21. Periodic Functions and Fourier Series

1 Periodic Functions

A real-valued function $f(x)$ of a real variable is called *periodic of period* $T > 0$ if $f(x+T) = f(x)$ for all $x \in \mathbf{R}$.

For instance the functions $\sin(x), \cos(x)$ are periodic of period 2π. It is also periodic of period $2n\pi$, for any positive integer n. So, there may be infinitely many periods. If needed we may specify the *least period* as the number $T > 0$ such that $f(x+T) = f(x)$ for all x, but $f(x+s) \neq f(x)$ for $0 < s < T$.

For later convenience, let us consider piecewise C^1 functions $f(x)$ which are periodic of period $2L > 0$ where L is a positive real number. Denote this class of functions by $Per_L(\mathbf{R})$.

Note that for each integer n, the functions $\cos(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L})$ are in examples of such functions. Also, note that if $f(x), g(x) \in Per_L(\mathbf{R})$, and α, β are constants, then $\alpha f + \beta g$ is also in $Per_L(\mathbf{R})$.

In particular, any finite sum

$$\frac{a_0}{2} + \sum_{m=1}^{k} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

is in $Per_L(\mathbf{R})$. Here the numbers a_0, a_m, b_m are constants.

2 Fourier Series

The next result shows that in many cases the infinite sum

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right) \tag{1}$$

determines a well-defined function $f(x)$ which again is in $Per_L(\mathbf{R})$.

An infinite sum as in formula (1) is called a Fourier series (after the French engineer Fourier who first considered properties of these series).

Fourier Convergence Theorem. *Let $f(x)$ be a piecewise C^1 function in $Per_L(\mathbf{R})$. Then, there are constants a_0, a_m, b_m (uniquely defined by f) such that at each point of continuity of $f(x)$ the expression on the right side* *of (1) converges to $f(x)$. At the points y of discontinuity of $f(x)$, the series converges to*

$$\frac{1}{2}(f(y-)+f(y+)).$$

The values $f(y-), f(y+)$ denote the left and right limits of f as $x \to y$, respectively.

That is,

$$f(y-) = lim_{x\to y, x<y} f(x), \quad f(y+) = lim_{x \to y, x>y} f(x).$$

Since the expression on the right side of (1) does not always converge to the value of f at each x, one often writes

$$f \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right) \tag{2}$$

and calls (2) the *Fourier expansion* of f.

It turns out that the constants a_0, a_m, b_m above are determined by the formulas

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{3}$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{m\pi x}{L})dx, \text{ and} \tag{4}$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})dx. \tag{5}$$

We will justify this a bit later, but for now, let us use these formulas to compute some Fourier series. The constants a_0, a_m, b_m are called the *Fourier coefficients* of f.

Example 1. Let $f(x)$ be defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x, & 0 \le x < 2 \end{cases}$$

$$f(x+4) = f(x) \quad \text{for all } x.$$

Determine the Fourier coefficients of f.

Note that the graph of this function $f(x)$ looks like a "triangular wave." Here $L = 2$, and we compute

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{0}(-x)dx + \frac{1}{2}\int_{0}^{2} xdx \\ &= 1 + 1 = 2, \end{aligned}$$

and, for $m > 0$,

$$\begin{aligned} a_m &= \frac{1}{2}\int_{-2}^{0}(-x)\cos(\frac{m\pi x}{2})dx + \frac{1}{2}\int_{0}^{2} x\ \cos(\frac{m\pi x}{2})dx \\ b_m &= \frac{1}{2}\int_{-2}^{0}(-x)\sin(\frac{m\pi x}{2})dx + \frac{1}{2}\int_{0}^{2} x\ \sin(\frac{m\pi x}{2})dx. \end{aligned}$$

To compute these integrals, we note that, integration by parts gives the formulas

$$\begin{aligned} \int x\ \cos(ax)dx &= \frac{x}{a}\sin(ax) - \int \frac{\sin(ax)}{a}dx \\ &= \frac{x}{a}\sin(ax) + \frac{\cos(ax)}{a^2} \\ \int x\ \sin(ax)dx &= -\frac{x}{a}\cos(ax) + \frac{\sin(ax)}{a^2} \end{aligned}$$

After some calculation, we get

$$a_m = \begin{cases} -\frac{8}{(m\pi)^2}, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}$$

and $b_m = 0$ for all m. We will see later that this last fact follows from the fact that $f(-x) = f(x)$ for all x.

