Hyperbolic planar diffeomorphisms from hyperbolic interval maps

There is a simple construction, which we call the unfolding construction which converts a map on the line to a planar diffeomorphism. This gives us many examples of hyperbolic planar maps. In fact, a structurally stable polynomial map of the line carries over to a structurally stable polynomial diffeomorphism of the plane. Here we will use the natural topology induced by the coefficients of the polynomials.

Consider a map \( \phi : \mathbb{R} \to \mathbb{R} \) defined on all of \( \mathbb{R} \). Let \( b \neq 0 \) and consider the map \( H(x, y) = (\phi(x) - y, bx) \). We call \( H \) a \( b \)-unfolding of \( \phi \).

**Remark.** We have written the map this way so that \( \det(DH) = b \). Sometimes one uses the map \((x, y) \mapsto (\phi(y) + y, bx)\). In this case \( \det(DH) = -b \). The following exercise shows that these two ways of writing \( H \) give affinely conjugate maps provided the Jacobian determinants are the same.

**Exercise:** Consider the maps \( H(x, y) = (\phi(x) + y, bx) \) and \( H_1(x, y) = (\phi(x) - y, -bx) \). Show there is an affine automorphism \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( AHA^{-1} = H_1 \).

We will see that many of the properties of \( \phi \) carry over to analogous properties of \( H \) is \( b \) is sufficiently small.

Let us consider some of the properties of \( H \).

**Fact 1.** If \( \phi \) is a \( C^r \) map, then \( H \) is a \( C^r \) diffeomorphism.

The map \( H \) is clearly \( C^r \) it suffices to show that it is bijective, and \( \det(DH(x, y)) \neq 0 \) for each \((x, y)\).

**Step 1.** \( H \) is bijective

We let \((x_1, y_1) = H(x, y) = (\phi(x) - y, bx)\). It suffices to solve for \((x, y)\) uniquely in terms of \((x_1, y_1)\).

But, it is easy to see that \((x, y) = (y_1/b, \phi(y_1/b) - x_1)\).

**Step 2.** \( \det(DH) \neq 0 \). We compute

\[
DH(x, y) = \begin{pmatrix}
D\phi(x) & -1 \\
b & 0
\end{pmatrix}.
\]

So, \( \det(DH) = b \neq 0 \).

**Fact 2.** If \( x_0 \) is a periodic sink of \( \phi \) of period \( \tau \), and \( |b| \) is small enough, then there is a periodic sink \((u_0, v_0)\) for \( H \) near \((x_0, 0)\) of period \( \tau \) for \( H \).

To see this, we first assume that \( \tau = 1 \); i.e.; \( x_0 \) is a fixed sink of \( \phi \). Then, \( 0 < |D\phi(x_0)| < 1 \).
Let \( E_1 = \{(x, y) : y = 0\} \).
Write \( H_b = (\phi(x) - y, bx) \).

If \( b = 0 \), then \( H_0(E_1) = E_1 \) and \( H_0 \mid E_1 \) is just the map \( \phi \) with an extra 0 in the second coordinate. The point \((x_0, 0)\) is a fixed point for \( H_0 \) with eigenvalues \( D\phi(x_0) \) and 0. Let \( \epsilon > 0 \) be such that \( H_0(B_\epsilon((x_0, 0))) \subset B_{\epsilon/2}((x_0, 0)) \). Also, there is a positive integer \( N \) such that for each \((x, y) \in B_\epsilon((x_0, 0))\), \(|DH_0^N(x, y)| < \frac{1}{2}\).

There is a \( \delta > 0 \) such that if \(|b| < \delta\), then

\[
H_b(B_\epsilon(x_0, 0)) \subset B_{\epsilon/2}((x_0, 0)),
\]

and for each \((x, y) \in B_\epsilon((x_0, 0))\), \(|DH_b^N(x, y)| < \frac{1}{2}\).

