9. Particular Solutions of Non-homogeneous second order equations–Undetermined Coefficients

We have seen that in order to find the general solution to the second order differential equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

where p, q, g are continuous functions in an interval I, it suffices to find two linearly independent solutions to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

and one particular solution to (1).

Here we will describe some methods for finding particular solutions.

Method 1: Undetermined coefficients

This method is useful when the the differential equation has *constant coefficients* and the function $g(t)$ has a special form: some linear combination of functions of the form

$$t^n, e^{\alpha t}, e^{\alpha t}cos(\beta t), e^{\alpha t}sin(\beta t). \tag{3}$$

In each of the above cases, one can assume a particular form for the solution $y(t)$ involving a polynomial $U(t)$ with unknown coefficients.

Plugging in $y(t)$ to the equation (1) then gives a system of linear equations for the coefficients of $U(t)$ which can be solved.

It is important to know the correct form to be assumed for the solution $y(t)$.

We consider the four cases separately.

Let $P(t)$ denote a polynomial of degree n in the following. That is,

$$P(t) = \sum_{i=0}^{n} a_i t^i$$

where $a_n \neq 0$.

$P(t)$ will be known from $g(t)$.

In each of the following cases, $U(t)$ will be an unknown polynomial

$$U(t) = \sum_{i=0}^{n} A_i t^i$$

of the same degree as $P(t)$, possibly with complex coefficients, and the coefficients A_i are to be determined.

Case 1: $g(t) = P(t)$

Write the characteristic polynomial as

$$z(r) = ar^2 + br + c.$$

If $z(0) \neq 0$, (i.e., $c \neq 0$) then one assumes

$$y(t) = U(t) = \sum_{i=0}^{n} A_i t^i$$

where $U(t)$ is a polynomial of the same degree as $P(t)$.

If $z(0) = 0$ and $z'(0) \neq 0$ (i.e, 0 is a root of multiplicity 1 of $z(r)$), then one assumes

$$y(t) = tU(t).$$

If $z(0) = 0 = z'(0)$, but $z''(0) \neq 0$, (i.e., 0 is a root of multiplicty 2 of $z(r)$), then one assumes

$$y(t) = t^2 U(t).$$

Case 2: $g(t) = e^{\alpha t} P(t)$

If $z(\alpha) \neq 0$, then one assumes $y(t) = U(t)e^{\alpha t}$.

If $z(\alpha) = 0$, but $z'(\alpha) \neq 0$, then one assumes

$$y(t) = tU(t)e^{\alpha t}.$$

If $z(\alpha) = 0, z'(\alpha) = 0$, but $z''(\alpha) \neq 0$, then one assumes

$$y(t) = t^2 U(t)e^{\alpha t}.$$

Case 3: $g(t) = P(t)e^{\alpha t} cos(\beta t)$

Here we reduce to Case 2 using complex methods. We can write $g(t)$ as the real part of the complex function $g_c(t) = P(t)exp((\alpha + i\beta)t)$

We find a complex solution $y_c(t) = t^s U_c(t) exp((\alpha + i\beta)t)$ where s is the multiplicity of $\alpha + i\beta$ as a root of $z(r)$.

We then take the real part of $y_c(t)$ to get our particular solution.

Case 4: $g(t) = P(t)e^{\alpha t} sin(\beta t)$

This is just like Case 3 except that we replace the words *real part* by *imaginary part*.

Remark. As an alternative to using complex methods in Cases 3 and 4, one may assume a solution $y(t)$ of the form

$$y(t) = t^s [U(t)e^{\alpha t} cos(\beta t) + V(t)e^{\alpha t} sin(\beta t)]$$

where s is the multiplicity of $\alpha + i\beta$ as a root of $z(r)$ and $U(t)$ and $V(t)$ are two real polynomials of the same degree as $P(t)$. Note that in this case one has to use both the unknown polynomials $U(t), V(t)$ even in Case 3 or Case 4.

As we will see, the complex method is more efficient because we replace the $2n$ real linear equations for the unknowns to n complex linear equations.

Let us consider some examples.

Example 1.

$$y'' + 2y' - 3y = 2t^2.$$

Let us try to find a solution of the form

$$y = At^2 + Bt + C.$$

Plugging in, we get

$$2A + 2(2At + B) - 3(At^2 + Bt + C) = 2t^2.$$

The coefficients of $t^2, t, 1$ have to be equal on both sides, so we get

$$-3A = 2,\ 4A - 3B = 0,\ 2A + 2B - 3C = 0.$$

Solving these equations for A, B, C we get

$$A = -\frac{2}{3},\ B = -\frac{8}{9},\ C = -\frac{4}{9} - \frac{16}{27}.$$

Example 2.