You will be asked to find various Fourier series in the homework.

3 Justification of the Fourier coefficient formulas

We need the following basic facts about the integrals of certain products of sines and cosines.

$$\int_{-L}^{L} \cos(\frac{m\pi x}{L}) cos(\frac{n\pi x}{L}) dx = \begin{cases} 0, & m \neq n, \\ L, & m = n \neq 0, \\ 2L, & m = n = 0 \end{cases} \tag{6}$$

$$\int_{-L}^{L} \cos(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L}) dx = 0 \text{ for all } m, n; \tag{7}$$

$$\int_{-L}^{L} \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L}) dx = \begin{cases} 0, & m \neq n, \\ L, & m = n \neq 0, \\ 0, & m = n = 0 \end{cases} \tag{8}$$

We justify formula (8), leaving the other similar calculations to the reader. First recall some formulas related to the sine and cosine functions.

The sum and difference formulas are:

$$\cos(\alpha + \beta) = \cos(\alpha)(\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{9}$$

$$\cos(\alpha - \beta) = \cos(\alpha)(\cos(\beta) + \sin(\alpha)\sin(\beta). \tag{10}$$

Applying the first formula with $\alpha = \beta$ gives

$$\cos(2\alpha) = \cos(\alpha)^2 - \sin(\alpha)^2$$

This implies that

$$\begin{aligned} 1 + \cos(2\alpha) &= \cos(\alpha)^2 + \sin(\alpha)^2 + \cos(\alpha)^2 - \sin(\alpha)^2 \\ &= 2\cos(\alpha)^2 \end{aligned}$$

or the so-called *cosine half-angle formula*

$$\cos(\alpha)^2 = \frac{1}{2}(1 + \cos(2\alpha)).$$

Similarly, the *sine half-angle formula* is

$$\sin(\alpha)^2 = \frac{1}{2}(1 - \cos(2\alpha)).$$

Formulas (9) and (10) imply that

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin(\alpha)\sin(\beta).$$

Using $\alpha = nx,\ \beta = mx$ then gives

$$\cos((n-m)x) - \cos((n+m)x) = 2\sin(nx)\sin(mx). \tag{11}$$

This implies, for $m \neq n$ and both positive,

$$\begin{aligned}
\int_{-L}^{L} \sin(\frac{m\pi x}{L})\sin(\frac{n\pi x}{L})dx &= \frac{1}{2}\int_{-L}^{L}\cos(\frac{(m-n)\pi x}{L})dx \\
&\quad -\frac{1}{2}\int_{-L}^{L}\cos(\frac{(m+n)\pi x}{L})dx \\
&= \frac{L}{2\pi}\left[\frac{\sin(\frac{(m-n)\pi x}{L})}{m-n} - \frac{\sin(\frac{(m+n)\pi x}{L})}{m+n}\right]_{-L}^{L} \\
&= 0.
\end{aligned}$$

If $m = n = 0$, then

$$\int_{-L}^{L} \sin(\frac{m\pi x}{L})\sin(\frac{m\pi x}{L})dx = \int_{-L}^{L} 0\ dx = 0,$$

while if $m = n \neq 0$, we have

$$\begin{aligned}
\int_{-L}^{L} \sin(\frac{m\pi x}{L})\sin(\frac{m\pi x}{L})dx &= \int_{-L}^{L}\left(\sin(\frac{m\pi x}{L})\right)^2 dx \\
&= \frac{1}{2}\int_{-L}^{L}\left[1 - \cos(\frac{2m\pi x}{L})\right] dx \\
&= \frac{1}{2}\left[x - \frac{\sin(\frac{2m\pi x}{L})}{\frac{2m\pi}{L}}\right]_{-L}^{L} \\
&= L.
\end{aligned}$$