It follows that there is a unique fixed sink for \( H_b \) in \( B_\epsilon((x_0, 0)) \).

Now, suppose that \( \tau > 1 \). For \(|b| \) small, each of the maps \( H_b^j \) are \( C^r \) near \( H_\tau \) for \( 0 \leq j \leq \tau \). The preceding argument can be applied to the map \( H^\tau \) to give that for \( b \) small enough, the map \( H_b^\tau \) has a fixed sink near \((x_0, 0)\).

Exercise. If \( x_0 \) is a hyperbolic expanding periodic point of \( \phi \) of period \( \tau > 0 \), then for small \( b \) there is a unique hyperbolic saddle point of period \( \tau \) for \( H_b \) near \((x_0, 0)\).

It follows from the preceding that hyperbolicity of periodic orbits of a fixed period for \( \phi \) carry over to hyperbolic periodic orbits for \( H_b \) with small \( b \).

Now, we shall see that the same property holds for hyperbolic sets for \( \phi \). This gives many examples of hyperbolic sets for planar diffeomorphisms. In particular, if we consider the logistic map \( f_r(x) = rx(1 - x) \) with \( r > 5 \) (actually \( r > 4 \) will work but that is harder to prove), then we have shown earlier that the set \( B \) of bounded orbits forms a hyperbolic Cantor set in the unit interval \( I = [0, 1] \). It will follow that if \(|b| > 0\) is small enough, then the map \( H_b(x, y) = (rx(1 - x) - y, bx) \) will have a hyperbolic set \( B_b \) near \( B \). It is not too hard to show that \((H_b, B_b)\) is topologically conjugate to the Smale horseshoe diffeomorphism we described previously.

**Theorem 0.1** Let \( \Lambda \) be an expanding hyperbolic set for a \( C^r \) map \( \phi : \mathbb{R} \to \mathbb{R} \). Then, there is an \( \epsilon > 0 \) such that if \(|b| < \epsilon\), then, there is a hyperbolic invariant set \( \Lambda_b \) for \( H_b \) near \( \Lambda \times \{0\} \).

Remark.
1. The proof actually shows more. For $b$ sufficiently small, there is a neighborhood $U$ of $\Lambda \times \{0\}$ such that the set $V = \cap_{j \in \mathbb{Z}} H_b^j(U)$ is a hyperbolic isolated set of $H_b$. The set $\Lambda_b$ will then be defined to be this $V$.

2. It is actually true that the sets $\Lambda_b$ can be shown to be isolated invariant sets. However, the proof is somewhat technical and will not be given in these notes. Here is a sketch. One first proves that $\Lambda$ is an isolated invariant set for $\phi$ (this is fairly easy). It must be at most a Cantor set union some isolated periodic points, so there is a neighborhood $U$ of $\Lambda$ in $\mathbb{R}$ so that if $z \in U \setminus \Lambda$, then some positive iterate of $z$ escapes $U$. Then, one uses the stable manifold theorem for the set $\Lambda_b$ and a small thickening of $U$ in $\mathbb{R}^2$ to construct an adapted neighborhood of $\Lambda_b$.

3. Extending the techniques below, one can prove the following statements.

   (a) Let $\phi$ be a structurally stable map from a real interval $I$. There is an $\epsilon > 0$ such that for $0 < |b| < \epsilon$, the unfolding map $H_b(x, y) = (\phi(x) - y, bx)$ is a structurally stable diffeomorphism from the rectangle $I \times [-1, 1]$ into itself.

   (b) Let $\phi$ be a structurally stable polynomial map on the reals $\mathbb{R}$. Then, there is a $\epsilon > 0$ such that, for $0 < |b| < \epsilon$, the unfolding map $H_b(x, y) = (\phi(x) - y, bx)$ is a structurally stable polynomial diffeomorphism of the plane $\mathbb{R}^2$. Here a polynomial diffeomorphism is a bijective polynomial map whose inverse is also a polynomial map.

