3.1 Unitary Representations

Def Let G be a locally compact group. A (unitary) representation of G is a continuous homomorphism $\pi: G \to \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ is the group of unitary operators on the Hilbert space $\mathcal{H}$ equipped with the strong operator topology. $\mathcal{H}$ is called the representation space of π and $\dim(\mathcal{H})$ is the degree of π. $\square$

Note that the continuity condition on π is simply that for all $\xi \in \mathcal{H}$

$$G \ni x \mapsto \pi(x)\xi \in \mathcal{H}$$

is continuous when $\mathcal{H}$ is equipped with its usual norm topology.

Exercise Let $\mathcal{H}$ be a Hilbert space. Show that a net $(u_i)_{i\in I} \subset \mathcal{U}(\mathcal{H})$ converges to $u \in \mathcal{H}$ in the strong operator topology iff for all $\xi, \eta \in \mathcal{H}$

$$\lim_{i\to I} \langle (u-u_i)\xi, \eta\rangle = 0$$

(That is, iff $u_i \to u$ in the weak operator topology). $\square$

Ex Let G be a locally compact group and let S be a locally compact Hausdorff. Suppose $G \curvearrowright S$ as homeomorphisms. For $f \in C_c(S)$ define

$$[\pi(x)f](s) := f(x^{-1}\cdot s) \qquad x \in G,\ s \in S$$

Then $\pi(x) f \in C_c(S)$. Then by the same argument as in the proof of Proposition 2.5 one has

$$\lim_{x\to 1} \|\pi(x) f - f\|_\infty = 0 \qquad \forall f \in C_c(S). \quad *$$

Now suppose S admits a Radon measure μ such that there exists a continuous function

$$\phi: G\times S \longrightarrow (0,\infty)$$

satisfying

$$d\mu(x\cdot s) = \phi(x,s)\, d\mu(s). \qquad (\phi \equiv 1 : \mu \text{ is } G\text{-invariant})$$

Define

$$[\tilde{\pi}(x) f](s) := \phi(x, x^{-1}\cdot s)^{-\frac{1}{2}} f(x^{-1}\cdot s) \quad (= \phi(x,x^{-1}s)^{-1/2}[\pi(x) f](s))$$

Then

$$\|\tilde{\pi}(x) f\|_2^2 = \int_S \phi(x, x^{-1}\cdot s)^{-1} |f(x^{-1}\cdot s)|^2\, d\mu(s)$$

$$= \int_S \phi(x,s)^{-1} |f(s)|^2\, d\mu(x\cdot s) \quad = \|f\|_2^2$$

So $\tilde{\pi}(x)$ extends uniquely to a unitary on $L^2(S,\mu)$. Note that

$$\phi(xy,s) = \frac{d\mu((xy)\cdot)}{d\mu}(s)$$

$$= \frac{d\mu((xy)\cdot)}{d\mu(y\cdot)}(s)\,\frac{d(y\cdot)}{d\mu}(s)$$

$$= \phi(x, y\cdot s)\,\phi(y,s)$$

So, one can check $\tilde{\pi}(x)\tilde{\pi}(y)=\tilde{\pi}(xy)$ and $\tilde{\pi}(x^{-1})=\tilde{\pi}(x)^*$ by verifying it on $C_c(S)$. The continuity of $\tilde{\pi}$ follows from $(*)$ and the same argument as in Proposition 2.27 □

We can consider $S=G$ with $G \curvearrowright G$ by left translation and μ a left Haar measure on G. In this case one has $\phi\equiv 1$ and $\tilde{\pi}=\pi$.

Def The <u>left regular representation</u> of a locally compact group G with left Haar measure μ is

$$\lambda: G \longrightarrow \mathcal{U}(L^2(G,\mu))$$
$$x \longmapsto [f \mapsto f(x^{-1}\cdot)]$$

The <u>right regular representation</u> G with right Haar measure ν is

$$\rho: G \longrightarrow \mathcal{U}(L^2(G,\nu))$$
$$x \longmapsto [f \mapsto f(\cdot\, x)]$$

□

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The following provides a way to compare representations.

