Hyperbolic Sets

We now extend the structure of the horseshoe to more general kinds of invariant sets.

Let $r \geq 1$, and let $f \in \mathcal{D}^r(M)$ where M is a Riemannian manifold. A compact $f-$invariant set is called *hyperbolic* if there are constants $C > 0, \lambda > 1$ and a splitting $T_xM = E^u_x \oplus E^s_x$ for each $x \in \Lambda$ such that

1. $Df_x(E^\sigma_x) = E^\sigma_{fx}$ for $\sigma = s, u$,
2. for $n \geq 0$ and $v \in E^u_x$, we have $\mid Df^n_x(v) \mid \geq \lambda^n \mid v \mid$, and
3. for $n \geq 0$ and $v \in E^s_x$, we have $\mid Df^{-n}_x(v) \mid \geq \lambda^n \mid v \mid$.

It can be shown that the conditions above are independent of the choice of Riemannian norm on M, and that there is a special Riemannian norm so that the constant $C = 1$. That is, we have $Df-$expansion by the factor λ for each vector in E^u_x and $Df-$contraction by the factor λ^{-1} for each vector in E^s_x.

An example of a hyperbolic set is the set $\Lambda = \bigcap_n f^n(Q)$ in the horseshoe diffeomorphism. Another example is a single hyperbolic periodic orbit.

Recall that we have defined the sets

$$W^s(x, f) = \{y \in M : d(f^n y, f^n x) \to 0 \text{ as } n \to \infty\},$$

and

$$W^u(x, f) = W^s(x, f^{-1}).$$

The first main theorem about hyperbolic sets is the following.

Theorem 0.1 *(Invariant manifold theorem for hyperbolic sets). Let $r \geq 1$, and let Λ be a hyperbolic set for a C^r diffeomorphism f with hyperbolic splitting $T_xM = E^u_x \oplus E^s_x$ for each $x \in \Lambda$. Then, the sets $W^u(x, f)$ and $W^s(x, f)$ are C^r injectively immersed copies of Euclidean spaces which are tangent at x to E^u_x and E^s_x, respectively. We also have dim $W^u(x, f) =$ dim E^u_x and dim $W^s(x, f) =$ dim E^s_x.*

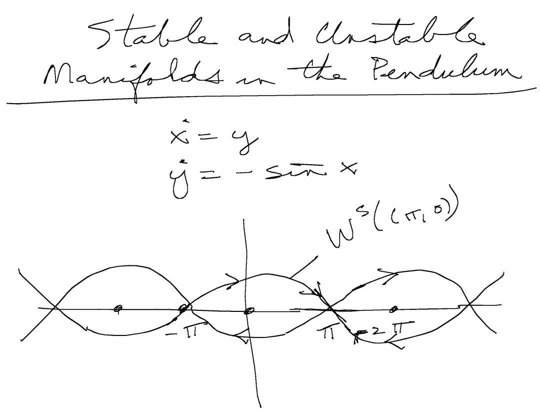
The manifolds $W^u(x,f)$ and $W^s(x,f)$ are called the ustable and stable manifolds of x in M.

If $M = \mathbf{R}^2$ and the subspaces E_x and E_u are one-dimensional, then these manifolds are C^r injectively immersed curves. This is the case in the horseshoe diffeomorphism.

Example. Consider the time-one map of the first order planar system associated to the pendulum equation $\ddot{x} + sin(x) = 0$.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -sin(x) \end{aligned}$$

Let p be a saddle critical point (for instance the point $(0,\pi)$). The stable and unstable manifolds of p are the orbits which are forward and backward asymptotic to p together with the point p itself.



An invariant set Λ for a diffeomorphism f is called an *isolated invariant set* if there is a compact neighborhood U of Λ in M so that

$$\bigcap_{n\in\mathbf{Z}} f^n(U) = \Lambda.$$

Such a neighborhood is called an *adapted* or *isolating* neighborhood of Λ.

Again, the set Λ of bounded orbits in the horseshoe diffeomorphism is a hyperbolic isolated invariant set.

One of the hall-marks of chaotic motion is the concept of *sensitive dependence on initial conditions.* One says that s map of diffeomorphism exhibits sensitive dependence on initial conditions on a set Λ if there is an $\epsilon > 0$ such that for each $x \in \Lambda$, there are points $y \in \Lambda$ arbitrarily close to x such that for some positive integer $n > 0$, we have $d(f^n x, f^n y) > \epsilon$. Thus, some points y arbitrarily close to x have their iterates getting a fixed distance apart from the iterates of x.

It turns out that any hyperbolic set with periodic points dense exhibits sensitive dependence on initial conditions at each of its points.

We will call an invariant set exhibiting sensitive dependence on initial conditions a *chaotic invariant set.*

There are certain hyperbolic chaotic invariant sets which have a describable structure. We define these next.

Let Λ be a hyperbolic isolated invariant set for an $f \in \mathcal{D}^1(M)$. We say that Λ is a *hyperbolic basic set* if the periodic orbits are dense in Λ and f is topologically transitive on Λ.

The next result states that hyperbolic basic sets are *robust* under perturbations of the underlying diffeomorphism f.

Theorem 0.2 *(Local Stability of Hyperbolic Basic Sets). Let $f \in \mathcal{D}^r(M)$ with $r \geq 1$, and let Λ be a hyperbolic basic set for f with adapted neighborhood U. Then, there is a neighborhood $\mathcal{N}^r$ of f in $\mathcal{D}^r(M)$ such that if $g \in \mathcal{N}^r$, then the set $\Lambda_g = \bigcap_{n\in\mathbf{Z}} g^n(U)$ is a hyperbolic basic set for g and there is a homeomorphism $h : \Lambda \to \Lambda_g$ such that $gh(x) = hf(x)$ for all $x \in \Lambda$.*

Thus, the whole orbit structure of hyperbolic basic sets is preserved under small perturbations of f.

