5. Exact Equations, Integrating Factors, and Homogeneous Equations

Exact Equations

A region D in the plane is a connected open set. That is, a subset which cannot be decomposed into two non-empty disjoint open subsets.

The region D is called *simply connected* if it contains no "holes." Alternatively, if any two continuous curves in D can be continuously deformed into one another. We do not make this precise here, but rely on standard intuition.

A differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

is called *exact* in a region D in the plane if the we have equality of the partial derivatives

$$M_y(x,y) = N_x(x,y)$$

for all $(x,y) \in D$.

If the region D is simply connected, then we can find a function $f(x,y)$ defined in D such that

$$f_x = M, \text{ and } f_y = N.$$

Then, we say that the general solution to (1) is the equation

$$f(x,y) = C.$$

This is because the differential equation can be written as

$$df = 0.$$

Here we will not develop the complete theory of exact equations, but will simply give examples of how they are dealt with.

Example.

Find the general solution to

$$(3x^2y^2 - 3y^2)dx + (2x^3y - 6xy + 3y^2)dy = 0.$$

Step 1: Check to see if $M_y = N_x$.

$$M = 3x^2y^2 - 3y^2,\ N = 2x^3y - 6xy + 3y^2$$

$$M_y = 6x^2y - 6y,\ N_x = 6x^2y - 6y$$

So, it is exact.

Then,

$$f = \int M dx + g(y) = x^3y^2 - 3xy^2 + g(y)$$

$$f_y = N = 2x^3y - 6xy + 3y^2 = 2x^3y - 6xy + g'(y)$$

$$3y^2 = g'(y), \; g(y) = y^3$$

So, we get

$$x^3y^2 - 3xy^2 + y^3 = C$$

as the general solution.

Integrating Factors

Sometimes a d.e. $Mdx + Ndy = 0$ is not exact, but can be made exact by multiplying by a non-zero function.

Let us see when this can be done with functions of x or y alone.

Consider a non-zero function $\mu(x)$ which is a function of x alone such that

$$(\mu M)_y = (\mu N)_x$$

We get

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x, \; \mu_y = 0$$

So,

$$\mu M_y = \mu_x N + \mu N_x$$

$$\mu(M_y - N_x) = \mu_x N$$

$$\frac{M_y - N_x}{N} = \frac{\mu_x}{\mu}$$

Now, if the Left Hand Side is a function of x alone, say $h(x)$, we can solve for $\mu(x)$ by

$$\mu(x) = e^{\int h(x)dx},$$

and reverse the above arguments to get an integrating factor.

Similarly, if

$$\frac{N_x - M_y}{M} = g(y)$$

is a function of y alone, we can find an integrating factor of the form

$$\nu(y) = e^{\int g(y)dy}.$$

Example:

Consider the equation

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

$$M = 3xy + y^2,\ N = x^2 + xy$$

Step 1: Check if exact

$$M_y - N_x = 3x + 2y - 2x - y = x + y$$

So, not exact.
Step 2: Compute

$$\frac{M_y - N_x}{N} = \frac{x+y}{x^2+xy} = \frac{1}{x}$$

So, get integrating factor of the form

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{logx} = x$$

So,

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$$

is exact.

$$f_x = M = 3x^2y + xy^2, f = x^3y + x^2y^2/2 + g'(y)$$

$$f(x,y) = x^3y + x^2y^2/2 = C$$

is the general solution.

Homogeneous equations

A function $f(x,y)$ is called homogeneous (or order p) if

$$f(tx, ty) = t^p f(x,y)$$

for all $t > 0$.

If M and N are homogeneous of the same degree, then the differential equation

$$\frac{dy}{dx} = M(x, y)/N(x, y)$$

can be reduced to a separable one for $v(x)$ by the change of variable

$$y(x) = xv(x).$$

To see this we calculate:

$$\begin{aligned} y' = v + xv' &= \frac{M(x, xv)}{N(x, xv)} \\ &= \frac{x^p M(1, v)}{x^p N(1, v)} \\ &= \frac{M(1, v)}{N(1, v)} \end{aligned}$$

So,

$$v + xv' = H(v), \quad \text{where } H(v) = \frac{M(1, v)}{N(1, v)}$$

This makes

$$xv' = H(v) - v$$

which is separable.