Def For unitary representations $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$, $i=1,2$, we say $T\in B(\mathcal{H}_1,\mathcal{H}_2)$ is an <u>intertwining operator</u> for π_1 and π_2 if

$$T\pi_1(x) = \pi_2(x)T \qquad \forall x\in G$$

We denote the set of all such operators by $C(\pi_1,\pi_2)$. We say π_1 and π_2 are (<u>unitarily</u>) <u>equivalent</u> if $C(\pi_1,\pi_2)$ contains a unitary operator. □

Note that π_1 and π_2 being equivalent is equivalent to $\pi_2(x) = U\pi_1(x)U^{-1}$ for some unitary $U: \mathcal{H}_1 \to \mathcal{H}_2$ and all $x\in G$.

Ex Let μ and ν be left and right Haar measures on G, respectively. Define $U: L^2(G,\mu) \to L^2(G,\nu)$ by $Uf = \check{f}$. Then

$$[U\lambda(x)U^* f](y) = [\lambda(x)U^*f](y^{-1}) = [U^*f](x^{-1}y^{-1}) = f(yx) = \rho(x)$$

That is, the left and right regular representations are equivalent. We can also define $W: L^2(G,\nu) \to L^2(G,\mu)$ by $Wf = \Delta^{-1/2} f$. Then

$$
\begin{aligned}
[W\rho(x)W^* f](y) &= \Delta^{-1/2}(y)\,[\rho(x) W^* f](y) \\
&= \Delta^{-1/2}(y)\,[W^* f](yx) \\
&= \Delta^{-1/2}(y)\,\Delta^{1/2}(yx)\, f(yx) = \Delta^{1/2}(x)\, f(yx)
\end{aligned}
$$

So ρ is equivalent to $\tilde{\rho}: G \to L^2(G,\mu)$ defined by $[\tilde{\rho}(x) f](y) = \Delta^{1/2} f(yx)$. Note that λ and $\tilde{\rho}$ commute on $L^2(G,\mu)$. □

Exercise Suppose $T \in C(\pi_1, \pi_2)$ has trivial kernel and dense range. Show π_1 and π_2 are unitarily equivalent. In particular, if $C(\pi_1, \pi_2)$ contains an invertible element then π_1 and π_2 are unitarily equivalent. [Hint: consider the polar decomposition of T.] □

Def Let $\pi: G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. The commutant of π is

$$C(\pi) := C(\pi,\pi) = \{T \in B(\mathcal{H}) : T\pi(x) = \pi(x) T \quad \forall x \in G\}.$$
□

Observe that $C(\pi)$ is a von Neumann algebra. Indeed, $1 \in C(\pi)$ and for $T, S \in C(\pi)$ and $x \in G$ we have

$$(T+S)\pi(x) = T\pi(x) + S\pi(x) = \pi(x) T + \pi(x) S = \pi(x)(T+S)$$

$$(TS)\pi(x) = T\pi(x) S = \pi(x)(TS)$$

$$T^*\pi(x) = (\pi(x^{-1}) T)^* = (T\pi(x^{-1}))^* = \pi(x) T^*$$

So $C(\pi)$ is a unital $*$-algebra, and if $(T_i)_{i\in I} \subset C(\pi)$ converges in the weak operator topology to $T \in B(\mathcal{H})$ then for all $\xi, \eta \in \mathcal{H}$

$$
\begin{aligned}
\langle (T\pi(x) - \pi(x) T)\xi, \eta\rangle &= \langle T\pi(x)\xi, \eta\rangle - \langle T\xi, \pi(x)^*\eta\rangle \\
&= \lim_{i\to\infty} \langle T_i \pi(x)\xi, \eta\rangle - \langle T_i \xi, \pi(x)^*\eta\rangle \\
&= \lim_{i\to\infty} \langle (T_i\pi(x) - \pi(x) T_i)\xi, \eta\rangle = 0
\end{aligned}
$$

So letting $\eta = (T\pi(x) - \pi(x) T)\xi$ and allowing ξ to vary over all of $\mathcal{H}$ we obtain $T\pi(x) = \pi(x) T$. Thus $T \in C(\pi)$ and $C(\pi)$ is a von Neumann algebra.

