The Schwarzian Derivative

There is a very useful quantity Sf defined for a C^3 one-dimensional map f, called the *Schwarzian derivative* of f.

Here is the definition.

Set

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \qquad (1)$$

Here we use $f'(x), f''(x), f'''(x)$ to denote the first, second, and third derivatives of f at x, respectively.

Using Df to denote the derivative function of f, we can also write Sf as

$$Sf(x) = D^2[(\log\ Df)] - \frac{1}{2}(D\log Df)^2. \qquad (2)$$

Definition. A map C^3 f has *negative Schwarzian derivative* if the quantity $Sf(x)$ is negative at any point x for which $f'(x) \neq 0$.

In this case, we write $Sf < 0$.

It turns out that maps with negative Scwarzian derivative have many useful and interesting properties.

Before proceeding, not that any quadratic map has negative Schwarzian derivative.

There is a simpler quantity N_f which is related to the Scwarzian derivative. Let us define the *nonlinearity* of f at x to be the quantity

$$\begin{aligned} N_f(x) &= (DlogDf)(x) \\ &= \frac{f''(x)}{f'(x)}. \end{aligned}$$

With these definitions, it is easy to compute that

$$Sf(x) = N_f'(x) - \frac{1}{2}(N_f(x))^2.$$

Observe that N_f gives a measure of how far the map f is from being affine.

For instance, if I is an interval in which $f'(x)$ does not vanish, then $N_f(x)$ vanishes in I if and only if $f''(x)$ vanishes in I. This means that $f(x)$ is an affine function in I; i.e., $f(x) = ax + b$ for some constants a and b.

For an affine function $f(x)$, the ratio of derivatives $\frac{f'(x)}{f'(y)}$ equals 1 in any interval where f' never vanishes.

The next proposition shows that an upper bound for $N_f(x)$ gives an upper bound on the ratio of derivatives $\frac{f'(x)}{f'(y)}$ on any interval in which f' never vanishes.

Proposition 0.1 *Let f be a C^2 function defined in an interval I and assume that $f'(x) \neq 0$ for every $x \in I$. Suppose there is a constant $K > 0$ such that $\mid N_f(x) \mid \leq K$ for every $x \in I$.*

Then, for any two points $x, y \in I$, we have

$$\frac{f'(x)}{f'(y)} \leq exp(K \mid x - y \mid). \tag{3}$$

Proof.

Let x and y be in I. Since

Since f' never vanishes in I, we have that $f'(x)$ and $f'(y)$ have the same sign in I.

Since

$$\frac{f'(x)}{f'(y)} = \frac{-f'(x)}{-f'(y)},$$

we may assume that $f'(x) > 0$.

Using the Mean Value Theorem for $\log(f'(x))$, there is a point θ bewteen in the interval between x and y such that

$$\begin{aligned} \log \frac{f'(x)}{f'(y)} &= \log f'(x) - \log f'(y) \\ &= \frac{f''(\theta)}{f'(\theta)} \mid x - y \mid \\ &\leq K \mid x - y \mid. \end{aligned}$$

Taking exponentials of both sides gives (3) as required. QED.

The next propositions give some of the basic properties of the Schwarzian derivative.

Proposition 0.2 (Composition Rule) *Let f and g be two C^3 self-maps of an interval I. Then, for each $x \in I$,*

$$S(f \circ g)(x) = (Sf \circ g)(x)(Dg(x))^2 + Sg(x). \tag{4}$$

Corollary 0.3 *Suppose the $Sf < 0$. Then, for any positive integer n, we also have $Sf^n < 0$.*

Exercises:

1. Prove Proposition 0.2

2. Let $f(x)$ be a real polynomial of degree n with n distinct real zeroes. Prove that f has negative Schwarzian derivative.

3. Consider the function $f(x) = a\ x^n e^{-bx}$ where a and b are positive real numbers and n is a positive integer. With the help of Mathematica (or directly if you prefer), give a proof that $Sf < 0$.

4. Using Mathematica, investigate the orbit structures of the maps in the previous exercise. Make plots of the maps, and several iterates. Check for attracting, expanding periodic points of low period, etc., for various $a's$ and $b's$.

Remark. The program Mathematica can be useful for actually proving various things. For instance, we illustrate this with a short notebook "Schwarzian-comp-1.nb." which you can download and experiment with.

In all cases, when we write Sf for a function f, we assume that f is of class C^k with $k \geq 3$.

Lemma 0.4 (Minimum Principle) *Suppose $Sf < 0$ in a interval I and $f'(x) \neq 0$ for each x in I. Then, $\mid f'(x) \mid$ has no local minimum in the interior of I.*

Proof.

Since $Sf = S(-f)$, we may assume that $f'(x) > 0$ in I. Suppose there were a point x_0 in the interior of I which is a local minimum of f'. Then,

$f''(x_0) = 0$. Since, $Sf < 0$, this gives that $f'''(x_0)f'(x_0) < 0$. Since $f'(x_0) > 0$, we have that $f'''(x_0) < 0$. This means that the second derivative of f' at x_0 is negative, so f' has a local maximum at x_0. The only way f' could have both a local maximum and a local minimum at x_0 is to be constant in a neighborhood of x_0. But then Sf would be zero in a neighborhood of x_0 which would contradict $Sf < 0$ in I. This proves the Lemma. QED.

