14. Initial Value Problems and the Laplace Transform

We first consider the relation between the Laplace transform of a function and that of its derivative.

Theorem. *Suppose that $f(t)$ is a continuously differentiable function on the interval $[0,\infty)$. Then,*

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0). \tag{1}$$

Proof.
We integrate the Laplace transform of $f(t)$ by parts to get

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)dt \\
&= -\frac{1}{s}e^{-st}f(t)\Big|_{t=0}^{t=\infty} - \int_0^\infty -\frac{1}{s}e^{-st}f'(t)dt \\
&= \frac{1}{s}f(0) + \frac{1}{s}\mathcal{L}(f'(t)).
\end{aligned}$$

Multiplying both sides by s and moving the first term on the right side of this equation to the left side gives the theorem. QED.

Applying this formula twice to a C^2 function $f(t)$ gives the expression

$$\begin{aligned}
\mathcal{L}(f'') &= s\mathcal{L}(f') - f'(0) \\
&= s(s\mathcal{L}(f) - f(0)) - f'(0) \\
&= s^2\mathcal{L}(f) - sf(0) - f'(0).
\end{aligned}$$

Repeating this for a C^n function gives

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

Now, consider the following initial value problem.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0,\ y'(0) = y_0'. \tag{2}$$

with characteristic equation $z(r) = ar^s + br + c$.
Taking the Laplace transform we get

$$a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = \mathcal{L}(g)$$

or

$$a(s^2\mathcal{L}(y) - sy(0) - y'(0)) + b(s\mathcal{L}(y) - y(0)) + c\mathcal{L}(y) = \mathcal{L}(g)$$

$$z(s)\mathcal{L}(y) = (as + b)y(0) + ay'(0) + \mathcal{L}(g)$$

$$\mathcal{L}(y) = \frac{(as + b)y_0 + ay_0' + \mathcal{L}(g)}{z(s)}.$$

If we knew how to take the inverse transform of the right hand side we would then have the solution to (2).

The table at the end of this section summarizes useful formulas for this purpose.

Let's take an example.

Example 1.

Use the Laplace transform to find the unique solution to

$$y'' - y' - 2y = 0,\ y(0) = 1, y'(0) = 2.$$

We have

$$\begin{aligned} \mathcal{L}(y) &= \frac{(s-1)+2}{s^2 - s - 2} \\ &= \frac{(s+1)}{(s-2)(s+1)} \\ &= \frac{1}{s-2} \end{aligned}$$

and we need to find the inverse Laplace transform of the right last term.
This is just $y(t) = e^{2t}$.
Next, suppose that we have the same equations with the initial conditions

$$y(0) = 1,\ y'(0) = 0.$$

This gives

$$\begin{aligned}\mathcal{L}(y) &= \frac{(s-1)}{s^2 - s - 2} \\ &= \frac{(s-1)}{(s-2)(s+1)}\end{aligned}$$

For this we use partial fractions.

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{1}{3}(\frac{1}{s-2}) + \frac{2}{3}(\frac{1}{s+1}).$$

So,

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}(\frac{1}{3}(\frac{1}{s-2}) + \frac{2}{3}(\frac{1}{s+1})) \\ &= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}\end{aligned}$$

Example 2.
Use the Laplace transform to find the unique solution to

$$y'' - 3y' + 2y = 3cos(3t),\ y(0) = 2, y'(0) = 3.$$

We have

$$\mathcal{L}(y'') - 3\mathcal{L}(y') + 2\mathcal{L}(y) = 3\mathcal{L}(cos(3t)),\ y(0) = 2, y'(0) = 3.$$