EX Recall that $\tilde{\rho}$ was a version of the right regular representation that commutes with the left regular representation λ:

$$[\tilde{\rho}(x) f](y) = \Delta^{1/2}(x) f(yx).$$

It turns out that $C(\lambda) = \overline{\tilde{\rho}(G)}^{WOT}$ and $C(\tilde{\rho}) = \overline{\lambda(G)}^{WOT}$. The fact that these representations commute gives the inclusions "$\supseteq$". The reverse inclusion is more subtle and requires one to study which $f \in L^2(G,\mu)$ define bounded operators on $L^2(G,\mu)$ via left and right convolution. □

We now discuss a few ways to build new representations from existing ones.

Def Let $\pi: G \to \mathcal{U}(\mathcal{H})$ be a representation. We say a closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is *invariant* for π if

$$\pi(x)\mathcal{K} \subseteq \mathcal{K} \qquad \forall x \in G.$$

For $\mathcal{K} \neq \{0\}$ invariant, $\pi^{\mathcal{K}}: G \to \mathcal{U}(\mathcal{K})$ defined by $\pi^{\mathcal{K}}(x) := \pi(x)|_{\mathcal{K}}$ is called a *subrepresentation* of π. Finally, we say π is *reducible* if it admits an invariant subspace $\{0\} \neq \mathcal{K} \neq \mathcal{H}$, and otherwise we say π is *irreducible*. □

Exercise Let $\pi_i: G \to \mathcal{U}(\mathcal{H}_i)$, $i=1,2$, be representations. Suppose $V \in C(\pi_1, \pi_2)$ is an isometry and denote $\mathcal{K} := V\mathcal{H}_1$. Show that $\mathcal{K}$ is is invariant for π_2 and that π_1 is unitarily equivalent to $\pi_2^{\mathcal{K}}$. In this case we say π_1 is subequivalent to π_2. □

We can characterize invariant subspaces in terms of the projections of the commutant. First, we require a lemma.

Lemma 3.1 Let $\pi: G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. If $\mathcal{K} \subseteq \mathcal{H}$ is invariant for π, then so is $\mathcal{K}^\perp$.

Proof For $\xi \in \mathcal{K}$, $\eta \in \mathcal{K}^\perp$, and $x \in G$ we have

$$\langle \pi(x)\eta, \xi \rangle = \langle \eta, \pi(x)^*\xi \rangle = \langle \eta, \overbrace{\pi(x^{-1})\xi}^{\mathcal{K}} \rangle = 0.$$

Since $\xi \in \mathcal{K}$ was arbitrary, we have $\pi(x)\eta \in \mathcal{K}^\perp$ and hence $\mathcal{K}^\perp$ is invariant for π. □

Proposition 3.2 Let $\pi: G \to \mathcal{U}(\mathcal{H})$ be a unitary representation, let $\mathcal{K} \subseteq \mathcal{H}$ be a closed subspace, and let $p \in B(\mathcal{H})$ be the projection onto $\mathcal{K}$. Then $\mathcal{K}$ is invariant for π if and only if $p \in C(\pi)$.

Proof ($\Rightarrow$) Suppose $\mathcal{K}$ is invariant for π. Then $\mathcal{K}^\perp$ is also invariant for π by Lemma 3.1, and we recall that $1-p$ is the projection onto $\mathcal{K}^\perp$. For $\xi \in \mathcal{H}$ and $x \in G$ we have

$$\pi(x)p\xi = (p + (1-p))\pi(x)p\xi = p\pi(x)p\xi + \cancelto{0}{(1-p)\pi(x)p\xi} = p\pi(x)p\xi$$

and

$$p\pi(x)\xi = p\pi(x)(p + (1-p))\xi = p\pi(x)p\xi + \cancelto{0}{p\pi(x)(1-p)\xi} = p\pi(x)p\xi.$$

Thus $\pi(x)p = p\pi(x)$ and $p \in C(\pi)$.