Now, suppose that

$$f(x) = \frac{a_0}{2} + \sum_{m\geq 1} a_m\cos(\frac{m\pi x}{L}) + b_m\sin(\frac{m\pi x}{L}). \tag{12}$$

Since the integrals of cosine and sine functions over intervals of lengths equal to their periods vanish, we have

$$\begin{aligned}
\int_{-L}^{L} f(x)dx &= \int_{-L}^{L} (\frac{a_0}{2} + \sum_{m\geq 1} a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}))dx \\
&= \frac{a_0}{2}(2L) + \sum_{m\geq 1} \int_{-L}^{L} a_m \cos(\frac{m\pi x}{L})dx \\
&\quad + \sum_{m\geq 1} \int_{-L}^{L} b_m \sin(\frac{m\pi x}{L})dx \\
&= a_0 L
\end{aligned}$$

Analogously, using the orthogonality relations above, we have that, for $n \geq 1$,

$$\begin{aligned}
\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx &= \int_{-L}^{L} \cos(\frac{n\pi x}{L})(\frac{a_0}{2} + \sum_{m\geq 1} a_m \cos(\frac{m\pi x}{L}))dx \\
&\quad + \int_{-L}^{L} \cos(\frac{n\pi x}{L})(\sum_{m\geq 1} b_m \sin(\frac{m\pi x}{L}))dx \\
&= a_n L
\end{aligned}$$

which gives (4). Formula (5) is justified in a similar way.

4 Even and Odd functions

A function $f(x)$ is called *even* if $f(-x) = f(x)$ for all x. Analogously, a function $f(x)$ is called *odd* if $f(-x) = -f(x)$ for all x. For example, $\cos(x)$ is even, and $\sin(x)$ is odd.

Also, one sees easily that linear combinations of even (odd) functions are again even (odd).

The following facts are useful.

1. The product of two odd functions is even.

2. The product of two even functions is even.

3. The product of and even function and an odd function is odd.

Now, let F be an even function in $Per_L(\mathbf{R})$, and let G be an odd function in $Per_L(\mathbf{R})$.

It follows that, for $n \geq 0$, we have

4. $F(x)cos(\frac{n\pi x}{L})$ is even,
5. $F(x)sin(\frac{n\pi x}{L})$ is odd,
6. $G(x)cos(\frac{n\pi x}{L})$ is odd, and
7. $G(x)sin(\frac{n\pi x}{L})$ is even.

Let us compute the Fourier coefficients of the even function F and the odd function G.

Then, using the change of variables $u = -x$, we see that

$$\int_{-L}^{0} F(x)\ dx = \int_{0}^{L} F(x)dx$$

and

$$\int_{-L}^{0} G(x)\ dx = -\int_{0}^{L} G(x)dx,$$

Hence,

$$\int_{-L}^{L} F(x)\ dx = \int_{-L}^{0} F(x)\ dx + \int_{0}^{L} F(x)dx = 2\int_{0}^{L} F(x)dx, \tag{13}$$

and

$$\int_{-L}^{L} G(x)\ dx = \int_{-L}^{0} G(x)\ dx + \int_{0}^{L} G(x)dx = -\int_{0}^{L} G(x)dx + \int_{0}^{L} G(x)dx = 0 \tag{14}$$

As a consequence, we get the following simplified formulas of the Fourier coefficients of even and odd functions.

Let F be an even function with Fourier coefficients a_n for $n \geq 0$ and b_n for $n \geq 1$.