   (c) Consider the Henon map $H_{a,b}(x, y) = (a - x^2 - by, x)$. For any $b \neq 0$ there is a constant $A(b) > 0$ such that if $a \geq A(b)$, the set $B$ of bounded orbits of $H_{a,b}$ is a hyperbolic set, and the map pair $(H_{a,b}, B)$ is topologically conjugate to the full two-sided two-shift $(\sigma, \Sigma)$. This theorem was first proved by Devaney and Nitecki [1]. A simpler proof which also includes the case of complex variables is in [2].

In the proof of Theorem 0.1 we want to use the cone conditions for hyperbolicity. It turns out that one does not need Riemannian norms. Finslers (i.e., continuously varying norms in the tangent spaces) are sufficient. In
what follows, one could actually obtain Riemannian norms, but that involves several extra steps in the proof, so we will just work with Finslers.

Recall that a subset \( \Lambda \subset \mathbb{R} \) is an expanding hyperbolic set for a \( C^r \) map \( \phi : \mathbb{R} \to \mathbb{R} \) if it is compact and there are constants \( C > 0 \), \( \lambda > 1 \) such that for each \( n \geq 0 \), we have

\[
| D\phi^n_x | \geq C \lambda^n.
\]

The next lemma says that we can find a new Finsler on \( \Lambda \) such that the constant \( C \) can be chosen to be 1.

**Lemma 0.2** Let \( \Lambda \) be an expanding hyperbolic set for a \( C^r \) map \( \phi : \mathbb{R} \to \mathbb{R} \). Then, there are a smooth Finsler \( x \to | \cdot |'_x \) on \( \mathbb{R} \) and a constant \( \lambda_1 > 1 \) such that for each \( x \in \Lambda \) and each tangent vector \( v \in T_x \mathbb{R} \), we have

\[
| D\phi_x(v) |_{\phi_x} \geq \lambda_1 | v |'_x.
\]  

**Proof.**

Let \( \tau > 1 \) and \( N > 0 \) be such that \( C \lambda^N > \tau \), and, for \( x \in \Lambda \), set

\[
| v |'_x = \sum_{j=0}^{N-1} | D\phi^j_x(v) |_{\phi^j_x}.
\]

It is easy to verify that \( | \cdot |'_x \) is a Finsler on \( \Lambda \). That is, each \( | \cdot |'_x \) is a norm on \( T_x \mathbb{R} \) varying continuously with \( x \in \Lambda \). Let us remark that this Finsler can be extended to each \( y \in \mathbb{R} \). To see this, note that \( | \cdot |'_x \) is simply defined by a strictly positive function \( a(x) \) on \( \Lambda \). Indeed, for a non-zero vector \( v \in T_x \mathbb{R} \), let \( a(x) = \frac{| v |'_x}{| v |} \). Now, we extend \( a(x) \) to a continuous function \( b(x) \) on all of \( \mathbb{R} \), and we get the extended Finsler \( | \cdot |'_x = b(x) | \cdot | \) for all \( x \in \mathbb{R} \).

Now, we proceed to show that \( | \cdot |' \) has the required property of expansion at the first iterate for \( x \in \Lambda \).

Because any two norms on \( \mathbb{R} \) are boundedly related and the maps \( x \to | \cdot |_x \) and \( x \to | \cdot |'_x \) are continuous, there is a constant \( C_1 > 0 \) such that, for any \( v \) and any \( x \in \Lambda \), we have

\[
| v |_x \geq C_1 | v |'_x.
\]

Now, using the Chain Rule, we have
Hence, we can take $\lambda_1 = 1 + (\tau - 1)C_1$.

This gives us a continuous Finsler $| \cdot |'$ on $\mathbb{R}$ satisfying (1). Since the Finslers on $\mathbb{R}$ are simply determined by strictly positive real functions, we may approximate $| \cdot |'$ by a smooth Finsler and keep the property (1) for a slightly smaller $\lambda_1 > 1$. QED.