($\Leftarrow$) If $p \in C(\pi)$, then for $\xi \in K$ and $x \in G$ we have

$$\pi(x)\xi = \pi(x)p\xi = p\pi(x)\xi \in K$$

so that K is invariant for π. $\square$

The structure of von Neumann algebras, such as $C(\pi)$, implies they have lots of projections. In particular, a von Neumann algebra M has a non-trivial projection $0 \neq p \neq 1$ if and only if $M \neq \mathbb{C}1$. This fact is at the heart of the next result.

$\boxed{\text{Theorem 3.3}}$ (Schur's Lemma) Let G be a locally compact group.

(a) A unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is irreducible if and only if $C(\pi) = \mathbb{C}1$.

(b) For irreducible unitary representations $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$, $i = 1, 2$, if π_1 and π_2 are unitarily equivalent then $\dim C(\pi_1, \pi_2) = 1$. Otherwise $C(\pi_1, \pi_2) = \{0\}$.

Proof (a): By the discussion preceeding the theorem, $C(\pi) = \mathbb{C}1$ iff $C(\pi)$ has no non-trivial projections, and by Proposition 3.2 this is equivalent to π being irreducible.

(b): Let $T \in C(\pi_1, \pi_2)$. Then for $x \in G$ we have

$$T^*\pi_2(x) = [\pi_2(x^{-1})T]^* = [T\pi_1(x^{-1})]^* = \pi_1(x)T^*$$

Hence $T^* \in C(\pi_2, \pi_1)$, and it follows that $T^*T \in C(\pi_1)$ and $TT^* \in C(\pi_2)$. The irreducibility of π_1 and π_2, along with (a), imply $T^*T = \alpha I_{\mathcal{H}_1}$ and $TT^* = \beta I_{\mathcal{H}_2}$. Note that for a unit vector $\xi \in \mathcal{H}_1$ we have

$$0 \leq \|T\xi\|^2 = \langle T^*T\xi, \xi\rangle = \langle \alpha\xi, \xi\rangle = \alpha$$

and similarly $\beta \geq 0$. Also $\alpha = \|T^*T\| = \|TT^*\| = \beta$. So if $\alpha \neq 0$ then $U := \alpha^{-1/2}T$ is a unitary operator, and hence if $C(\pi_1, \pi_2) \neq \{0\}$ then π_1 and π_2 are unitarily equivalent. Suppose this is the case and let $S \in C(\pi_1, \pi_2)$. Then $U^*S \in C(\pi_1)$ so that $U^*S = \gamma 1_{\mathcal{H}_1}$, and consequently $S = UU^*S = \gamma U$. Therefore $\dim C(\pi_1, \pi_2) = 1$ $\square$

$\boxed{\text{Corollary 3.4}}$ Let G be an abelian locally compact group. If $\pi : G \to \mathcal{U}(\mathcal{H})$ is an irreducible unitary representation, then $\dim \mathcal{H} = 1$ so that $\pi : G \to \mathbb{T}$.

Proof Since G is abelian, $\pi(G) \subseteq C(\pi) = \mathbb{C}I_{\mathcal{H}}$. But then every subspace of $\mathcal{H}$ is invariant for π, so to prevent non-trivial invariant subspaces we must have $\dim \mathcal{H} = 1$. $\square$

The other standard way to build new representations out of old ones is via direct sums. For a family $\{\mathcal{H}_i : i\in I\}$ of Hilbert spaces, recall that their direct sum is given by

$$\bigoplus_{i\in I} \mathcal{H}_i := \{ \xi : I \to \bigsqcup_{i\in I} \mathcal{H}_i \ : \ \xi(i)\in\mathcal{H}_i \text{ and } \sum_{i\in I} \|\xi(i)\|^2 < \infty \}$$

with inner product

$$\langle \xi, \eta \rangle = \sum_{i\in I} \langle \xi(i), \eta(i) \rangle$$

For finite $I = \{1,\ldots,n\}$, note that $\bigoplus_{i\in I} \mathcal{H}_i$ can be identified with $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$.

