86.70 The Use of Hyperbolic Cosines in Solving Cubic Polynomials
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Reviewed work(s):
Source: *The Mathematical Gazette*, Vol. 86, No. 507 (Nov., 2002), pp. 473-477
Published by: The Mathematical Association
Stable URL: http://www.jstor.org/stable/3621149
Accessed: 20/11/2011 10:53

with $\lambda = 1$ when **a** is $\mathbf{b} \times \mathbf{c}$. The argument above now shows that (5) holds with $\lambda = 1$ for any **a** and this is (1).

References

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86.70 The use of hyperbolic cosines in solving cubic polynomials

Introduction

The method of extracting one real root of a cubic polynomial has been known for some hundreds of years, and has been widely documented. However, the extraction of the remaining roots is invariably treated in a less satisfactory manner, the implication often being that the corresponding quadratic equation should be formed and solved in the conventional way. This article improves on this technique by deriving a set of compact algebraic formulae based on hyperbolic functions which will evaluate *all* the roots of a cubic polynomial *directly*, regardless of whether they are real, imaginary, complex, or repeated.

Background

Given a cubic polynomial of the form

$$f(x) = ax^3 + bx^2 + cx + d$$

the standard method of solving $f(x) = 0$ is to form the reduced (sometimes referred to as 'depressed') cubic equation

$$az^3 + mz + n = 0$$

where $z = x + b/3a$, then to use one of a number of substitutions to evaluate z. The classical substitution is $z = p + q$, which enables one real root, α, to be evaluated using cube roots. However, the evaluation of the remaining roots using this technique always involves manipulating complex numbers, and is rarely discussed; the easiest solution is to solve the quadratic

$$\frac{f(x)}{(x - \alpha)} = 0.$$

Nickalls [1] describes a trigonometric technique to solve a cubic with three real roots using a substitution based on the cosine function, although this yields complex angles when the technique is applied to cubics with complex roots. Nickalls doesn't explore this area, as it's impossible to illustrate complex angles graphically—however, an extension of his technique is here shown to provide simple algebraic solutions to any cubic, regardless of the nature of its roots.

Solving the cubic

It is necessary to define the following quantities (similarly to [1]):

$$x_N = -\frac{b}{3a}, \qquad y_N = f(x_N), \qquad \delta^2 = \frac{b^2 - 3ac}{9a^2}, \qquad h = 2a\delta^3.$$

The reduced cubic is then of the form

$$az^3 - 3a\delta^2 z + y_N = 0$$

where $z = x - x_N$. Nickalls showed geometrically that, for $h \neq 0$,* the relationship between y_N and h determines the nature of the roots of the original cubic as shown in the following table:

Relationship	form of the cubic
$\lvert y_N / h \rvert \leqslant 1$	three real roots (may include a co-incident pair)
$\lvert y_N / h \rvert > 1$	one real root, two turning points
(y_N / h) imaginary (δ^2 negative)	one real root, no turning points

This article will be primarily concerned with those cases yielding only one real root.

The case where $(y_N / h) < -1$

We define the solution of the reduced cubic equation to be $z = 2\delta \cosh t$. This gives

$$2a\delta^3 \left(4 \cosh^3 t - 3 \cosh t\right) + y_N = 0.$$

Crucially, we allow t to be complex, namely $3t = u + iv$. Then since $h = 2a\delta^3$, we get

$$\cosh(u + iv) = -\frac{y_N}{h}.$$

Using the identity

$$\cosh(u + iv) \equiv \cosh u \cos v + i \sinh u \sin v$$

this becomes

$$\cosh u \cos v + i \sinh u \sin v = -\frac{y_N}{h}.$$

* If $h = 0$, then $\delta = 0$ by definition, and the reduced equation becomes trivial. It is worth noting that, if y_N is also zero, the original cubic will have three co-incident real roots.