Then, $b_n = 0$ for all $n \geq 1$, and

$$a_n = \frac{2}{L}\int_{0}^{L} F(x)\cos\left(\frac{n\pi x}{L}\right) dx \text{ for all } n \geq 0 \tag{15}$$

Similarly, if $G(x)$ is an odd function with Fourier coefficients a_n for $n \geq 0$ and b_n for $n \geq 1$, then $a_n = 0$ for all $n \geq 0$, and

$$a_n = \frac{2}{L} \int_0^L G(x) \sin\left(\frac{n\pi x}{L}\right) dx \text{ for all } n \geq 0 \tag{16}$$

In particular, *the fourier series of an even function only has cosine terms* and *the fourier series of an odd function only has sine terms.*

5 The Fourier Series of Even and Odd extensions

For each real number α we define the $\alpha-$translation function T_α by $T_\alpha(x) = x + \alpha$ for all x.

Let $L > 0$, and let $I = [-L, L)$. Notice that the collection of $2nL$ translates of I as n goes through the integers gives a disjoint collection of intervals, each of length $2L$, which cover the whole real line $\mathbf{R}$.

That is, if $\mathbf{Z}$ is the set of integers $\{0, 1, -1, 2, -2, \ldots\}$, then

$$\mathbf{R} = \bigcup_{n \in \mathbf{Z}} T_{2nL}(I)$$

Another way to say this is that, for each $x \in \mathbf{R}$, there is a unique integer n_x and a unique point $y_x \in I$ such that $x = y_x + 2n_x L$.

Now, consider a real-valued function f defined on the interval $I = [-L, L)$. There is a unique function F of period $2L$ defined on all of $\mathbf{R}$ obtained by taking any $x \in \mathbf{R}$ and setting $F(x)$ to be $f(y_x)$. This function F is called the *periodic* $2L-$ *extension* of f. Sometimes, we leave out the L and call F simply the *periodic extension* of f.

If f is piecewise C^1, then F is in $Per(L)$ and has a Fourier series.

Now, consider a piecewise C^1 function f defined on $[0, L)$.

The *even extension* F of f to $[-L, L)$ is the function defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in [0, L) \\ f(-x) & \text{if } x \in [-L, 0) \end{cases}$$

and the *odd extension* G of f to $[-L, L)$ is the function defined by

$$G(x) = \begin{cases} f(x) & \text{if } x \in [0, L) \\ -f(-x) & \text{if } x \in [-L, 0) \end{cases}$$

From formulas (15) and (16) we obtain the following formulas for the Fourier coefficients of the even and odd extensions of f.

Even case:

$$F \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

$$b_m = 0,\ a_m = \frac{2}{L} \int_0^L f(x) cos\left(\frac{2\pi m x}{L}\right) dx$$

Odd case:

$$G \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

$$a_m = 0,\ b_m = \frac{2}{L} \int_0^L f(x) sin\left(\frac{2\pi m x}{L}\right) dx$$

Example 1.

Compute the Fourier Series of the even extension $F(x)$ of the function $f(x)$ such that

$$f(x) = \begin{cases} 3 & \text{if } 0 \leq x < 2 \\ 6 & \text{if } 2 \leq x < 5 \end{cases}$$

and $F(x+10) = F(x)$ for all x.

Solution

Since F is even, $b_n = 0$ for all $n \geq 1$, and, for $n \geq 0$,

$$\begin{aligned} a_n &= \frac{2}{5} \int_0^5 f(x) cos(\frac{n\pi x}{5}) dx \\ &= \frac{2}{5} \left(\int_0^2 3 cos(\frac{n\pi x}{5}) dx + \int_2^5 6 cos(\frac{n\pi x}{5}) dx \right) \\ &= \frac{2}{5} \left(\frac{15}{n\pi} \left[sin(n\pi x/5) \mid_{x=0}^{x=2} \right] + \frac{30}{n\pi} \left[sin(n\pi x/5 \mid_{x=2}^{x=5} \right] \right) \\ &= \frac{2}{5} \left(\frac{15}{n\pi} [sin(n\pi 2/5)] + \frac{30}{n\pi} [sin(n\pi) - sin(n\pi 2/5)] \right) \\ &= \frac{2}{5} \left(\frac{15}{n\pi} [sin(n\pi 2/5)] + \frac{30}{n\pi} [-sin(n\pi 2/5)] \right) \end{aligned}$$

Remark. In submitting answers to the WebWork problems on Fourier series, you must remove expressions like $sin(n*\pi)$ or $cos(\pi*(2n-1)/2)$. When

n is an integer these are equal to 0, but WebWork checks these functions at non-integral points and does not give the zero value to them. Hence, e.g., if the $sin(n * \pi)$ of the last example is left in, WebWork will mark the answer as incorrect.

