8. Reduction of Order and more on complex roots

Reduction of Order:

Suppose we are given a general homogeneous second order d.e.

$$L(y) = y'' + p(t)y' + q(t)y = 0. \tag{1}$$

We know that, in order to find the general solution, it suffices to find two linearly independent solutions. It turns out that, if we can find one non-zero solution, then a second independent solution can always be found as usual up to integration), by a method called *reduction of order.*

Here is how it works,

Suppose y_1 is one non-zero solution to (1). Let us try to find a second solution $y_2 = y_1 v$ where v is a non-constant function.

For y_2 to be a solution, we have

$$(y_1 v)'' + p(y_1 v)' + q y_1 v = 0$$

or

$$\begin{aligned} y_1'' v + 2y_1' v' &+ y_1 v'' \\ &+ p y_1' v + p y_1 v' + q y_1 v = 0 \end{aligned}$$

$$v(y_1'' + py_1' + qy_1) + v'(2y_1' + py_1) + y_1v'' = 0.$$

The first term in this expression is 0 since $L(y_1) = 0$, so we are left with

$$v'(2y_1' + py_1) + y_1v'' = 0. \tag{2}$$

Now, y_1 and p are known, so we get a first order linear d.e. for v'. We solve this for v', then integrate to get v, and then go back to get an actual solution $y_2 = y_1v$ of $L(y) = 0$.

Since v is not constant, we clearly get that $y_2 = y_1v$ and y_1 are linearly independent functions.

Let us be more specific here about solving for v.

Equation (2) can be re-wiritten as

$$\frac{v''}{v'} = \frac{-2y_1' - p\ y_1}{y_1} \tag{3}$$

$$d\ log(v') = \frac{-2y_1' - p\ y_1}{y_1}$$

$$d\ log(v') = \frac{-2y_1'}{y_1} - p$$

$$log(v') = \int \left(\frac{-2y_1'}{y_1} - p \right) dt$$

$$v' = exp(\int \left(\frac{-2y_1'}{y_1} - p \right) dt)$$

$$v = \int \left[exp(\int \left((\frac{-2y_1'}{y_1} - p)dt \right) \right] dt \tag{4}$$

Example 1: The function $y(t) = t^3 + t$ is a solution to the d.e.

$$y'' + 3ty' - 9y = 0.$$

Find the general solution.
Write $y_2 = yv = (t^3 + t)v$ for some unknown v.
Then,

$$\begin{aligned} y_2'' + 3ty_2' &- 9y_2 \\ &= (6tv + 2(3t^2 + 1)v' + (t^3 + t)v'' \\ &\quad +3t((3t^2 + 1)v + (t^3 + t)v') - 9(t^3 + t)v \\ &= (6t^2 + 2)v' + (3t^4 + 3t^2)v' + (t^3 + t)v'' = 0 \end{aligned}$$

or

$$v'(2 + 9t^2 + 3t^4) + t(t^2 + 1)v'' = 0$$

$$\frac{v''}{v'} = -\frac{2 + 9t^2 + 3t^4}{t(t^2 + 1)}$$

$$\log(v') = \int -\frac{2 + 9t^2 + 3t^4}{t(t^2+1)} dt$$

$$v' = exp(\int -\frac{2 + 9t^2 + 3t^4}{t(t^2+1)} dt)$$

$$v = \int exp(\int -\frac{2 + 9t^2 + 3t^4}{t(t^2+1)} dt) dt$$

General solution:

$$y(t) = c_1(t^3 + t) + c_2(t^3 + t)v(t)$$

Example 2: The function $y_1(t) = \frac{1}{t}$ is a solution of the differential equation

$$2t^2y'' + 3ty' - y = 0, \quad t > 0. \tag{5}$$

Find a second linearly independent solution.

Set $y = vy_1 = vt^{-1}$.