$\boxed{\text{Def}}$ Let $\{\pi_i : i\in I\}$ be a family of unitary representations $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$. The direct sum representation

$$\bigoplus_{i\in I} \pi_i : G \to \mathcal{U}(\bigoplus_{i\in I} \mathcal{H}_i)$$

is defined by

$$[(\bigoplus_{i\in I} \pi_i)(x)\xi](j) = \pi_j(x)\xi(j)$$

for all $x\in G$ and $j\in I$. □

$\boxed{\text{Exercise}}$ (a) Show $\bigoplus_i \pi_i$ is a continuous unitary representation of G.
(b) For each $i\in I$ define $\iota_i : \mathcal{H}_i \to \bigoplus_i \mathcal{H}_i$ by

$$[\iota_i(\xi)](j) = \delta_{i=j}\,\xi.$$

Show that $K_j := \iota_j(\mathcal{H}_j)$ is invariant for $\bigoplus_i \pi_i$ and that π_j is subequivalent to $\bigoplus_i \pi_i$. □

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The following gives a converse to part b of the previous exercise.

$\boxed{\text{Proposition 3.5}}$ Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. If $K \leq \mathcal{H}$ is invariant for π, then π is unitarily equivalent to $\pi^K \oplus \pi^{K^\perp}$.

Proof: Recall that $K^\perp$ is also invariant for π by Lemma 3.1. Define

$$\begin{aligned} U : K\oplus K^\perp &\longrightarrow \mathcal{H} \\ (\xi,\eta) &\longmapsto \xi+\eta. \end{aligned}$$

Note that $U^*\xi = (p\xi, (1-p)\xi)$ where $p\in B(\mathcal{H})$ is the projection onto K. Then

$$\begin{aligned} U(\pi^K\oplus\pi^{K^\perp})(x)U^*(\xi) &= U(\pi^K\oplus\pi^{K^\perp})(x)(p\xi, (1-p)\xi) \\ &= U(\pi(x)p\xi, \pi(x)(1-p)\xi) = \pi(x)\xi \end{aligned}$$

So $U\in C(\pi^K\oplus\pi^{K^\perp}, \pi)$ and hence these representations are unitarily equivalent. □

More generally, we have the following, whose proof is left as an exercise.

Proposition 3.6 Let $\pi: G \to \mathcal{U}(\mathcal{H})$ be a unitary representation and let $\{K_i \leq \mathcal{H} : i \in I\}$ be a pairwise orthogonal family of closed subspaces each invariant for π. Then

$$K := \overline{\operatorname{span}}\Big(\bigcup_{i\in I} K_i\Big)$$

is invariant for π and π^K is unitarily equivalent to $\bigoplus_i \pi^{K_i}$.

Def Let $\pi: G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. For $\xi \in \mathcal{H}$ we call

$$\pi(G)\xi = \{\pi(x)\xi : x \in G\}$$

its <u>orbit</u> under G. We say ξ is <u>cyclic</u> if $\operatorname{span}(\pi(G)\xi)$ is dense in $\mathcal{H}$, and in this case we say π is a <u>cyclic representation</u>. □

If G is discrete then the left regular representation λ on $L^2(G,\#) = \ell^2(G)$ is cyclic where any $\delta_x \in \ell^2(G)$ is a cyclic vector. Similarly for the right regular representation. This can fail for non-discrete groups, but one still has the following:

Proposition 3.7 Every unitary representation is a direct sum of cyclic representations.