$$y'' + 2y' - 3y = 4e^{3t}.$$

Here we try $y = Ae^{3t}$.
We get

$$9Ae^{3t} + 6Ae^{3t} - 3Ae^{3t} = 4e^{3t}.$$

$$12A = 4, A = 1/3.$$

It is worthwhile to notice something about this example.
Consider the general situation in which we have

$$y'' + py' + qy = Ae^{bt} \tag{4}$$

in which

$$b \textit{ is not a root of } z(r) = r^2 + pr + q. \tag{5}$$

Let us assume a solution $y_p(t)$ of the form $y_p(t) = Ce^{bt}$.
Plugging into the equation (4), we get

$$(Ce^{bt})'' + p(Ce^{bt})' + qCe^{bt} = Ae^{bt}$$

or

$$(Cb^2e^{bt}) + p(Cbe^{bt}) + qCe^{bt} = Ae^{bt}$$

$$Ce^{bt}(b^2 + pb + q) = Ae^{bt}$$

$$Ce^{bt}z(b) = Ae^{bt}$$

or

$$C = \frac{A}{z(b)}$$

Thus, under the assumption (5), we have a simple formula for the particular solution:

$$y_p(t) = \frac{A}{z(b)} e^{bt}$$

Example 3.

$$y'' + 2y' - 3y = 2t^2 + 4e^{3t}. \tag{6}$$

Here we use the *superposition principle.*

If we have the equation

$$L(y) = g_1(t) + g_2(t) \tag{7}$$

and $y_1,\ y_2$ are solutions to $L(y) = g_1,\ L(y) = g_2$, respectively, then $y_1 + y_2$ is a solution to (7).

So, we get the particular solution

$$y(t) = \frac{1}{3}e^{3t} - \frac{2}{3}t^2 - \frac{8}{9}t - \frac{4}{9} - \frac{16}{27}$$

to (6).

Example 4.

$$y'' + 2y' - 3y = 4e^{-3t}. \tag{8}$$

We again try $y = Ae^{-3t}$.

We get

$$9Ae^{-3t} - 6Ae^{-3t} - 3Ae^{-3t} = 4e^{-3t}.$$

This has no solution. The problem is that -3 is a root of the characteristic polynomial $r^2 + 2r - 3$.

In this case, we try $y = At(e^{-3t})$.

We get

$$y' = Ae^{-3t} - 3Ate^{-3t}$$

$$\begin{aligned} y'' &= -3Ae^{-3t} - 3A(e^{-3t} - 3te^{-3t}) + 2Ae^{-3t} \\ &\quad -6Ate^{-3t} - 3Ate^{-3t} = 4e^{-3t}. \end{aligned}$$

$$-4A = 4, \;\; A = -1.$$

Hence, we get the solution $y = -te^{-3t}$.

What is going on with the method?

Here it helps to go to the abstract situation.

Consider

$$y'' + py' + qy = Be^{\alpha t}$$

We try to find a solution of the form $y = Ae^{\alpha t}$.

We get

$$A(\alpha^2 + p\alpha + q)e^{\alpha t} = Be^{\alpha t}.$$

If α is not a root of $z(r) = r^2 + pr + q$, then we get A as

$$A = \frac{B}{z(\alpha)}.$$

If α is a root of multiplicity one, and we try $y = Ate^{\alpha t}$, we find

$$Ate^{\alpha t}[\alpha^2 + p\alpha + q] + Ae^{\alpha t}(2\alpha + p) = Be^{\alpha t}.$$

This is the same as

$$Ate^{\alpha t}z(\alpha) + Ae^{\alpha t}z'(\alpha) = Be^{\alpha t}.$$

So, we get

$$A = \frac{B}{z'(\alpha)}.$$

Similarly, if α is a root of multiplicity 2 to $z(r)$, we get

$$y = At^2e^{\alpha t}$$

where $A = \frac{B}{2} = \frac{B}{z''(\alpha)}$.

In general, we try a function of the form $y = At^se^{\alpha t}$ where s is the multiplicity of α as a root of $z(r)$.

Example 5. As we saw above, similar methods apply to right hand sides of the form

$$e^{\alpha t}sin(\beta t), e^{\alpha t}cos(\beta t)$$

making use of complex arithmetic methods and taking real and imaginary parts.

For instance, consider

$$y'' + 2y' - 3y = 3cos(2t) \tag{9}$$

We recognize that this equation is the real part of the differential equation

$$y'' + 2y' - 3y = 3e^{2it}. \tag{10}$$

The characteristic polynomial is $z(r) = r^2 + 2r - 3$.

Since $2i$ is not a solution of $z(r) = 0$, we get a complex solution of the form $y(t) = Ae^{2it}$. Its real part will turn out to be solution to (9).

Let us be more specific.

Using the method in the first part of Example 4 (which also works for complex differential equations) , and the fact that $2i$ is not a root of $z(r)$, we can find a particular complex solution to (10) as

$$y_c(t) = \frac{3}{z(2i)} e^{2it}$$

Since we equation (9) was the real part of equation (10), we take the real part of the solution to (10) to get a particular solution to (9).

Let us carry this out.

We have

$$\begin{aligned} y_c(t) &= \frac{3}{z(2i)} e^{2it} \\ &= \frac{3}{4i-7} e^{2it} \\ &= \frac{3}{-7+4i} e^{2it} \\ &= \frac{3(-7-4i)}{7^2+4^2} e^{2it} \\ &= \frac{-21-12i}{65} e^{2it} \\ &= \left(-\frac{21}{65} - \frac{12}{65} i\right) e^{2it} \\ &= \left(-\frac{21}{65} - \frac{12}{65} i\right)(cos(2t) + i sin(2t)) \\ &= -\frac{21}{65} cos(2t) + \frac{12}{65} sin(2t) + i\left(-\frac{12}{65} cos(2t) - \frac{21}{65} sin(2t)\right) \end{aligned}$$

Let us use $Re(w)$ and $Im(w)$ to denote the real and imaginary parts of the complex number w.