Definition. Let p be an attracting fixed point of a map $f : I \to I$ where I is some interval. The *basin* of p is the set of points $y \in I$ such that $f^n(y) \to p$ as $n \to \infty$. The *immediate basin* of p is the connected component of the basin of p containing p. If $O(p)$ is an attracting periodic orbit, its *basin* is the set of points y such that $\omega(y) = O(p)$. Its *immediate basin* is the union of the connected components of the basin which contain a point of $O(p)$. A periodic point p of period n is called *neutral* if $\mid Df^n(p) \mid = 1$.

A neutral fixed point p is called *isolated* if there is a neighborhood U of p such that $f(x) \neq x$ for $x \in U \setminus \{p\}$. A neutral periodic point p of period n is said to be *isolated* if it is isolated as a fixed point of f^n. Note that if p is an isolated neutral periodic point, then there is a neighborhood U of p such that if $x \in U \setminus \{p\}$, then either $\omega(x) = O(p)$ or $\alpha(x) = O(p)$. If there is a neighborhood U of p and an $x \in U$ such that $\omega(x) = O(p)$, then we say that p is *attracting from one at least one side*. It may be that a neutral periodic point is attracting from one, both, or neither side. If it is attracting from one side, we define the basin of $O(p)$ to be $B(O(p)) = \{x \in N : \omega(x) = O(p)$. The *immediate basin* of p is the union of the connected components of $B(O(p))$ containing a point of $O(p)$.

Note that basins of attracting fixed points are open sets.

Since non-empty open sets in $\mathbf{R}$ are at most countable disjoint unions of open intervals, it follows that the immediate basin of an attracting fixed point is an open interval.

The following proposition is one of the main reasons for the importance of the property of negative Schwarzian derivative.

Theorem 0.5 *(D. Singer) Suppose $f : I \to I$ is a C^3 self-map of a non-trivial closed interval in the real line with $Sf < 0$ in I. Then,*

1. *the immediate basin of any attracting periodic orbit contains either a critical point of f or a boundary point of the interval I,*

2. *each neutral periodic point is attracting at least from one side, its immediate basin contains either a critical point or a boundary point of I,*

and

3. there exists no interval of periodic points.

In particular, the number of non-repelling periodic orbits is bounded by two plus the number of critical points.

Proof.

We begin with the first statement in the Theorem.

Let p be an attracting periodic point which has no boundary point in its immediate basin. Let n be the period of p and let T be the connected component of its basin containing p.

Let $B(p)$ be the basin of p for f^n.

Note that T is the union of the open intervals in $B(p)$ which contain p. (Exercise).

Then, $f^n(T) \subseteq T$. It follows that $f^n(\partial T) \subseteq Closure(T)$. If $f^n(\partial T) \bigcap T \neq \emptyset$, then some boundary point of I would be in T. Since, we have assumed that this is not true, we must have that $f^n(\partial T) \subseteq \partial T$.

Claim. There is a point $x \in T$ such that $Df^n(x) = 0$.

Assume the claim for the moment. By the Chain Rule, we have

$$Df^n(x) = Df(f^{n-1}x)Df(f^{n-1}x) \ldots Df(x),$$

so there must be a $j \in [0, n-1)$ such that $Df(f^j x) = 0$. Thus, $f^j(x)$ is a critical point in $f^j(T)$ which is the immediate basin of $f^j(p)$.

Hence, to prove the first statement of the theorem, we must prove the Claim.

Suppose, by way of contradiction, that the Claim is false. Then, $Df^n(x) \neq 0$ for all $x \in T$.

We either have f $Df^n(x) > 0$ for all $x \in T$ or $Df^n(x) < 0$ for all $x \in T$. In any case

$$Df^{2n}x > 0 \text{ for each } x \in T. \tag{5}$$

Also, f^{2n} fixes both boundary points of T.

Note that the boundary points of T cannot be attracting periodic points of f^{2n} because then there would be points in T which are not in the basin of p. Hence, $Df^{2n}(x) \geq 1$ at both boundary points of T.

Since $0 < Df^{2n}(p) < 1$, Moving x across T from left to right, we find that $Df^{2n}(x)$ goes from being greater than or equal to one, to being less than one, and back to being greater than or equal to one.

Hence, $\mid Df^{2n} \mid$ must have a local minimum in T. Since $Sf^{2n} < 0$, this is impossible, and the Claim is proved.

For the first part of statement 2, assume that there is a neutral periodic point p of period n which is not attracting from either side for f^n. Then, $Df^{2n}(x) \geq 1$ for all x close to p. But, then Df^{2n} must have a minimum at p which is impossible.

Now assume that p is a neutral periodic point of period n which is attracting from at least one side. As in the attracting case, if T is the immediate basin of $O(p)$, then either T contains a boundary point of I, or f^{2n} fixes ∂T. Now, the proof that T contains a critical point is similar to the case of an attracting periodic point. If not, then Df^{2n} would have a local minimum in T which is impossible. the

The same reasoning (existence of a local minimum of $\mid Df^{2n} \mid$ for some $n > 0$) applies if there is an interval of periodic points. QED.