Proof Let $\pi: G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Fix $\xi_0 \in \mathcal{H}\setminus\{0\}$ and observe that

$$K := \overline{\operatorname{span}}(\pi(G)\xi_0)$$

is invariant under π. If $K = \mathcal{H}$, we are done. Otherwise, using Zorn's Lemma we can extend ξ_0 to a maximal family $\{\xi_i \in \mathcal{H}\setminus\{0\} : i \in I\}$ such that $K_i := \overline{\operatorname{span}}(\pi(G)\xi_i)$ are pairwise orthogonal. Then for

$$K := \overline{\operatorname{span}}\Big(\bigcup_{i\in I} K_i\Big),$$

we have that π^K is unitarily equivalent to $\bigoplus_i \pi^{K_i}$ by Proposition 3.6, and each π^{K_i} is cyclic by construction. It therefore suffices to show $K = \mathcal{H}$ so that $\pi^K = \pi$. Suppose, towards a contradiction that is not the case. Then there exists non-zero $\eta \in K^\perp$. Note that $K^\perp$ is invariant under π by Lemma 3.1 and hence

$$L := \overline{\operatorname{span}}(\pi(G)\eta) \leq K^\perp.$$

But then $\{\xi_i : i \in I\} \cup \{\eta\}$ contradicts the maximality of $\{\xi_i : i \in I\}$. Hence $K = \mathcal{H}$. □

Observe that if $|I| > 1$ in the above proof, then π is <u>not</u> irreducible. It follows that every irreducible representation is cyclic. In fact, <u>every</u> non-zero vector is a cyclic vector in this case.

3.2 Group and Group Algebra Representations

Recall that $L^1(G,\mu)$ forms a Banach $*$-algebra under convolution and the involution $f^*(x) := \Delta(x)^{-1}\overline{f(x^{-1})}$, where μ is a left Haar measure and Δ is the modular function.

Def Let A be a Banach $*$-algebra and let $\mathcal{H}$ be a Hilbert space. A <u>$*$-representation</u> of A on $\mathcal{H}$ is a $*$-homomorphism $\pi : A \to B(\mathcal{H})$ satisfying

$$\|\pi(a)\| \le \|a\| \qquad \forall a \in A$$

We say π is <u>non-degenerate</u> if

$$\bigcap_{a\in A} \ker(\pi(a)) = \{0\}.$$

□

If A is unital we would also demand $\pi(1) = 1$ for a $*$-representation, in which case it is automatically non-degenerate.

In this section, we show there is a one-to-one correspondence between unitary representations of G and nondegenerate $*$-homomorphisms $\pi : L^1(G,\mu) \to B(\mathcal{H})$.

Suppose $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation. For $f \in L^1(G,\mu)$ observe that for each $\xi, \eta \in \mathcal{H}$ one has

$$\left| \int_G f(x) \langle \pi(x)\xi, \eta\rangle \, d\mu(x) \right| \le \int_G |f(x)| \, |\langle \pi(x)\xi, \eta\rangle| \, d\mu(x) \le \|f\|_1 \cdot \|\xi\| \cdot \|\eta\|.$$

That is,

$$(\xi, \eta) \longmapsto \int_G f(x) \langle \pi(x)\xi, \eta\rangle \, d\mu(x)$$

is a bounded sesquilinear form on $\mathcal{H}$. By fixing ξ and applying the Reisz representation theorem for Hilbert spaces ($\mathcal{H}^* \cong \mathcal{H}$), we see that $\exists\, \xi(f) \in \mathcal{H}$ satisfying

$$\langle \xi(f), \eta\rangle = \int_G f(x) \langle \pi(x)\xi, \eta\rangle \, d\mu(x) \qquad \forall \eta \in \mathcal{H}$$

