Quick Power Series Recurrence Formulas

Consider the initial value problem for the second order homogeneous linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,\ y(0) = c_0,\ y'(0) = c_1 \tag{1}$$

where $P(x), Q(x), R(x)$ are real analytic functions near $x = 0$ and $P(0) \neq 0$.

Thus, we can represent $P(x), Q(x), R(x)$ as power series

$$P(x) = p_0 + p_1 x + p_2 x^2 + \dots$$

$$Q(x) = q_0 + q_1 x + q_2 x^2 + \dots$$

$$R(x) = r_0 + r_1 x + r_2 x^2 + \dots$$

where each series converges absolutely for x in some interval $(-\alpha, \alpha)$ for some positive real number α and $p_0 \neq 0$.

It follows then that there is a unique solution $y(x)$ to (1) which can be expressed as a power series

$$y(x) = c_0 + c_1 x + c_2 x^2 \dots \tag{2}$$

which also converges absolutely in $(-\alpha, \alpha)$.

Our goal is to find the constants $c_0, c_1, c_2, \dots$ from the coefficients of the power series p_i, q_i, r_i.

It turns out that this can always be done through the use of *recurrence formulas* in which one successively determines a c_n in terms of $c_1, c_2, \dots, c_{n-1}$.

In this section, we will describe how to get these recurrence formulas when the functions $P(x), Q(x), R(x)$ are polynomials of degree two or less.

Thus, we can write the differential equation in (1) as

$$(a + bx + cx^2)y'' + (d + ex + fx^2)y' + (g + hx + kx^2)y = 0 \tag{3}$$

where $a, b, c, d, e, f, g, h, k$ are constants with $a \neq 0$.

Let us assume that we have

$$y(x) = \Sigma_{n \geq 0} c_n x^n \tag{4}$$

From the initial conditions we already know c_0 and c_1.

Differentiating (4), we get

$$y'' = \sum_{n\geq 0} n(n-1)c_n x^{n-2},\ y' = \sum_{n\geq 0} nc_n x^{n-1}$$

Plugging this in to the equation, and temporarily ignoring where the indices of summation go, we have

$$a \ \sum n(n-1)c_n x^{n-2} + b \ \sum n(n-1)c_n x^{n-1} + c \ \sum n(n-1)c_n x^n +$$

$$d \sum nc_n x^{n-1} + e \ \sum nc_n x^n + \ f \sum nc_n x^{n+1} +$$

$$g \sum c_n x^n + \ h \sum c_n x^{n+1} + \ k \sum c_n x^{n+2} = 0$$

Next, we rewrite this sum by putting all of the powers of x equal to n giving

$$a \ \sum (n+2)(n+1)c_{n+2} x^n + b \ \sum (n+1)nc_{n+1} x^n + c \ \sum n(n-1)c_n x^n +$$

$$d \sum (n+1)c_{n+1} x^n + e \ \sum nc_n x^n + \ f \sum (n-1)c_{n-1} x^n +$$

$$g \sum c_n x^n + \ h \sum c_{n-1} x^n + \ k \sum c_{n-2} x^n = 0$$

From this last equation, we must have the sum of all the multipliers of a given power of x equal to 0, so we can write the

General Recurrence Formula:

$$\begin{array}{c} a \ (n+2)(n+1)c_{n+2} + b \ (n+1)nc_{n+1} + c \ n(n-1)c_n + \\ d \ (n+1)c_{n+1} + e \ nc_n + f \ (n-1)c_{n-1} + \\ g \ c_n + h \ c_{n-1} + k \ c_{n-2} = 0 \end{array} \tag{5}$$

Now, we plug in the recurrences for $n = 0, n = 1$, etc. ignoring the terms with subscripts less than 0, to get the

First and Second Recurrence Formulas

$$\begin{aligned} &n = 0: \; 2 * 1 * a \; c_2 + d \; c_1 + g \; c_0 = 0 \\ &n = 1: \; 3 * 2 * a \; c_3 + 2(b + d) \; c_2 + (e + g) \; c_1 + h \; c_0 = 0 \end{aligned} \tag{6}$$

and the table (5) for $n \geq 2$.

The procedure can be automated somewhat by introducing the following *Recurrence Table* of the equation (3).

Recurrence Table

$(n+2)(n+1)$	$(n+1)n$	$n(n-1)$
ac_{n+2}	bc_{n+1}	cc_n
$n+1$	n	$n-1$
dc_{n+1}	ec_n	fc_{n-1}
gc_n	hc_{n-1}	kc_{n-2}

The expressions involving c_j for various $j's$ are called the *unknowns* of the table. The expressions $a, b, c, d, e, f, g, h, k$ are the coefficients of the table, and the expressions involving n are called the *multiplers* of the table.

The general recurrence relation can then be gotten by taking each entry of the table involving a c_j, multiplying it by the multiplier just above it, and setting the sum of all of the obtained expressions equal to 0.