Then, we see that the real part of $y_c(t)$ is

$$Re(y_c(t)) = -\frac{21}{65} cos(2t) + \frac{12}{65} sin(2t)$$

This is our particular solution to equation (9).

If we had started with

$$y'' + 2y' - 3y = 3sin(2t), \tag{11}$$

then the same method applies. We simply take the imaginary part of $y_c(t)$ to get our particular solution to (11) as

$$Im(y_c(t)) = -\frac{12}{65}cos(2t) - \frac{21}{65}sin(2t)$$

As we saw in this example, it is usually necessary to divide by complex numbers and then take real or imaginary parts.

For instance, we may have the number

$$\frac{a+bi}{c+di},$$

and we have to take the real or imaginary part.

As in example 4 in Section 8, it is useful to transform the division into a multiplation using the formula for the inverse

$$\frac{1}{c+di} = \frac{c-di}{c^2+d^2}.$$

This gives

$$\begin{aligned} \frac{a+bi}{c+di} &= (a+bi)\frac{c-di}{c^2+d^2} \\ &= \frac{ac+bd+(bc-ad)i}{c^2+d^2}, \end{aligned}$$

and we can read the real and imaginary parts off easily as we did in the above example.

Another method to solve equations (9) or (11), which does not use complex methods is to assume a solution of the form $Acos(t)+Bsin(t)$, plug in and solve for A and B. As we mentioned above, the advantage of complex arithmetic is that it allows us to keep track of coefficients two at a time, and thus only do half the number of the calculations.

Example 6. Let us consider another example.

Solve the IVP

$$y'' + 5y' + 2y = 3sin(4t), y(0) = 1, y'(0) = 2 \tag{12}$$

Solution:

Step 1. Find the general solution to the associated homogeneous equation.

The characteristic equation is $z(r) = r^2 + 5r + 2$ which has the roots

$$r_1 = \frac{-5+\sqrt{17}}{2},\ r_2 = \frac{-5-\sqrt{17}}{2}$$

The general solution to this homogeneous equation is

$$y(t) = c_1 exp(r_1 t) + c_2 exp(r_2 t)$$

(We do not need this here, but let us observe that the first fundamental solution of the homogeneous equation is $y_1(t) = exp(r_1 t)$, and the second fundamental solution is $y_2(t) = exp(r_2 t)$).

Step 2. Find a particular solution.

We observe that the given ODE is the imaginary part of the complex ODE

$$y'' + 5y' + 2y = 3e^{4it}$$

Since, $4i$ is not a root of $z(r)$, we get a particular complex solution of the form

$$\begin{aligned} y_c(t) &= \frac{3}{z(4i)} e^{4it} \\ &= \frac{3}{(4i)^2 + 20i + 2}(cos(4t) + isin(4t)) \\ &= \frac{3}{-16 + 20i + 2}(cos(4t) + isin(4t)) \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{-14+20i}(cos(4t)+isin(4t)) \\
&= \frac{3(-14-20i)}{14^2+20^2}(cos(4t)+isin(4t))
\end{aligned}
$$

To get a particular solution of (12), we take the imaginary part of y_c to get

$$
y_p = \frac{3}{14^2+20^2}(-14sin(4t) - 20cos(4t))
$$

The general solution to (12) is

$$
y(t) = c_1 exp(r_1 t) + c_2 exp(r_2 t) + y_p(t) \tag{13}
$$

For the IVP, we need

$$
y(0) = 1, \; y'(0) = 2
$$

This leads to

$$
\begin{aligned}
c_1 + c_2 + y_p(0) &= 1 \\
r_1 c_1 + r_2 c_2 + y_p'(0) &= 2
\end{aligned}
$$

or

$$
\begin{aligned}
c_1 + c_2 + \tfrac{3*(-20)}{14^2+20^2} &= 1 \\
r_1 c_1 + r_2 c_2 + \tfrac{3*(-14)*4)}{14^2+20^2} &= 2
\end{aligned}
$$

We re-write this system as

$$
\begin{aligned}
c_1 + c_2 &= A \\
r_1 c_1 + r_2 c_2 &= B
\end{aligned}
$$

with

$$
A = 1 - \frac{3*(-20)}{14^2+20^2}, \; B = 2 - \frac{3*(-14)*4)}{14^2+20^2}
$$

We solve this system as usual

$$c_1 = A - c_2,\ r_1(A - c_2) + r_2 c_2 = B$$

or

$$c_2 = \frac{B - r_1 A}{r_2 - r_1},\ c_1 = A - \frac{B - r_1 A}{r_2 - r_1}.$$

Then, we get the solution $y(t)$ by plugging everything into equation (13).

While this is computationally complicated, the procedure is straightforward.