2. First Order Linear Equations and Bernoulli's Differential Equation

First Order Linear Equations

A differential equation of the form

$$y' + p(t)y = g(t) \tag{1}$$

is called a first order scalar linear differential equation. Here we assume that the functions $p(t), g(t)$ are continuous on a real interval $I = \{t : \alpha < t < \beta\}$.

We will discuss the reason for the name *linear* a bit later.

Now, let us describe how to solve such differential equations.

There is a theorem which says that under these continuity assumptions, if $t_0 \in (\alpha, \beta)$, then, for any real number y_0, there is a unique solution $y(t)$ to the initial value problem

$$y' + p(t)y = g(t), \ \ y(t_0) = y_0 \tag{2}$$

which is defined on the whole interval I.

Now that we know there is a solution, we can use various methods to try to find it.

There is a useful trick (or observation) for this.

Assuming y is a non-zero solution to (1), suppose there was a non-zero function μ such that

$$(\mu y)' = \mu g$$

Then, we would have

$$\mu' y + \mu y' = \mu g$$

$$\mu' y + \mu(g - py) = \mu g$$

$$\mu' y = \mu p y$$

$$\mu' = \mu p$$
$$\frac{d\ log\mu}{dt} = p$$

Since $p = p(t)$ is a continuous function of t, we can integrate both sides to find log μ, and then take the exponential to find μ.

$$\log\ \mu(t) = \int_{t_0}^{t} p(s)ds$$

$$\mu(t) = e^{\int_{t_0}^{t} p(s)ds}.$$

Now, *define* $\mu(t)$ by the last formula. Going backwards through the previous equations, we obtain the formula

$$(\mu y)' = \mu g.$$

Since the right side is now a known function of t, we can integrate again and get

$$(\mu y)(t) = \int_{t_0}^{t} \mu(s)g(s)ds + c \tag{3}$$

for some constant c.

The function μ is called an *integrating factor* for the equation (2) (we will see its connection to integrating factors for making equations *exact* later).

Note that equation (3) can be written in the form

$$\psi(t, y) = c$$

where

$$\psi(t, y) = \mu y - \int_{t_0}^{t} \mu(s)g(s)ds$$

The function $\psi(t, y)$ is called a *potential function* for the equation (2). Any two such potential functions for the same first order differential equation differ by a constant.

Returning to equation (3), we get

$$y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^{t} \mu(s) g(s) ds + c \right] \tag{4}$$

Notice that $\mu(t_0) = e^0 = 1$, and that if we evaluate $y(t_0)$, the integral vanishes and we get

$$y(t_0) = \frac{c}{\mu(t_0)} = c.$$

To summarize, the solution to the initial value problem (2) is given by

$$y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^{t} \mu(s) g(s) ds + y_0 \right] \tag{5}$$

This involves taking two integrals.

The general solution to (2) is given by leaving the constant c in the previous formula and taking the indefinite integral

$$y(t) = \frac{1}{\mu(t)} \left[\int^{t} \mu(s) g(s) ds + c \right] \tag{6}$$

Examples:

1. Find the general solution to the d.e.

$$y' + \frac{1}{2} y = \frac{3}{2}$$

Here

$$\mu(t) = e^{\int^t \frac{1}{2} dt} = e^{\frac{t}{2}},$$

so, the general solution has the form

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left(\int^t \mu(t)\frac{3}{2}dt + c\right) \\ &= e^{-\frac{t}{2}}(3e^{\frac{t}{2}} + c) \\ &= 3 + ce^{-\frac{t}{2}} \end{aligned}$$

2. In the preceding d.e. find the solution whose graph passes through the point $(0, 2)$.

   Here $y(0) = 2$, so

$$3 + c = 2, \quad c = -1$$

3. Find the solution of the initial value problem

$$y' - \frac{y}{2} = e^{-t}, \; y(0) = -1.$$

   Let

$$\mu = exp\left(\int_0^t -\frac{1}{2}dt\right) = e^{\frac{-t}{2}}$$

   The solution is

$$\begin{aligned} y(t) &= e^{\frac{t}{2}}\left(\int_0^t e^{\frac{-t}{2}}e^{-t}dt - 1\right) \\ &= e^{\frac{t}{2}}\left(\int_0^t e^{\frac{-3t}{2}}dt - 1\right) \\ &= e^{\frac{t}{2}}\left(\left[\frac{-2}{3}e^{\frac{-3t}{2}}\right]_0^t - 1\right) \\ &= e^{\frac{t}{2}}\left(\frac{-2}{3}(e^{\frac{-3t}{2}} - 1) - 1\right) \\ &= \frac{-2}{3}e^{-t} - \frac{1}{3}e^{\frac{t}{2}} \end{aligned}$$

4. Find the solution of the IVP

$$y' + 2ty = t, \; y(0) = 0$$

$$\mu = e^{\int_0^t 2tdt} = e^{t^2}$$

$$\begin{aligned} y(t) &= \frac{1}{e^{t^2}} \int_0^t te^{t^2} dt \\ &= \frac{1}{e^{t^2}} \left[\frac{1}{2} e^{t^2} \right]_0^t \\ &= \frac{1}{2e^{t^2}} (e^{t^2} - 1) \\ &= \frac{1}{2} - \frac{1}{2e^{t^2}} \end{aligned}$$

The reason for the name *linear* is as follows.