Equating the real and imaginary parts, $u = 0$ or $v = n\pi$. If $u = 0$, then $\cos v = -y_N/h > 1$, which is impossible, so $v = n\pi$. Hence $(-1)^n \cosh u = -(y_N/h) > 1$, but $\cosh u > 0$ and so n is even. Let $n = 2m$. Hence $v = 2m\pi$ and $u = \pm \operatorname{Cosh}^{-1}(-y_N/h)$. Then

$$t = \pm\tfrac{1}{3}\operatorname{Cosh}^{-1}\left(-\frac{y_N}{h}\right) + \frac{2m\pi i}{3} \quad \text{and } z = 2\delta\cosh\left(\pm\tfrac{1}{3}\operatorname{Cosh}^{-1}\left(-\frac{y_N}{h}\right) + \frac{2m\pi i}{3}\right).$$

This has three different values according as $m \equiv 0,\ 1,\ 2 \pmod 3$, as cosh is an even function. Hence the roots of the original cubic equation are

$$\alpha = x_N + 2\delta\cosh\left(\tfrac{1}{3}\cosh^{-1}\left(-\frac{y_N}{h}\right)\right)$$

$$\beta, \gamma = x_N - \delta\left[\cosh\left(\tfrac{1}{3}\cosh^{-1}\left(-\frac{y_N}{h}\right)\right) \pm i\sqrt{3}\sinh\left(\tfrac{1}{3}\cosh^{-1}\left(-\frac{y_N}{h}\right)\right)\right].$$

Thus this technique provides the two complex roots *directly*.

The case where $(y_N/h) > +1$

The case of $(y_N/h) > +1$ is dealt with in a similar way, and yields the values of u and v as

$$u = \cosh^{-1}\left(\frac{y_N}{h}\right), \qquad v = \pi,\ 3\pi,\ 5\pi,\ \ldots$$

giving the original cubic's roots as

$$\alpha = x_N - 2\delta\cosh\left(\tfrac{1}{3}\cosh^{-1}\left(\frac{y_N}{h}\right)\right)$$

$$\beta, \gamma = x_N + \delta\left[\cosh\left(\tfrac{1}{3}\cosh^{-1}\left(\frac{y_N}{h}\right)\right) \pm i\sqrt{3}\sinh\left(\tfrac{1}{3}\cosh^{-1}\left(\frac{y_N}{h}\right)\right)\right].$$

The case where (y_N/h) is imaginary

This case is a little more tricky, although the technique is the same. Since $\delta^2 < 0$, δ is imaginary, as is h and thus y_N/h. If δ is written as $i\Delta$ where Δ is real, then

$$\begin{aligned}\frac{-y_N}{h} &= \frac{-y_N}{2a\delta^3} \\ &= \frac{-iy_N}{2a\Delta^3}.\end{aligned}$$

Further, if we define $H = 2a\Delta^3$, then

$$-\frac{y_N}{h} = -\frac{iy_N}{H}.$$

To solve for z, we use the usual substitution of $z = 2\delta\cosh t$, where $3t = u + iv$, and proceed thus:

$$\cosh u \cos v + i \sinh u \sin v = -\frac{iy_N}{H}.$$

Equating coefficients:

$$u = \sinh^{-1}\left(\frac{y_N}{H}\right), \qquad v = \tfrac{3}{2}\pi, \tfrac{7}{2}\pi, \tfrac{11}{2}\pi, \ldots$$

from which

$$z = 2i\Delta \cosh\left(\tfrac{1}{3}\sinh^{-1}\left(\frac{y_N}{H}\right) + \tfrac{1}{2}i\pi\right)$$

$$z = 2i\Delta \cosh\left(\frac{1}{3}\sinh^{-1}\left(\frac{y_N}{H}\right) + \frac{7}{6}i\pi\right)$$

$$z = 2i\Delta \cosh\left(\frac{1}{3}\sinh^{-1}\left(\frac{y_N}{H}\right) + \frac{11}{6}i\pi\right)$$

hence the roots of the original cubic equation are

$$\alpha = x_N - 2\Delta \sinh\left(\tfrac{1}{3}\sinh^{-1}\left(\frac{y_N}{H}\right)\right)$$

$$\beta, \gamma = x_N + \Delta\left[\sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{y_N}{H}\right)\right) \pm i\sqrt{3}\cosh\left(\frac{1}{3}\sinh^{-1}\left(\frac{y_N}{H}\right)\right)\right].$$

The case of three real roots

For completion, it's worth mentioning that the above technique *can* be applied where $|y_N / h| \leqslant 1$, and merely reveals Nickall's solutions:

$$\alpha = x_N + 2\delta \cos\left(\tfrac{1}{3}\cos^{-1}\left(\frac{y_N}{h}\right)\right)$$

$$\beta, \gamma = x_N + 2\delta \cos\left(\tfrac{1}{3}\cos^{-1}\left(-\frac{y_N}{h}\right) \pm \tfrac{2}{3}\pi\right)$$