We now proceed to the proof of Theorem 0.1.

We will make use of a smooth Finsler $| \cdot |'$ on $\Lambda$ as in Lemma 0.2.

Let $\Lambda_0$ be the set $\Lambda \times \{0\}$ in $\mathbb{R}^2$. We define a smooth Finsler on $\Lambda_0$ as follows. For $(x, y) \in \Lambda_0$ and $v = (v_1, v_2) \in T_{(x,y)}\mathbb{R}^2$, set

$$| v |'_{x,y} = \max( | v_1 |'_{x}, | v_2 |).$$

Let $\lambda_1 > \lambda_2 > 1$, and let $1 > \epsilon > 0$ be such that $| D\phi(x) |'_{x} - \epsilon > \lambda_2$ for $x \in \Lambda$.

For simplicity of notation, we now drop the "primes" in the use of this Finsler, and simply denote it by $| \cdot |$.

Let $U$ be a small neighborhood of $\Lambda_0$ so that for $z \in U$ and $v = (v_1, 0) \in T_z\mathbb{R}^2$, we have
\[ |DH_b(z)(v)| \geq \lambda_2|v|. \]

We consider the cone \(C_z\) at \(z \in U\) defined by
\[
C_z = \{v = (v_1, v_2) : |v_2| \leq \epsilon |v_1| \}\.
\]

This defines a cone field on \(U\). Note that it is actually dependent on \(z\) in \(U\) since \(|\cdot| = |\cdot|_z\) depends on \(z\).

Let \(\tilde{U} = \tilde{U}_b = U \cap H_b(U) \cap H^{-1}_b(U)\).

We want to show that if \(|b|\) is appropriately small, then \(C = \{C_z\}\) is an \((H_b, 1)\) hyperbolic cone field on \(\tilde{U}\). Here we make the obvious changes in the definition of hyperbolicity to non-invariant sets (e.g., \(U\) in the present case). The hyperbolicity estimates hold as long as the orbits remain in \(U\). Then, the largest invariant set in \(U\) will be a hyperbolic set in the sense of the definition given earlier.

Let \(\lambda_3\) be such that \(\lambda_2 > \lambda_3 > 1\).

**Claim.** If \(|b| < \frac{1}{2}\epsilon\) and \(|\lambda_2 - \frac{|b|}{\epsilon}| > \lambda_3\), then \(DH_b\) is \(\lambda_3\)-expanding and \(\lambda_3\)-co-expanding on \(C\) on \(U_1 \cap H_b(U_1) \cap H^{-1}_b(U_1)\).

That is,
\[
z \in \tilde{U}, \ v \in C_z \Rightarrow |DH_b(v)| \geq \lambda_3 |v|,
\]

and
\[
z \in \tilde{U}, \ v \in C^c_z \Rightarrow |DH^{-1}_b(v)| \geq \lambda_3 |v|.
\]

**Proof of the Theorem assuming the Claim.**

Let
\[
\Lambda_b = \bigcap_{j \in \mathbb{Z}} H^j_b(U) = \bigcap_{j \in \mathbb{Z}} H^j_b(\tilde{U}).
\]

Let \(N > 0\) be such that \(\lambda_3^N > 2\). Then, set \(\tilde{U}_N = \cap_{-N \leq j \leq N} H^j_b(\tilde{U})\).

It follows from the Claim that if \(z \in \tilde{U}_N\), then \(C\) is \((H_b, N)\) hyperbolic at \(z\). That is, \(DH_b\) expands and co-expands \(C_z\) by a factor of 2.

Hence, \(C\) is a hyperbolic cone field on \(\Lambda_b\), so \(\Lambda_b\) is hyperbolic.