Example 2.

Compute the Fourier Series of the odd extension $F(x)$ of the function $f(x)$ such that

$$f(x) = \begin{cases} 3 & \text{if } 0 \leq x < 4 \\ -2 & \text{if } 4 \leq x < 6 \end{cases}$$

and $F(x+12) = F(x)$ for all x.

Since this is similar to Example 1, we only set up the necessary integrals and leave their computation to the reader.

Since F is odd, $a_n = 0$ for all $n \geq 0$, and, for $n \geq 0$,

$$\begin{aligned} b_n &= \frac{2}{6}\int_0^6 f(x) sin(\frac{n\pi x}{6})dx \\ &= \frac{2}{6}\left(\int_0^4 3cos(\frac{n\pi x}{6})dx + \int_4^6 (-2)cos(\frac{n\pi x}{6})dx\right) \end{aligned}$$

6 Orthogonal Functions

Let $v = (a_1, a_2, \ldots, a_n),\ w = (b_1, b_2, \ldots, b_n)$ be vectors in $\mathbf{R}^n$. The standard dot product of v and w is the number

$$v \cdot w = \sum_{i=1}^{n} a_i b_i$$

Let us also denote this by $< v, w >$ and call it the standard *inner product* of v and w.

This has the properties that

1. $< v, v > \geq 0$ for all vectors v and $< v, v >= 0$ if and only if $v = 0$
2. $< v, w >=< w, v >$ for all vectors v, w
3. $< av + bw, u >= a < v, u > +b < w, u >$ for all vectors u, v, w and scalars a, b.

The *norm* or *length* of $v = \sqrt{< v, v >}$.

Definition. Let $1 \leq k \leq n$ be a collection $\{v_1, v_2, \ldots, v_k\}$ of vectors in $\mathbf{R}^n$. The collection is called an *orthogonal* set of vectors in $\mathbf{R}^n$ if $< v_i, v_j >= 0$ for all $i \neq j$.

The collection $V = \{v_1, v_2, \ldots, v_k\}$ is called an *orthonomal* set of vectors if it is an orthogonal set and each vector has length 1.

If w can be expressed as a linear combination

$$w = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$$

and $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal set, then we can determine the coefficients a_i of w as follows. Using orhogonality, we have

$$\begin{aligned} < w, v_i > &= < a_1 v_1 + a_2 v_2 + \ldots + a_k v_k, v_i > \\ &= < a_1 v_1, v_i > + < a_2 v_2, v_i > + \ldots + < a_k v_k, v_1 > \\ &= a_i < v_i, v_i > \end{aligned}$$

Thus,

$$a_i =< w, v_i > / < v_i, v_i > . \tag{17}$$

In the case that our orthogonal set V contains n vectors, then *any* vector w can be uniquely expressed as $w = \sum_{i=1}^{n} a_i v_i$ and the coefficients a_i can be determined from (17).

Definition. An orthogonal set V of vectors in $\mathbf{R}^n$ is called *complete* if every vector w in $\mathbf{R}^n$ can be written uniquely as a linear combination of elements in V.

The previous comments state that *any* orthogonal set of n vectors in $\mathbf{R}^n$ is complete.

We wish to apply these concepts to function spaces.

Let $L > 0$ and let $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}([-L, L])$ denote the set of piecewise continuous functions on $[-L, L]$.

We define an inner product on $\mathcal{F}$ by

$$< f, g >= \frac{1}{L} \int_{-L}^{L} f(x) g(x) dx$$

This has some of the usual properties of the dot product on $\mathbf{R}^n$.

