The suspension of a diffeomorphism

We write $\mathcal{D}^r(M)$ for the set of C^r diffeomorphisms from the manifold M to itself. Also, we write $\mathcal{X}^r(M)$ for the C^r vector fields on M.

Let $f \in \mathcal{D}^r(M)$ and $g \in \mathcal{D}^r(N)$. A C^r conjugacy from f to g is a C^r diffeomorphism $h : M \to N$ such that $g = hfh^{-1}$. Thus, a C^r conjugacy is simply a topological conjugacy which happens to be a C^r diffeomorphism. As with all topological conjugacies, the *dynamical properties* are preserved.

Let M be a C^r manifold, and let $f \in \mathcal{D}^r(M)$. We will show that f is C^r conjugate to the time-one map of a C^r flow on a manifold $\tilde{M}$ where $dim\ \tilde{M} = dim\ M + 1$.

We first consider the product $M_1 = \mathbf{R} \times M$. There is a simple flow $\eta(t,(s,x)) = (t+s,x)$ on $\mathbf{R} \times M$ called the *horizontal flow.*

We now obtain our manifold $\tilde{M}$ by identifying the points $(t+1,x)$ with $(t, f(x))$. That is, we define an equivalence relation $\sim$ in M_1 by

$$(t,x) \sim (t',x') \text{ iff } t' = t+n \text{ and } x' = f^n(x)$$

for some integer n.

The quotient space $\tilde{M} = M/\sim$ has the structure of a C^r manifold and $dim\tilde{M} = dim\ M + 1$. The horizontal flow η pushes down to a flow ϕ on $\tilde{M}$. We leave it as an exercise that if ϕ_1 is the time-one map of the flow ϕ, then the pair $(\phi_1, \tilde{M})$ is C^r conjugate to the pair (f, M).

The manifold $\tilde{M}$ is called the *suspension* or *mapping cylinder* of the pair (f, M). The flow ϕ_1 is called the *suspension* of f.

We can describe all suspension flows. Let $\tilde{M}$ be a C^r manifold and $\phi(\cdot,\cdot)$ be a C^{r-1} flow on $\tilde{M}$. A C^r global cross-section to ϕ is a C^r submanifold M of $\tilde{M}$ with the following properties.

1. For each $x \in M$, there are real numbers $\tau_-(x) < 0, \tau_+(x) > 0$ such that $\phi(\tau_-(x),x) \in M$, $\phi(t,x) \notin M$ for $(\tau_-(x) < t < 0$, $\phi(\tau_+(x),x) \in M$, and $\phi(t,x) \notin M$ for $0 < t < \tau_+(x)$. That is, the orbit through $x \in \tilde{M}$ has a first return to x in the forward and backward directions.

2. The flow ϕ is transverse to M. That is, for each $x \in M$, the tangent vector to $t \to \phi(t,x)$ at x is not in the tangent space T_xM.

If ϕ has a C^r global cross-section, then it is C^r-conjugate to the suspension flow of the first return map $f : M \to M$.

Let us give some examples of this.

1. Quasi-perodic flow on the two-torus. Let $\alpha \in (0,1)$, and let $f = R_\alpha(z) = e^{2\pi i\alpha}z$ be the rotation through angle $2\pi i\alpha$ on the unit circle $M = S^1$. The suspension $\tilde{M}$ is diffeomorphic to the torus $\mathbf{T}^2$ and the flow ϕ on $\tilde{M}$ is, up to a C^r change of coordinated, obtained in the following way.

   Let $X(x, y)$ be the constant vector field on the plane defined by $X(x, y) = (1, \alpha)$ for each $(x, y) \in \mathbf{R}^2$.

   Let's write the associated system of differential equations given by X.

$$\begin{aligned} \dot{x} &= 1 \\ \dot{y} &= \alpha \end{aligned}$$

   The general solution is given by $\psi(t, (x, y)) = (x + t, y + \alpha t)$.

   Thus, the orbits are lines with slope α. The time-one map is the map $(x, y) \to (x + 1, y + \alpha)$. This maps the line $(x = 0)$ to the line $(x = 1)$ with the $y-$coordinate being $y \to y + \alpha$. Consider the equivalence relation $\sim_1$ on $\mathbf{R}^2$ defined by $(x, y) \sim_1 (x', y')$ iff and only if $x' = x + n, y' = y$ for some integer n. The quotient space $\mathbf{R}^2/\sim_1$ has a C^∞ structure which makes it diffeomorphic to the cylinder $S^1 \times \mathbf{R}$. Let $\eta : \mathbf{R}^2 \to \mathbf{R}^2/\sim_1$ be the natural projection. The flow ψ induces a flow $\tilde{\psi} = \eta_\star\psi$ on $\mathbf{R}^2/\sim_1$ which has the set $\Sigma = \eta((x = 0))$ as a global cross-section. The first return map to Σ may be thought of as our original rotation f. More precisely, with the induced natural smooth structures, the map f is C^∞ conjugate to the first return map to Σ.

   Any flow induced by the constant vector field $(1, \alpha)$ with α an irrational number is called a *quasi-periodic* flow (or *irrational flow*) on $\mathbf{T}^2$. It has many interesting properties. For instance, every orbit is dense in $\mathbf{T}^2$. It can be shown that every C^r flow $(r \geq 2)$ on $\mathbf{T}^2$ such that every orbit is dense is topologically conjugate to a quasi-periodic flow.

2. Forced Oscillations of second order differential equations.

   Consider the second order differential equation

$$\ddot{x} + f(x)\dot{(x)} + g(x) = F(t)$$

where f, g, F are C^r functions and $F(t+1) = F(t)$ for all t.

Such an equation is called a *forced oscillation* with periodic forcing function F.

We can form an equivalent first order autonomous system in $\mathbf{R}^3$ by taking

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -f(x)\dot{x} - g(x) + F(t) \\
\dot{t} &= 1.
\end{aligned}$$

Assume that every solution is defined for all time.

Identifying the planes $t = 0$ with $t = 1$, each orbit defines a *first return map* f from the $t = 0$ plane to itself. Standard results in ODE show that the map f is a C^r diffeomorphism from $\mathbf{R}^2$ to itself.

Thus, a forced oscillation as above gives us a diffeomorphism $f \in \mathcal{D}^r(\mathbf{R}^2)$. It can be shown that f preserves orientation on $\mathbf{R}^2$. That is, $det(Df(x)) > 0$ for each $x \in \mathbf{R}^2$.

Conversely, suppose $f \in \mathcal{D}^r(\mathbf{R}^2)$ preserves orientation. Then, the suspension manifold $\tilde{M}$ is a C^r 3-manifold which is diffeomorphic to $\mathbf{R}^2 \times S^1$, and the map f is C^r conjugate to the time-one map of a C^r flow on $\mathbf{R}^2 \times S^1$.

Thus, up to questions of global existence of solutions, the dynamics (i.e., orbit structure) of forced oscillations with periodic forcing functions is equivalent to the dynamics of planar diffeomorphisms.