Then,

$$y' = v't^{-1} - vt^{-2}, \; y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$$

Substituting into the differential equation we get

$$\begin{aligned} & 2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} \\ = \; & 2tv'' + (-4 + 3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v \\ = \; & 2tv'' - v' = 0 \end{aligned}$$

Letting $w = v'$, this is

$$2tw' - w = 0, \quad 2tw' = w$$

which we solve as

$$\frac{w'}{w} = \frac{1}{2t}$$

$$dlog(w) = \frac{1}{2t}$$

$$log(w) = \frac{1}{2}log(t) = log(t^{\frac{1}{2}})$$

$$w = t^{\frac{1}{2}}$$

Now

$$v' = w = t^{\frac{1}{2}}$$

so,

$$v = \int t^{\frac{1}{2}} dt = \frac{2}{3}t^{\frac{3}{2}}$$

and

$$y = y_1 v = t^{-1}\frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}t^{\frac{1}{2}}$$

is a second linearly independent solution. Note that, since the differential equation is linear and homogeneous, any constant times $\frac{2}{3}t^{\frac{1}{2}}$ will also be a solution of (5). In particular, $t^{\frac{1}{2}}$ is also a solution which is independent of y_1.

Example 3:

Let us illustrate this with the constant coefficient case with a multiple root.

Consider

$$L_2(y) = y'' + py' + qy = 0 \tag{6}$$

where p, q are constants, $p \neq 0$ and $p^2 - 4q = 0$. Then, if $r_1 = \frac{-p}{2}$, we have

$$r_1^2 + pr_1 + q = 0 \text{ and } 2r_1 + p = 0. \tag{7}$$

We know that $y_1 = e^{r_1 t}$ is a solution to $L_2(y) = 0$.

Let us try to find another solution of the form $ve^{r_1 t}$.

From the above computation, we get

$$\begin{aligned} v((e^{r_1 t})'' + p(e^{r_1 t})' + qe^{r_1 t}) &+ v'(2(e^{r_1 t})' + pe^{r_1 t}) \\ &+ e^{r_1 t}v'' = 0 \end{aligned}$$

The first term of the left side of this equation vanishes since $y_1(t) = e^{r_1 t}$ is a solution to (6) and the second term (which equals $v'e^{r_1 t}(2r_1 + p)$ also vanishes by the second part of (7).

Hence, the equation simplifies to $e^{r_1 t}v'' = 0$, or, $v = ct + d$. We might as well take $c = 1, d = 0$, and get $v(t) = t$. Thus, we get the second linearly independent solution as $y_2(t) = te^{r_1 t}$ (which we only stated before).

Review of Complex Numbers:

A complex number is an expression of the form $z = a + bi$ where $i = \sqrt{-1}$. The number a is called the *real part* of z, and the number b is called the *imaginary part* of z. We can think of the complex number as a pair (a, b) of real numbers, and, hence, as a point in the plane $\mathbf{R}^2$.

We define addition and multiplication of complex numbers as follows.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

The usual rules of arithmetic hold; i.e., complex addition and multiplication are commutative. Note also that $i^2 = -1$.

Complex numbers may be thought of as vectors in the plane with $a + bi$ corresponding to the vector (a, b). Note that this makes $i = (0, 1)$.

Then, complex addition is simply vector addition. Complex multiplication is harder to see geometrically.

If $z = a + bi, w = c + di$ are non-zero complex numbers, we can write the vectors (a, b) and (c, d) in polar coordinates as

$z = r_1(cos(\theta_1), sin(\theta_1)), w = r_2(cos(\theta_2), sin(\theta_2))$. Then, it can be shown that zw has the polar form

$$zw = r_1r_2(cos(\theta_1 + \theta_2), sin(\theta_1 + \theta_2)).$$

Thus, writing z and w in polar coordinates, with magnitudes r_1, r_2 and arguments θ_1, θ_2, the *magnitude* of zw is the *product* r_1r_2, and the *argument* of zw is the *sum* $\theta_1 + \theta_2$.

In the polar coordinate representation $z = (r\ cos(\theta), r\ sin(\theta))$, we call r the *magnitude* of z, and θ the *argument* of z. The angle θ has to be chosen in some interval of length 2π. If it is chosen in the interval $-\pi \leq \theta < \pi$, then it is denoted by $Arg(z)$, and called the *principal value* of the argument of z. The magnitude $\mid z \mid$ of z can be found by the formula

$$\mid a + bi \mid = \sqrt{a^2 + b^2}$$

Another way to look at these expressions is in terms of *complex* exponentials.

Let us temporarily define the complex exponential function on complex numbers $z = a + bi$ by

$$e^z = e^{a+bi} = e^a(\cos(b) + i\sin(b)). \qquad (8)$$

(We will justify this at the end of this section).