Example 1

Solve the equation:

$$x^3 - 11x^2 + 35x - 49 = 0.$$

The parameters are:

$$x_N = \frac{11}{3}, \qquad y_N = \frac{-520}{27}, \qquad \delta = \frac{4}{3} \qquad \text{and} \qquad h = \frac{128}{27}.$$

Since $y_N / h < -1$, let $z = 2\delta \cosh t$, where $\cosh 3t = -y_N / h$. Evaluating $\frac{1}{3}\cosh^{-1}(-y_N / h)$ as 0·6931472, the roots are given by:

$$\alpha = \tfrac{11}{3} + 2\left(\tfrac{4}{3}\right)\cosh 0{\cdot}6931472 \qquad \approx 7$$

$$\beta, \gamma = \tfrac{11}{3} - \tfrac{4}{3}\left(\cosh 0{\cdot}6931472 \pm i\sqrt{3}\sinh 0{\cdot}6931472\right) \approx 2 \pm i\sqrt{3}.$$

Example 2

Solve the equation:

$$x^3 + x^2 - 11x + 45 = 0.$$

The parameters are:

$$x_N = \frac{-1}{3}, \qquad y_N = \frac{1316}{27}, \qquad \delta = \frac{\sqrt{34}}{3} \qquad \text{and} \qquad h = \frac{68\sqrt{34}}{27}.$$

Since $y_N/h > 1$, let $z = 2\delta \cosh t$, where $\cosh 3t = y_N/h$. Evaluating $\frac{1}{3}\cosh^{-1}(y_N/h)$ as 0·6231009, the roots are given by:

$$\begin{aligned} \alpha &= \tfrac{-1}{3} - 2\left(\tfrac{\sqrt{34}}{3}\right)\cosh 0{\cdot}6231009 && \approx -5 \\ \beta, \gamma &= \tfrac{-1}{3} + \left(\tfrac{\sqrt{34}}{3}\right)(\cosh 0{\cdot}6231009 \pm i\sqrt{3}\sinh 0{\cdot}6231009) && \approx 2 \pm i\sqrt{5}. \end{aligned}$$

Example 3

Solve the equation:

$$x^3 - 5x^2 + 19x - 15 = 0.$$

The parameters are:

$$x_N = \frac{5}{3}, \qquad y_N = \frac{200}{27}, \qquad \delta = i\frac{4\sqrt{2}}{3} \qquad \text{and} \qquad H = \frac{256\sqrt{2}}{27}.$$

Since $\delta^2 < 0$, let $z = 2\delta \cosh t$, where $\cosh 3t = -iy_N/H$. Evaluating $\frac{1}{3}\sinh^{-1}(y_N/H)$ as 0·1758687, the roots are given by:

$$\begin{aligned} \alpha &= \tfrac{5}{3} - 2\left(\tfrac{4\sqrt{2}}{3}\right)\sinh 0{\cdot}1758687 && \approx 1 \\ \beta, \gamma &= \tfrac{5}{3} + \left(\tfrac{4\sqrt{2}}{3}\right)(\sinh 0{\cdot}1758687 \pm i\sqrt{3}\cosh 0{\cdot}1758687) && \approx 2 \pm i\sqrt{11}. \end{aligned}$$

Reference

1. R. W. D. Nickalls, A new approach to solving the cubic: Cardan's solution revealed, *Math. Gaz.* **77** (November 1993) pp. 354-359.

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86.71 Yet another proof that $\sum \frac{1}{n^2} = \frac{1}{6}\pi^2$

The idea behind this proof is to evaluate a certain double integral in two ways, one of which gives a numerical answer and the other of which leads to an expansion in a series related to $\sum \frac{1}{n^2}$. The double integral involved is, in essence, one of those evaluated in an interesting recent article by Javad Mashreghi, [1]; my contribution here is merely to highlight the connection with $\sum \frac{1}{n^2}$. For brevity, I omit the entirely routine technical justifications of each step of the argument. There is, of course, a myriad of other proofs that $\sum \frac{1}{n^2} = \frac{1}{6}\pi^2$; Robin Chapman has collected 14 of them in a splendid article on his website [2].