$ is a Cantor set in I.

Now, we consider the tent map $f_A(x) = A(1 - 2| \frac{1}{2} - x |)$ for $A > 1$.

This map is piecewise linear, symmetric about $\frac{1}{2}$, has slope equal to $| 2A |$ at each point of differentiability, and has the value 0 at $x = 0, 1$.

Proposition 0.1 *For $A > 1$, the set of bounded orbits for the tent map $f_A(x) = A(1 - 2| x - \frac{1}{2} |)$ is a Cantor set in* $[0, 1]$.

Proof. Let us consider the branches f_1, f_2 of the inverse map defined on $[0, 1]$ by the formulas

$$f_1(x) = \frac{x}{2A},$$

and

$$f_2(x) = -\frac{x}{2A} + \frac{1}{2A}.$$

The maps $f_1,\ f_2$ are affine contractions (i.e. affine maps with derivative of absolute value less than 1). Also, f_1 maps $[0,1]$ bijectively onto the interval $[0, \frac{1}{2A}]$, and f_2 maps $[0,1]$ bijectively onto $[1 - \frac{1}{2A}, 1]$.

The images $f_1(I)$ and $f_2(I)$ are disjoint.

Define $I_{a_0 a_1 \ldots a_k} = f_{a_0 a_1 \ldots a_k}(I)$. It follows easily by induction that if $\mathbf{a} = (a_0 a_1 \ldots a_k) \neq \mathbf{b} = (b_0 b_1 \ldots b_k)$, then $I_{a_0 a_1 \ldots a_k} \bigcap I_{b_0 b_1 \ldots b_k} = \emptyset$. Moreover, the length of $I_{b_0 b_1 \ldots b_k}$ is $(\frac{1}{2A})^{k+1}$.

Let

$$F_k = \bigcup I_{a_0 \ldots a_k}.$$

Then,

$$F = \bigcap_k F_k$$

is a Cantor set.

Let B denote the set of points x whose forward orbits are bounded.

We claim

$$B = F. \tag{1}$$

If $x \in F$, then $f_A^{k+1}(x) \in I$ for each $k \geq 0$, so $F \subset B$.

Now, we prove the converse: $B \subset F$.

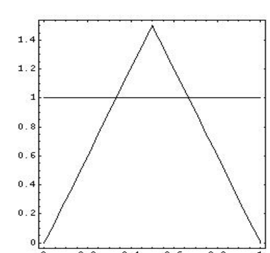


Figure 1: $f(x) = A(1 - 2|\ \frac{1}{2} - x\ |),\ A > 1$

Write $f = f_A$.

Note that if x is any point which is not in I, then $f^n(x) \to -\infty$.

So, any $x \in B$ must lie in I.

Let

$$J = I \setminus (I_1 \bigcup I_2).$$

Then, if any forward iterate $f^k(x)$ of x gets into J, then $f^{k+1}(x)$ is not in I, so $f^n(x) \to \infty$.

Now, let x be a point whose forward orbit is bounded. Then, first $x \in I$. If $x \notin F$, then it is in some interval in the complement of F in I.

Any such interval must have the form $f_{a_0 a_1 \ldots a_k}(J)$ for some finite sequence of $a_i = 1$ or 2.

If follows that $f^{k+1}(x) \in J$. Hence, $f^n(x) \to -\infty$ as $n \to \infty$.

This proves that $B \subset F$ as required. QED.