This is a striking result. Let us see what it means for the horseshoe diffeomorphism f.

We defined the map f to take Q into $\mathbf{R}^2$ with $\Lambda = \bigcap^n f^n(Q)$. Note that Q is not an adapted neighborhood of Λ. However, if we thicken Q slightly to the compact set $Q' = \beta Q$ with $1 < \beta < 1 + \epsilon$ for small ϵ, then Q' will be an adapted neighborhood of Λ. Thus, if g is C^1 near f, the whole orbit structure of f on Λ is available (up to topological conjugacy) for g on the nearby set $\bigcap_n g^n(Q')$.

The robustness of hyperbolic basic sets makes them essential for understanding smooth systems.

We now give some more examples of hyperbolic basic sets.

Anosov diffeomorphisms

When the whole manifold M is a hyperbolic set for the diffeomorphism f, one calls f an *Anosov* diffeomorphism. We now consider some examples of this.

Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ as a linear automorphism of $\mathbf{R}^2$. There are two real eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2}, \lambda_2 = \frac{3-\sqrt{5}}{2}$. Note that for a 2×2 matrix

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

with an eigenvalues λ, we can get an associated eigenvector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ from the equation

$$(\lambda - b_{11})v_1 - b_{12}v_2 = 0.$$

Taking $v_1 = 1$, we get $v_2 = \frac{\lambda - b_{11}}{b_{12}}$.

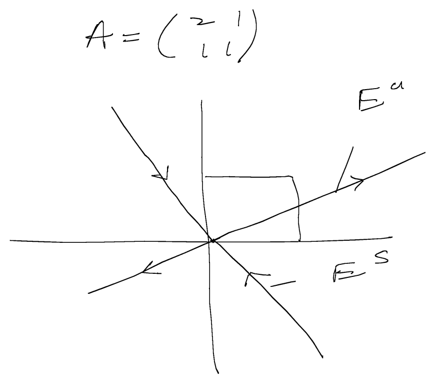
Applying this to the matrix A, and writing the eigenvalues and associated eigenvectors, we have

$$\lambda_1 = \frac{3+\sqrt{5}}{2}, \quad \begin{pmatrix} 1 \\ \frac{\lambda_1 - 2}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0.61 \end{pmatrix}$$

$$\lambda_2 = \frac{3-\sqrt{5}}{2}, \quad \begin{pmatrix} 1 \\ \frac{\lambda_2 - 2}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix} \approx \begin{pmatrix} 1 \\ -1.61 \end{pmatrix}$$

Observe that $\lambda_1 > 1 > \lambda_2 = \frac{1}{\lambda_1}$.

Let E^u be the subspace of real multiples of $\begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix}$, and let $E^s = (E^u)^\perp$ be the orthogonal complement of E^u. Then, E^s is the set of real multiples of $\begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix}$. We may think of these as subspaces of $T_x\mathbf{R}^2$ for any point $x \in \mathbf{R}^2$. We write these as E^u_x, E^s_x, respectively.



Let $\pi : \mathbf{R}^2 \to \mathbf{T}^2$ be the natural projection onto the two-torus. The map A has determinant 1, so it induces an automorphism $\tilde{A}$ of $\mathbf{T}^2$. The whole two-torus is a hyperbolic basic set for $\tilde{A}$ with splitting the projections $\tilde{E}^u_{\pi x} = D\pi_x(E^u_x)$, $\tilde{E}^s_{\pi x} = D\pi_x(E^s_x)$ for every $x \in \mathbf{R}^2$. The unstable manifold of the point $\pi(x) \in \mathbf{T}^2$ is the $\pi-$image of line in $\mathbf{R}^2$ through x and parallel to E^u. A similar statement holds for the stable manifold of πx. These lines have irrational slope in $\mathbf{R}^2$, so for each $y \in \mathbf{T}^2$, $W^u(y)$ and $W^s(y)$ are dense in $\mathbf{T}^2$.

The above map is called a linear hyperbolic toral automorphism.

The same construction can be give on the $n-$torus $\mathbf{T}^n$ starting with an $n \times n$ matrix A with integer entries, determinant 1, and eigenvalues of norm different from 1.

These hyperbolic toral automorphisms are, in some sense, diffeomorphic analogs of the expanding maps of the circle. It follows from the above theorem on local stability that they are structurally stable. In addition, they have dense orbits and a dense set of periodic orbits.

The Solenoid Attractor

Let $f \in \mathcal{D}^r(M)$ for $r \geq 1$. An *attractor* is a compact invariant topologically transitive set Λ such that there is a neighborhood U of Λ in M such that if $x \in U$, then $\omega(x) \subset \Lambda$.

The next example is a hyperbolic attractor in $\mathbf{R}^3$ known as the *solenoid*.

Let $\mathbf{C}^2$ be complex two-space, and Consider the map $F : \mathbf{C}^2 \to \mathbf{C}^2$ defined by

$$F(z, w) = (z^2, \frac{z}{2} + \frac{w}{4}).$$

Let

$$U = S^1 \times D^2 = \{(z, w) : \mid z \mid = 1 \text{ and } \mid w \mid \leq 1\}.$$

This may be thought of as a solid torus in $\mathbf{R}^3$. The map F is a diffeomorphism taking U onto a compact subset U' of its interior.

The set U' looks like a long thin torus winding twice around the central circle $S^1 \times \{\mathbf{0}\}$ in U. The successive interates $f^n(U)$ wind around more and more times and approach a set Λ which is locally the product of a Cantor set and an interval.

This set Λ is a hyperbolic attractor in $\mathbf{R}^3$.

See the figure on the next page.

Solenoid