3. Separable differential Equations

A differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

is called *separable* if the function $f(x, y)$ decomposes as a product $f(x, y) = \phi_1(x)\phi_2(y)$ of two functions ϕ_1 and ϕ_2.

Proceding formally we can rewrite this as

$$\frac{dy}{dx} = \phi_1(x)\phi_2(y)$$

$$\frac{dy}{\phi_2(y)} = \phi_1(x)dx.$$

Using the second formula, we can integrate both sides to get

$$\int^y \frac{dy}{\phi_2(y)} = \int^x \phi_1(x)dx + C$$

as the general solution.

Note that this is an implicit relation between $y(x)$ and x.

Indeed, the last integral formula has the form

$$F(y(x)) = G(x) + C$$

for some functions F and G. To find $y(x)$ as a function of x we would have to solve this implicit relationship.

This is frequently hard to do, so we will leave the solution in implicit form.

A more general version of this is the d.e.

$$M(x)dx + N(y)dy = 0 \tag{1}$$

We say that the general solution to this d.e. is an expression

$$f(x, y) = C$$

where $f_x = M(x)$, $g_y = M(y)$.

Since the family of the preceding equation as C varies is a family of curves, one sometimes says that this is the family of *integral curves* for the d.e. (1). ∎

Also, the initial value problem

$$\frac{dy}{dx} = \phi_1(x)\phi_2(y),\ \ y(x_0) = y_0$$

can be solved as

$$\int_{y_0}^{y} \frac{dy}{\phi_2(y)} = \int_{x_0}^{x} \phi_1(x)dx$$

This picks out a specific curve in the family of integral curves. ∎

Examples:

1. Find the general solution of the d.e.

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}.$$

Write this as

$$-x^2 dx + (1 - y^2) dy = 0$$

The general solution has the form

$$f(x, y) = C$$

where $f_x = -x^2$ and $f_y = 1 - y^2$.
Hence, we can take

$$\begin{aligned} f &= \int^x -x^2 dx + \int^y (1 - y^2) dy \\ &= -\frac{x^3}{3} + y - \frac{y^3}{3} \end{aligned}$$

and the general solution is

$$-\frac{x^3}{3} + y - \frac{y^3}{3} = C.$$

2. For the preceding d.e. find the integral curve passing through $(1, 3)$.

   We need to substitute $x = 1,\ y = 3$ in the above formula.

   We get
$$\frac{1}{3} + 3 - \frac{81}{3} = C,$$
   so the desired curve is
$$-\frac{x^2}{3} + y - \frac{y^3}{3} = \frac{10}{3} - \frac{81}{3}.$$

3. Solve the IVP
$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)},\ \ y(0) = -1.$$
   Write this as
$$-(3x^2 + 4x + 2)dx + 2(y-1)dy = 0.$$
   Integrate to

$$-x^3 - 2x^2 - 2x + y^2 - 2y = C,$$

and plug in $x = 0, y = -1$ to get $C = 3$.

So,

$$\text{ANS: } -x^3 - 2x^2 - 2x + y^2 - 2y = 3.$$

A difference between linear and non-linear first order scalar equations.

Given a first order linear d.e. of the form

$$y' + p(t)y = g(t)$$

with $p(t), g(t)$ continuous on an interval I, and a point t_0 in I, the solution to the IVP

$$y' + p(t)y = g(t), \ y(t_0) = y_0$$

exists on the whole interval I.

This fails for non-linear equations.

As an example, take

$$y' = y^2, y(0) = y_0$$

We solve this equation as

$$\frac{dy}{y^2} = dt$$

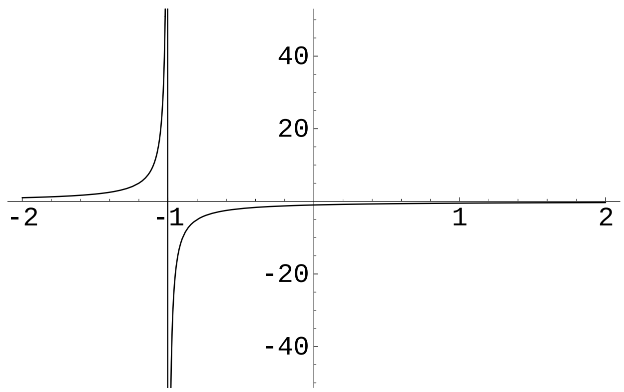
$$\int^y \frac{dy}{y^2} = \int_0^t dt$$

$$-\frac{1}{y} = t + C$$

$$y = -\frac{1}{t+C}, = y_0 = -\frac{1}{C}$$

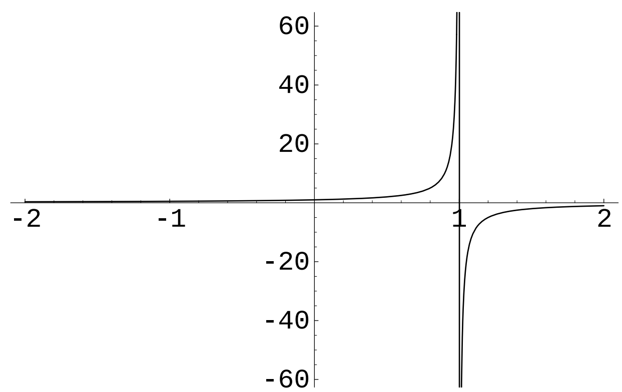
This solution blows up at the point $t = -C$. The graphs of solutions look like those in the following figure.

In[8]:= **C = 1; Plot**[−**1** / (**t** + **1**), {**t**, −**2**, **2**}]



Out[8]= - Graphics -

In[9]:= **C = −1; Plot**[−**1** / (**t** − **1**), {**t**, −**2**, **2**}]



Out[9]= - Graphics -