Using this definition and the fact that $\cos(b)^2+\sin(b)^2 = 1$, we see that if $z \neq 0$, then $\mid z \mid = e^a$ and

$$z = re^{i\theta} = \mid z \mid e^{iArg(z)}.$$

$$e^{a+bi} = e^a e^{bi} = e^a(cos(b) + i(sin(b))).$$

It turns out that the usual exponential rules hold: $e^{z+w} = e^z e^w,\ e^{zw} = (e^z)^w$.

Then, we can remember the product geometry as follows. Let $z = e^{a+bi}, w = e^{c+di}$.

Then,

$$zw = e^{a+bi}e^{c+di} = e^{a+c}e^{(b+d)i}.$$

There are some other useful concepts for performing various computations of complex numbers.

If $z = a + bi$, we define the *complex conjugate* of z, denoted $\bar{z}$ by

$$\bar{z} = a - bi$$

Note that $z\bar{z} = a^2 + b^2$. so, if $z = a + bi \neq 0$, then

$$\frac{1}{z} = \frac{\bar{z}}{\mid z \mid^2} = \frac{\bar{z}}{a^2 + b^2}$$

Example 4. Given $z = (2-3i)/(6+6i)$, write each of z, $\bar{z}$, and $\frac{1}{z}$ in the standard form $a+bi$. That is, determine the real and imaginary parts of these quantities.

First, we determine a, b such that $z = a + bi$.

We have

$$\begin{aligned}
z &= \frac{2-3i}{6+6i} \\
&= (2-3i)(\frac{6-6i}{72}) \\
&= \frac{1}{72}(12 - 18 + i(-12 - 18)) \\
&= \frac{1}{72}(-6 + i(-30)) \\
&= -\frac{1}{12} - \frac{15}{36}i \\
\\
\bar{z} &= -\frac{1}{12} + \frac{15}{36}i \\
\\
\frac{1}{z} &= \frac{-\frac{1}{12} + \frac{15}{36}i}{\frac{1}{144} + \frac{15^2}{36^2}}
\end{aligned}$$

Example 5. Express $z = (1+i)^{20}$ in the standard form $a + bi$.

Let $w = 1 + i$. We want w^{20}.

It is useful here to first express w in its polar form $w = \mid w \mid e^{iArg(w)} = re^{i\Theta}$.

If $w = x + iy$, we have

$$x = rcos(\Theta), \quad y = r\ sin(\Theta)$$

So,

$$r = \sqrt{x^2 + y^2},\ \Theta = arctan(y/x)$$

For $x = 1,\ y = 1$ we have

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

So,

$$(1 + i)^{20} = (\sqrt{2})^{20}e^{5\pi i} = 2^{10}e^{\pi i} = -1024$$

Example 6. Cube roots

We apply the use of complex exponentials to compute some cube roots.

Consider the problem of finding all cube roots of 1; i.e. solve $z^3 = 1$.

Since $\mid z^3 \mid = \mid z \mid^3 = 1$, we have that $\mid z \mid = 1$, so there is a θ such that $z = e^{i\theta}$.

It then follows that

$$z^3 = e^{i3\theta} = 1$$

so, $3\theta = 2\pi n$ for some integer n. This gives that $\theta = \frac{2\pi n}{3}$ for some integer n.

The numbers $\frac{2\pi n}{3}$ in the interval $[0, 2\pi)$ of this form occur for $n = 0, 1, 2$.

The Fundamental Theorem of Algebra says that a polynomial

$$a_0 + a_1 z + a_2 z^2 + \ldots a_d z^d$$

with $a_d \neq 0$ of degree d has at most d distinct roots (exactly d roots if we count repeated roots).

Applying that to $d = 3$, gives that $z^3 = 1$ has at most 3 roots

We have found the three roots $z = 1, z = e^{\frac{2\pi i}{3}}, z = e^{\frac{4\pi i}{3}}$, so this is all of them.

For later use, let us call $\omega = e^{\frac{2\pi i}{3}}$ the *primitive cube root* of 1.

From the geometry of complex numbers, it is easy to see that

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

General $d-$th Roots

The same idea used for solving $z^3 = 1$, can be used for solving $z^d = c$ for any positive integer d (to be interesting d should be greater than 3) and any non-zero complex number c.