Then fixing η we can see that $\xi \mapsto \xi(f)$ is a bounded linear transformation with norm at most $\|f\|_1$. We define $\int_G f(x)\pi(x)\, d\mu(x) \in B(\mathcal{H})$ by

$$\left(\int_G f(x)\pi(x)\, d\mu(x) \right)(\xi) := \xi(f).$$

That is, $\int_G f(x)\pi(x)\, d\mu(x)$ is the unique element of $B(\mathcal{H})$ satisfying

$$\left\langle \int_G f(x)\pi(x)\, d\mu(x)\, \xi, \eta \right\rangle = \int_G f(x) \langle \pi(x)\xi, \eta\rangle \, d\mu(x) \qquad \forall \xi, \eta \in \mathcal{H}$$

We will also write

$$\pi(f) := \int_G f(x)\pi(x)\, d\mu(x)$$

$\boxed{\text{Ex}}$ For the left regular representation $\lambda : G \to \mathcal{U}(L^2(G,\mu))$ and $f \in L^1(G,\mu)$ we have

$$\lambda(f)\, g = f * g \qquad \forall g \in L^2(G,\mu).$$

Indeed, for $g, h \in L^2(G,\mu)$ we have

$$\begin{aligned}
\langle \pi(f) g, h\rangle &= \int_G f(x) \langle \lambda(x) g, h\rangle \, d\mu(x) \\
&= \int_G f(x) \int_G g(x^{-1}y)\, \overline{h(y)}\, d\mu(y)\, d\mu(x) \\
&= \int_G \int_G f(x) g(x^{-1}y)\, d\mu(x)\, \overline{h(y)}\, d\mu(y) = \langle f * g, h\rangle
\end{aligned}$$

□

$\boxed{\text{Exercise}}$ For the right regular representation ρ, compute $\rho(f)$ in terms of convolution. □

$\boxed{\text{Theorem 3.8}}$ Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then

$$L^1(G,\mu) \ni f \mapsto \pi(f) \in B(\mathcal{H})$$

is a nondegenerate $*$-representation. Moreover, for $x \in G$ and $f \in L^1(G,\mu)$ we have

$$\pi(x)\pi(f) = \pi(L_x f) \qquad \text{and} \qquad \pi(f)\pi(x) = \Delta(x)^{-1} \pi(R_{x^{-1}} f)$$

Proof Linearity of $f \mapsto \pi(f)$ follows immediately from the definition of $\pi(f)$, and we have already seen $\|\pi(f)\| \le \|f\|_1$. Fix $f, g \in L^1(G,\mu)$ and $\xi, \eta \in \mathcal{H}$. Then

$$\begin{aligned}
\langle \pi(f*g)\xi, \eta\rangle &= \int_G f*g(x) \langle \pi(x)\xi, \eta\rangle \, d\mu(x) \\
&= \int_G \int_G f(y) g(y^{-1}x)\, d\mu(y) \langle \pi(x)\xi, \eta\rangle \, d\mu(x) \\
&= \int_G f(y) \int_G g(y^{-1}x) \langle \pi(x)\xi, \eta\rangle \, d\mu(x)\, d\mu(y) \\
&= \int_G f(y) \int_G g(x) \langle \pi(yx)\xi, \eta\rangle \, d\mu(x)\, d\mu(y) \\
&= \int_G f(y) \langle \pi(g)\xi, \pi(y)^*\eta\rangle \, d\mu(y) \\
&= \int_G f(y) \langle \pi(y)\pi(g)\xi, \eta\rangle \, d\mu(y) = \langle \pi(f)\pi(g)\xi, \eta\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle \pi(f^*)\xi, \eta\rangle &= \int_G \Delta(x)^{-1} \overline{f(x^{-1})} \langle \pi(x)\xi, \eta\rangle \, d\mu(x) \\
&= \int_G \Delta(x)^{-1} \overline{f(x^{-1})} \langle \xi, \pi(x^{-1})\eta\rangle \, d\mu(x)
\end{aligned}$$

$$= \int_G \overline{f(x)} \langle \xi, \pi(x)\eta \rangle \, d\mu(x)$$

$$= \overline{\int_G f(x) \langle \pi(x)\eta, \xi \rangle \, d\mu(x)}$$

$$= \overline{\langle \pi(f)\eta, \xi \rangle}$$

$$= \langle \xi, \pi(f)\eta \rangle \quad = \langle \pi(f)^* \xi, \eta \rangle.$$

