Problem 5.
A population P obeys the logistic model. It satisfies the equation

$$\frac{dP}{dt} = \frac{5}{900}P(9-P)$$

for $P > 0$.
(a) The population is increasing when ??

Ans : We need $\frac{dP}{dt} > 0$. This occurs when $P(9-P) > 0$. Note that this inequality divides $\mathbb{R}$ into 2 regions, namely $(0,9),(9,\infty)$, as P was assumed to be positive. Taking test points $P = 1, P = 10$ gives respectively $8, -10$. Thus, the interval that we want is $(0,9)$, since the derivative can only change sign at a zero.

(b) The population is decreasing when ??

Ans : By mutual exclusion, the population is decreasing when $P > 9$.

(c) Assume that $P(0) = 2$. Find $P(65)$.

Ans : First solve the ODE. This is a separable ODE. Rewrite as

$$\frac{dP}{P(9-P)} = \frac{5}{900}dt \quad (\text{label} - *)$$

Now integrate both sides. The left hand side, by partial fractions is

$$\int \frac{1}{9P} - \frac{1}{9(P-9)} = \frac{1}{9}(\log \frac{P}{9-P}) + C$$

so that from $(*)$

$$\log P - \log{(9-P)} = \frac{t}{20} + D$$

for some new constant D.
Using the initial condition gives $D = \log{(\frac{2}{7})}$ By basic algebra then, solving for P gives

$$P = \frac{18e^{\frac{t}{20}}}{2e^{\frac{t}{20}} + 7}$$

Plugging in $t = 65$ gives the answer

$$P(65) = \frac{18e^{\frac{13}{4}}}{2e^{\frac{13}{4}} + 7} \approx 7.9245$$

6. A tank contains 60 kg of salt and 1000 L of water. A solution of a concentration 0.03 kg of salt per liter enters a tank at the rate 8 L/min. The solution is mixed and drains from the tank at the same rate.

(a) What is the concentration of our solution in the tank initially?
Ans: This is initial volume / initial concentration $= \frac{60\text{kg}}{1000\text{L}} = .06\frac{\text{kg}}{\text{L}}$

(b) Find the amount of salt in the tank after 1.5 hours.
Ans : We use the formula from the notes. If $Q(t)$ is the amount of salt as a function of t, then with

$$Q' = Q'_{\text{in}} - Q'_{\text{out}} = 8(.03)\frac{\text{kg}}{\text{min}} - \frac{Q}{1000}8\frac{\text{kg}}{\text{min}}$$

Thus, we get the differential equation and initial value (simplifying and omitting the dimensions)

$$Q' + \frac{1}{125}Q = \frac{6}{25}, Q(0) = 60$$

Note that the initial condition is from part (a). The solution to the differential equation

$$Q' + AQ = B$$

is

$$Q = \frac{B}{A} + Ce^{-At}$$

for some constant C. Using the initial value condition gives $C = 60 - \frac{B}{A}$. So we get with $A = \frac{1}{125}, B = \frac{6}{25}$, the solution

$$Q = 30 + 30e^{-\frac{t}{125}}$$

With $t = 90$ (since 1.5 hours is 90 minutes to agree with our dimensions), we get the answer $t \approx 44.6025$ kg.

(c) Find the concentration of salt in the solution in the tank as time approaches infinity.

Ans: Taking the limit, we see that $Q(\infty) = 30$. Since there were 1000 liters initially, the answer is 30 $\frac{\text{kg}}{\text{L}}$.

Problem 10
A tank contains 60 kg of salt and 2000 L of water. Pure water enters a tank at the rate 10 L/min. The solution is mixed and drains from the tank at the rate 5 L/min.

(a) What is the amount of salt in the tank initially?
Ans : 60 kg

(b) Find the amount of salt in the tank after 4 hours?
Ans : In exactly the same process as problem 6, we get that the differential equation is

$$Q' = 0 - Q\frac{5}{2000 + 5t}$$

Note that for this problem, there is more water flowing in per minute than out. 10 liters in - 5 liters out per minute gives a +5 net liters per minute increase, which is where the $2000 + 5t$ is coming from. Separating the differential equation gives

$$\frac{dQ}{Q} = -\frac{5dt}{2000 + 5t}$$

Integrating gives

$$\log Q = -\log(400 + t) + C$$

Using the initial condition $Q(0) = 60$ and solving for C yields $C = \log 60 + \log 400$. The unique solution is then

$$\log Q = \log(400 + t) + \log 60 + \log 400$$

Written explicitly for Q, we get

$$Q = e^{-\log(400+t)+\log 24000}$$

Thus, after 4 hours, or 240 minutes, our answer is

$$Q(240) = e^{-\log(640+\log 24000)} = 37.5\text{kg}$$

Note this answer is exact.

(c) Find the concentration of salt in the solution in the tank as time approaches infinity. (Assume your tank is large enough to hold all the solution.)

Ans: Taking the limit in part (b), we easily see we get 0.