**Proof of the Claim.**

**Step 1.** If \(v = (v_1, v_2) \in C_z\) for \(z \in \tilde{U}\), then \(|DH_b(v)| \geq \lambda_2 |v|\).

Let
\[ w = DH_b(v) = \begin{pmatrix} D\phi(x) & -1 \\ b & 0 \end{pmatrix}, \]

so that \( w = (w_1, w_2) \) with
\[ w_1 = D\phi(x)v_1 - v_2, \quad w_2 = bv_1. \]

We make three statements:

1. \(|v| = |v_1|\),
2. \(|w| = |w_1|\), and
3. \(|w_1| > \lambda_2 |v_1|\).

This will finish Step 1.

The first statement is easy since \( \epsilon < 1 \), \(|v| = \max(|v_1|, |v_2|)\), and \(|v| = \max(|v_1|, |v_2|)\).

Similarly, for the second statement, we have \( nw_2 = b |w_1| \leq |w_1| \).

For the third statement, we have
\[
|w_1| = |D\phi(x)v_1 - v_2| \\
\geq |D\phi(x)||v_1| - \epsilon |v_1| \\
= (|D\phi(x)| - \epsilon)|v_1| \\
> \lambda_2 |v_1|. 
\]

**Step 2.** If \( v = (v_1, v_2) \in C_z^c \) for \( z \in \tilde{U} \), then \( |DH_b^{-1}(v)| \geq \lambda_2 |v| \).

Setting \( v = (v_1, v_2) \) and \( w = (w_1, w_2) = DH_b^{-1}(v) \) we have
\[
DH_b^{-1} = \frac{1}{b} \begin{pmatrix} 0 & 1 \\ -b & D\phi(x) \end{pmatrix},
\]
so,
\[
w_1 = \frac{v_2}{b}, \quad w_2 = -v_1 + \frac{D\phi(x)}{b}v_2.
\]

The corresponding three statements are the following.
1. \(| v | \leq \frac{|v_2|}{\epsilon}\),

2. \(| w | = | w_2 |\), and

3. \(| w_2 | > \lambda_2 \frac{|v_2|}{\epsilon}\).

For the first statement, we have \(| v_2 | > \epsilon |v_1|\), or, \(| v_1 | < \frac{|v_2|}{\epsilon}\). Since, \(\epsilon < 1\), we have \(| v_2 | < \frac{|v_2|}{\epsilon}\), so both \(| v_1 |\) and \(| v_2 |\) are bounded above by \(\frac{|v_2|}{\epsilon}\).

For the second statement, we show that \(| w_2 | \geq | w_1 |\).

We have

\[
| w_2 | = \left| \frac{D\phi(x)}{b} v_2 - v_1 \right|
\geq \left| \frac{D\phi(x)}{b} \right| | v_2 | - \frac{|v_2|}{\epsilon}
\geq \left| \frac{D\phi(x)}{b} \right| - \frac{1}{\epsilon} | v_2 |
= \left( \left| \frac{D\phi(x)}{b} \right| - \frac{1}{\epsilon} \right) |bw_1|
= \left( \lambda_2 - \frac{1}{\epsilon} \right) |bw_1|
= \lambda_3 |bw_1|
> |bw_1|
\]

Now, we go to the third statement.

Since \(| b | < \frac{1}{2} \epsilon\) and \(|D\phi(x)| \geq \lambda_2\), We have

\[
| w_2 | = \left| \frac{D\phi(x)}{b} v_2 - v_1 \right|
\geq \left| \frac{D\phi(x)}{b} \right| | v_2 | - \frac{|v_2|}{\epsilon}
\geq \left( \left| \frac{D\phi(x)}{b} \right| - 1 \right) \frac{|v_2|}{\epsilon}
\]
\[
\begin{align*}
\geq & \quad (2\lambda_2 - 1) \frac{|v_2|}{\epsilon} \\
\geq & \quad \lambda_2 \frac{|v_2|}{\epsilon}
\end{align*}
\]

QED.

References