This gives rise to

$$(s^2 - 3s + 2)\mathcal{L}(y) - (s-3)2 - 3 = 3(\frac{s}{s^2+9})$$

or,

$$(s^2 - 3s + 2)\mathcal{L}(y) - 2s + 3 = 3(\frac{s}{s^2+9})$$

$$(s^2 - 3s + 2)\mathcal{L}(y) = 2s - 3 + 3(\frac{s}{s^2 + 9})$$

$$\mathcal{L}(y) = \frac{2s - 3}{(s-2)(s-1)} + \frac{3s}{(s^2+9)(s-2)(s-1)}$$

We now find

$$\mathcal{L}^{-1}(\frac{2s-3}{(s-2)(s-1)}) = y_1(t)$$

and

$$\mathcal{L}^{-1}(\frac{3s}{(s^2+9)(s-2)(s-1)}) = y_2(t),$$

and then add them to get

$$y(t) = y_1(t) + y_2(t).$$

$\mathcal{L}^{-1}(\frac{2s-3}{(s-2)(s-1)})$**:**
Write

$$\frac{2s-3}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}$$

This gives the linear equations

$$\begin{aligned} A + B &= 2 \\ -A - 2B &= -3 \end{aligned}$$

which has the solutions $A = 1, B = 1$.
Hence,

$$y_1(t) = e^{2t} + e^t.$$

$\mathcal{L}^{-1}(\frac{3s}{(s^2+9)(s-2)(s-1)}):$
We write the right side as

$$\frac{A+Bs}{s^2+9}+\frac{C}{s-2}+\frac{D}{s-1} \tag{3}$$

or

$$\frac{A}{s^2+9}+\frac{Bs}{s^2+9}+\frac{C}{s-2}+\frac{D}{s-1}.$$

which we write as

$$\frac{A}{3}\frac{3}{s^2+9}+\frac{Bs}{s^2+9}+\frac{C}{s-2}+\frac{D}{s-1}.$$

From this we recognize the the inverse transform as

$$\frac{A}{3}sin(3t)+Bcos(3t)+Ce^{2t}+De^{t}.$$

To find A, B, C, D we put the fraction in (3) over the common denominator and note that the numerator equals $3s$. This gives the following system of linear equations for A, B, C, D.

$$\begin{aligned} 2A-9C-18D &= 0 \\ -3A+2B+9C+9D &= 3 \\ A-3B-C-2D &= 0 \\ B+C+D &= 0 \end{aligned}$$

The solution of this system is

$$A=-81/130, B=-21/130, C=6/13, D=-3/10$$

Remarks.

1. The use of the Laplace transform to solve initial value requires that the initial values $y(0),\ y'(0)$ be taken at time $t_0=0$. If one is given the IVP

$$ay''+by'+cy=g(t), y(t_0)=y_0,\ y'(t_0)=y_0',$$

then one simply translates the t variable by defining the function $v(t)=y(t+t_0)$. Since $y''(t+t_0)=v''(t)$ and $y'(t+t_0)=v'(t)$, we get the new IVP

$$av''(t) + bv'(t) + cv(t) = g(t + t_0),\ v(0) = y_0,\ v'(0) = y_0'.$$

We can use the Laplace transform to find $v(t)$. Then, we find $y(t)$ using the formula $y(t) = v(t - t_0)$.

2. So far, the Laplace transform simply gives us another method with which we can solve initial value problems for linear differential equations with constant coefficients. The possible advantages are that we can solve initial value problems without having first to solve the homogeneous equation and then finding the particular solution. The Laplace transform takes the differential equation for a function y and forms an associated algebraic equation to be solved for $\mathcal{L}(y)$. Then, one has to take the inverse Laplace transform to get y. A possible disadvantage is that the computations may be cumbersome, and we need to find the inverse transforms at the end of the process.

The real power of the Laplace transform will be seen later when we deal with differential equations with discontinuous right hand sides.

For convenience, we give a list of elementary functions and their Laplace transforms.

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2 + a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\dfrac{s}{s^2 + a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2 - a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2 - a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t) f(t-c)$	$e^{-cs} F(s)$	Sec. 6.3
14. $e^{ct} f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c} F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\displaystyle\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s) G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28