The first few recurrence relations can be obtained by setting $n = 0, n = 1, n = 2$, etc in the general recurrence relation.

The easiest way to construct the recurrence table from the equation is to do it in several steps:

Step 1. Put the coefficients and unknowns in first, leaving space above for the multipliers.

$$\text{Step 1}\quad \left|\begin{array}{ccc} a\ c_{n+2} & b\ c_{n+1} & c\ c_n \\ \hline \\ d\ c_{n+1} & e\ c_n & f\ c_{n-1} \\ \hline \\ g\ c_n & h\ c_{n-1} & k\ c_{n-2} \end{array}\right|$$

Step 2: Enter the corresponding multipliers r

$$\text{Step 2}\quad \left|\begin{array}{ccc} (n+2)(n+1) & (n+1)n & n(n-1) \\ a\ c_{n+2} & b\ c_{n+1} & c\ c_n \\ \hline n+1 & n & n-1 \\ d\ c_{n+1} & e\ c_n & f\ c_{n-1} \\ \hline \\ g\ c_n & h\ c_{n-1} & k\ c_{n-2} \end{array}\right|$$

Here is a way to remember the multipliers and unknowns c_j of the columns of the Recurrence Table.

The $n-$th terms of y'', y', y are, respectively,

$$n(n-1)\ c_n x^{n-2},\ n c_n x^{n-1},\ \ c_n x^n$$

Changing all the powers to x^n gives

$$(n+2)(n+1)c_{n+2}x^n,\ (n+1)c_{n+1}x^n,\ c_n x^n$$

So,

> *the coefficients of x^n form the multipliers and unknowns in the first column of the table.*

The second column is obtained from the first by replacing every n by an $n-1$, and the third column is obtained from the second by replacing each $n-1$ by $n-2$.

Example 1: Consider the equation

$$2y'' + (3 + 2x)y' + (5 - x)y = 0$$

Step 1:

$$\left|\begin{array}{ccc}
2c_{n+2} & 0 \cdot c_{n+1} & 0 \cdot c_n \\
\hline
\\
3c_{n+1} & 2c_n & 0 \cdot c_{n-1} \\
\hline
\\
5c_n & -c_{n-1} & 0 \cdot c_{n-2}
\end{array}\right|$$

Step 2:

$$\left|\begin{array}{ccc}
(n+2)(n+1) & (n+1)n & n(n-1) \\
2c_{n+2} & 0 \cdot c_{n+1} & 0 \cdot c_n \\
\hline
\\
n+1 & n & n-1 \\
3c_{n+1} & 2c_n & 0 \cdot c_{n-1} \\
\hline
\\
5c_n & -c_{n-1} & 0 \cdot c_{n-2}
\end{array}\right|$$

Now, we get the successive recurrence relations by adding the terms for $n = 0, n = 1$, etc. and neglecting terms c_m with m negative.

$n = 0$: $2 * 2 * 1 * c_2 + 3 * 1 * c_1 + 5c_0 = 0$

$n = 1$: $2 * 3 * 2 * c_3 + 3 * 2c_2 + (5 + 2 * 1)c_1 - c_0 = 0$

$n \geq 2$: $2(n + 2)(n + 1)\ c_{n+2} + 3(n + 1)c_{n+1} + (2n + 5)c_n - c_{n-1} = 0$

Example 2:

$$(5 - 4x + 3x^2)y'' + (6 + 3x + 4x^2)y' + (-3 + 5x - 4x^2)y = 0$$

Step 1:

$5c_{n+2}$	$-4 \cdot c_{n+1}$	$3 \cdot c_n$
$6c_{n+1}$	$3c_n$	$4 \cdot c_{n-1}$
$-3c_n$	$5c_{n-1}$	$-4 \cdot c_{n-2}$

Step 2:

$(n+2)(n+1)$ $5c_{n+2}$	$(n+1)n$ $-4 \cdot c_{n+1}$	$n(n-1)$ $3 \cdot c_n$
$n+1$ $6c_{n+1}$	n $3c_n$	$n-1$ $4 \cdot c_{n-1}$
$-3c_n$	$5c_{n-1}$	$-4 \cdot c_{n-2}$

Now, we get the successive recurrence relations again as above.

$n = 0$: $5 * 2 * 1 * c_2 + 6c_1 - 3c_0 = 0$

$n = 1$: $5 * 3 * 2 * c_3 - 4 * 2c_2 + 6 * 2c_2 + 3c_1 - 3c_1 + 5c_0 = 0$

$n \geq 2$: $$5(n+2)(n+1)c_{n+2} - 4(n+1)nc_{n+1} + 3n(n-1)c_n + 6(n+1)c_{n+1} + 3nc_n + 4(n-1)c_{n-1} - 3c_n + 5c_{n-1} - 4c_{n-2} = 0$$