Write $c = re^{i\theta_0}$ with $0 \leq \theta_0 < 2\pi$ and look for $z = r_1 e^{i\theta}$ such that $z^d = c$.

First, let's find the roots of $z^d = 1$ with $d \geq 3$. Proceding as we did for the case of $d = 3$, let $\omega_0 = 1$, and

$$\omega_k = e^{\frac{2k\pi i}{d}}$$

where $k = 1, 2, \ldots, d-1$.

These are all distinct and there are d of them, so they provide all of the solutions to $z^d = 1$.

Now, suppose that $c = re^{i\theta_0}$ with $0 \leq \theta_0 < 2\pi$. Let r_1 be the positive real number with $r_1^d = r$ and let $\theta_1 = \frac{\theta_0}{d}$.

Then, if $z_0 = r_1 e^{i\theta_1}$, we clearly have $z_0^d = r_1^d e^{di\theta_1} = re^{i\theta_0} = c$. So, z_0 is one of the roots of $z^d = c$.

But now, for each $k = 1, 2, \ldots, d-1$, we have $z_0\omega_k$ satisfies $(z_0\omega_k)^d = z_0^d\omega_k^d = c \cdot 1 = c$. Thus,the numbers

$$z_0, z_0\omega_1, z_0\omega_2, \ldots, z_0\omega_{d-1}$$

gives us all of the $d-$th roots of c.

Application of Complex Functions to Differential Equations

We can differentiate and integrate complex valued functions just as we do for real-valued functions, keeping track of $i^2 = -1$ when necessary.

Let us apply this concept.

Consider the d.e.

$$y'' + py' + qy = 0$$

where $p,\ q$ are real constants.

Assume that the characteristic polynomial $z(r) = r^2 + pr + c$ has roots of the form $r = \alpha \pm i\beta$ with $\beta \neq 0$.

We get a complex valued solution of the form

$$y_c(t) = e^{(\alpha + i\beta)t}$$

Its real and imaginary parts are

$$y_1(t) = e^{\alpha t} cos(\beta t),\ y_2(t) = e^{\alpha t} sin(\beta t).$$

In general, one can show that if $L(y)$ is a linear second order differential operator with real coefficients, and y_c is a non-zero complex solution to $L(y) = 0$, then the real and imaginary parts of y_c are linearly indepdent solutions to $L(y) = 0$. This is how one justifies the solutions we wrote down to the case of characteristic equations with non-real roots.

Justification of the complex exponentials

We will describe the techniques necessary for the justification, but leave the detailed proofs to a higher level course.

The standard way to define the complex exponential (e.g. see Ahlfors *Complex Analysis* 3rd edition) is by power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots$$

One can prove that, for any complex number z, this infinite series converges to a unique complex number (which we naturally call e^z or $exp(z)$.

The function satisfies the properties:

$$e^{z+w} = e^z e^w$$

and

$$(e^z)^w = e^{zw}$$

for any complex numbers z and w.

Writing $z = a + bi$, we then get

$$e^z = e^{a+bi} = e^a e^{bi}$$

Now, we have $i^{2n} = (-1)^n$ and $i^{2n+1} = (-1)^n i$ for positive integers n,

For $a = 0$, we then get

$$\begin{aligned}
e^{bi} &= 1 + bi + \frac{(bi)^2}{2!} + \frac{(bi)^3}{3!} + \frac{(bi)^4}{4!} + \frac{(bi)^5}{5!} + \dots \\
&= 1 + bi + \frac{b^2 i^2}{2!} + \frac{b^3 i^3}{3!} + \frac{b^4 i^4}{4!} + \frac{b^5 i^5}{5!} + \dots \\
&= 1 - \frac{b^2}{2!} + \frac{b^4}{4!} + \dots \\
&\qquad + i(b - \frac{b^3}{3!} + \frac{b^5}{5!} + \dots)
\end{aligned}$$

and we recognize the right side as the power series $cos(b) + isin(b)$.

Now, if a function $f(z)$ with z real or complex is expanded in a convergent poser series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

then the coefficients a_n of the series are uniquely determined by the formulas

$$a_n = \frac{Df^n(0)}{n!}$$

where $Df^n(0)$ denotes the $n-$th derivative of $f(z)$ at $z = 0$.

From this and the expansion above, one can prove the formula

$$e^{ib} = cos(b) + i\ sin(b)$$