Consider the space $\mathcal{C}^1 = C^1(\alpha, \beta)$ of continuously differentiable functions on the open interval $I = (\alpha, \beta)$, and

let $\mathcal{C}^0 = C^0(I)$ be the space of continous functions on I.

A function L from one function space to another is usually called an *operator*

We can define operations of addition and scalar multiplication on the spaces $\mathcal{C}^1$ and $\mathcal{C}^0$ as follows.

1. $(f + g)(t) = f(t) + g(t)$ for all t (pointwise addition)
2. $(c \cdot f)(t) = cf(t)$ for all t (pointwise scalarmultiplication)

An operator $L : \mathcal{C}^1 \to \mathcal{C}^0$ is called a *linear operator* if it preserves the operations of pointwise addition and scalar multiplication.

That is, for any two functions $f, g \in \mathcal{C}^1$ and $c \in \mathbf{R}$, we have

1. $L(f + g) = L(f) + L(g)$, and
2. $L(c \cdot f) = cL(f)$

Examples:

1. the operator $L(f) = f' = Df$ is linear
2. the operator $L(f) = f' + 1$ is not linear
3. the operator $L(f) = (f')^2$ is not linear
4. for any function $p(t)$, the operator defined by

$$L(y)(t) = y'(t) + p(t)y(t) \quad \forall t$$

   is linear.
5. If V and W are any spaces of functions we can similarly define linear operators from V to W.
6. Letting $\mathcal{C}^n(I)$ denote the space of $n-$times continuously differentiable functions on the interval I, one can check that the $n-$th derivative operator $y \to y^{(n)}$ from $\mathcal{C}^n(I)$ to $\mathcal{C}^0(I)$ is a linear operator.
7. Given continuous functions

$$p_0(t), p_1(t), \ldots, p_{n-1}(t)$$

   on an interval I, the operator

$$\begin{aligned} L(y)(t) &= y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots \\ &\quad \ldots + p_1(t)y' + p_0(t)y \end{aligned}$$

   is a linear operator from $\mathcal{C}^n(I)$ to $\mathcal{C}^0(I)$.

In general, a *linear differential equation* is one of the form

$$L(y)(t) = g(t)$$

where $L(y)$ is a linear operator from $\mathcal{C}^n$ to $\mathcal{C}^0$ involving sums of scalar multiples of $D^j y$ for $0 \le j \le n$ and $D^j y$ denotes the j-th derivative of $y(t)$ with respect to t.

Bernoulli's Differential Equation

A differential equation of the form

$$y' + p(t)y = g(t)y^n \tag{7}$$

is called Bernoulli's differential equation.

If $n = 0$ or $n = 1$, this is linear. If $n \neq 0, 1$, we make the change of variables $v = y^{1-n}$. This transforms (7) into a linear equation.

Let us see this.

We have

$$v = y^{1-n}$$

$$v' = (1-n)y^{-n}y'$$

$$y' = \frac{1}{1-n}y^n v'$$

and

$$y = y^n v$$

Hence,

$$y' + py = gy^n$$

becomes

$$\frac{1}{1-n}y^n v' + py^n v = gy^n$$

Dividing y^n through and multiplying by $1 - n$ gives

$$v' + (1-n)pv = (1-n)g. \tag{8}$$

We can then find v and, hence, $y = v^{\frac{1}{1-n}}$.

Example 1.

Find the general solution to

$$y' + ty = ty^3.$$

We put $v = y^{-2}$

From (8), we get

$$v' - 2tv = -2t$$

$$\mu = e^{-t^2}$$

$$\begin{aligned} v &= e^{t^2} \left(\int^t e^{-t^2}(-2t)dt + c \right) \\ &= e^{t^2} \left(e^{-t^2} + c \right) \\ &= 1 + ce^{t^2}, \end{aligned}$$

and,

$$y = v^{-\frac{1}{2}} = \left[1 + ce^{t^2} \right]^{-\frac{1}{2}}.$$

Example 2.
Solve the IVP

$$x\frac{dy}{dx} + 5y = 2x^2y^4, \ y(1) = 3$$

We first put this into the standard form as

$$y' + \frac{5}{x}y = 2xy^4, \ y(1) = 3$$

This is a Bernoulli equation with $n = 4$.
Setting $v = y^{1-n} = y^{-3}$ we get the first order linear equation

$$v' - \frac{15}{x}v = -6x$$

with $p = -\frac{15}{x}$ and $\mu = exp(\int pdx) = exp(-15log(x)) = x^{-15}$.
So,

$$\begin{aligned} v &= x^{15} \left[\int -6x^{-14}dx + C \right] \\ &= x^{15} \left(-6\frac{x^{-13}}{-13} + C \right) \\ &= \frac{6}{13}x^2 + Cx^{15} \end{aligned}$$

Now, $v(1) = y(1)^{-3} = 3^{-3} = \frac{1}{27}$, so

$$\frac{6}{13} + C = \frac{1}{27},$$

or

$$C = \frac{1}{27} - \frac{6}{13}$$

and

$$y = \left[\frac{6}{13}x^2 + (\frac{1}{27} - \frac{6}{13})x^{15}\right]^{-\frac{1}{3}}$$