1. $< f, f > \geq \ 0$ for $f \in \mathcal{F}$
2. $< af + bg, h >= a < f, g > +b < g, h >$ for $f, g, h \in \mathcal{F}$ and a, b constants

Note: Since f is only piecewise continuous, it does not follow that

$$< f, f >= 0$$

implies that f is the zero function. It is only zero off a finite set of points. If f were continuous and $< f, f >= 0$, then it would imply that $f(x) = 0$ for all $x \in [-L, L]$, but in applications, the condition that we only deal with continuous functions is too restrictive.

Now consider the functions $\cos(n\pi x/L), \sin(n\pi x/L)$ for $n = 0, 1, 2, \ldots$.

Note that if $n = 0$, then $\cos(n\pi x/L) = 1$ for all x, and $\sin(n\pi x/L) = 0$ for all x.

The justification of Fourier series in section 3 shows that the set of functions

$$\{\cos(n\pi x/L),\ n = 1, 2, \ldots\} \bigcup \{\sin(n\pi x/L),\ n = 1, 2, 3, \ldots\}$$

forms an orthonormal set of functions in $\mathcal{F}$ with respect the the inner product $< \cdot, \cdot >$ on $\mathcal{F}$ we have just defined.

If we add the constant function $\mathbf{1}$, then we can express every function $f \in \mathcal{F}$ (up to finitely many exceptional points) as (an infinite sum)

$$f(x) = c_0 \ cos(0\pi x/L) + \sum_{n=1}^{\infty} (a_n cos(n\pi x/L) + b_n sin(n\pi x/L)) \qquad (18)$$

Here the coefficients are the costants $c_0, a_1, a_2, \ldots, b_1, b_2, \ldots$. We have denoted the constant term by c_0 instead of $a_0/2$ for a reason which we now explain.

The expression (18) is just the Fourier series of f.

Let us determine the coefficients in the way we did for orthogonal sets in $\mathbf{R}^n$.

Since, for any $n > 0$ we have

$$< cos(0\pi x/L), cos(n\pi x/L) > = < cos(0\pi x/L), sin(n\pi x/L) >= 0,$$

we get

$$\begin{aligned} < f(x), cos(0\pi x/L) > &= \frac{1}{L}\int_{-L}^{L} f(x)cos(0\pi x/L)dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{L}\int_{-L}^{L} c_0 dx \\ &= 2c_0 \end{aligned}$$

Since

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx,$$

we have that $c_0 = \frac{a_0}{2}$.

Thus, we write the constant term in the Fourier series as $\frac{a_0}{2}$ so that the formulas in terms of integrals for the Fourier coefficients then always have the factor $\frac{1}{L}$ times an integral from $-L$ to L.

Similarly,

$$\begin{aligned} a_n &= < f(x), cos(n\pi x/L) > \\ &= \frac{1}{L}\int_{-L}^{L} f(x)cos(n\pi x/L)dx \end{aligned}$$

and

$$\begin{aligned} b_n &= < f(x), sin(n\pi x/L) > \\ &= \frac{1}{L}\int_{-L}^{L} f(x)sin(n\pi x/L)dx \end{aligned}$$

Thus, if we list the functions $cos(n\pi/x), n = 0, 1, 2, \ldots$ and $sin(n\pi x/L), n = 1, 2, \ldots$ in one single list as $\{f_0, f_1, f_2, \ldots\}$, where f_0 is the function which is equal to 1 everywhere, then the Fourier series for f can be expressed as

$$f(x) \sim c_0 f_0 + c_1 f_2 + \ldots + \qquad (19)$$

where

$$c_n = < f, f_n >$$

for all $n > 0$.

The reason we did not use an equality in (19) is that $f(x)$ is only equal to the right hand side off a (possibly empty) finite set of points in each closed interval. As we said before, we do not have equality unless f is continuous.

Remark Note that the set $\{\mathbf{1}, cos(n\pi x/L), sin(n\pi x/L), n = 1, 2, 3, \ldots\}$ is almost orthonormal. It fails to be an orthonormal set only because the constant function **1** is not a unit vector. Its length is 2. This is not very important for our purposes.