Hence $f \mapsto \pi(f)$ is a $*$-representation. Next, for fixed $x \in G$ we have

$$\langle \pi(x)\pi(f)\xi, \eta \rangle = \int_G f(y) \langle \pi(x)\pi(y)\xi, \eta \rangle \, d\mu(y)$$

$$= \int_G f(y) \langle \pi(xy)\xi, \eta \rangle \, d\mu(y)$$

$$= \int_G (L_x f)(y) \langle \pi(y)\xi, \eta \rangle \, d\mu(y) = \langle \pi(L_x f)\xi, \eta \rangle$$

and

$$\langle \pi(f)\pi(x)\xi, \eta \rangle = \int_G f(y) \langle \pi(y)\pi(x)\xi, \eta \rangle \, d\mu(y)$$

$$= \int_G f(y) \langle \pi(yx)\xi, \eta \rangle \, d\mu(y)$$

$$= \int_G \Delta(x)^{-1} (R_{x^{-1}} f)(y) \langle \pi(y)\xi, \eta \rangle \, d\mu(y) = \langle \pi(\Delta(x)^{-1} R_x f)\xi, \eta \rangle.$$

Finally, fix $\xi \in \mathcal{H} \setminus \{0\}$. Let U be a compact neighborhood of $1 \in G$ such that $\|\pi(x)\xi - \xi\| < \|\xi\|$ for all $x \in U$. For $f := \frac{1}{\mu(U)} \cdot 1_U$ we have

$$\|\pi(f)\xi - \xi\| \leq \frac{1}{\mu(U)} \int_U \|\pi(x)\xi - \xi\| \, d\mu(x) < \|\xi\|.$$

Thus $\|\pi(f)\xi\| \geq \|\xi\| - \|\pi(f)\xi - \xi\| > 0$. Since $\xi \in \mathcal{H} \setminus \{0\}$ was arbitrary, we see that $f \mapsto \pi(f)$ is non-degenerate. □

Next we show all non-degenerate $*$-representations of $L^1(G,\mu)$ are derived from a unitary representation of G. The idea is that for an approximate unit $(\psi_u)_{u \in \mathcal{U}} \subset L^1(G,\mu)$, $\pi(\psi_u)$ should converge to $\pi(e) = 1$ and more generally $\pi(L_x \psi_u)$ to $\pi(x)$.

Theorem 3.9 Suppose $\pi : L^1(G,\mu) \longrightarrow B(\mathcal{H})$ is a non-degenerate $*$-representation. Then there exists a unique unitary representation $\pi_0 : G \to \mathcal{U}(\mathcal{H})$ such that

$$\pi(f) = \int_G f(x) \pi_0(x) \, d\mu(x) \qquad \forall f \in L^1(G,\mu)$$

Proof Define

$$D := \operatorname{span}\{\pi(f)\xi : f \in L^1(G,\mu) \text{ and } \xi \in \mathcal{H}\}$$

If $\eta \in D^{\perp}$ then for all $f \in L^1(G,\mu)$

$$\|\pi(f)\eta\|^2 = \langle \eta, \overbrace{\pi(f * f^*)\eta}^{\in D} \rangle = 0.$$

So $\eta = 0$ since π is non-degenerate, and hence D is dense in $\mathcal{H}$. Let $(\psi_u)_{u\in\mathcal{U}} \subset L^1(G,\mu)$ be an approximate unit. Then for $\pi(f)\xi \in D$ we have

$$\pi(L_x \psi_u)\pi(f)\xi = \pi(L_x \psi_u * f)\xi \longrightarrow \pi(L_x f)\xi$$

since $\|\pi(L_x(\psi_u * f)) - \pi(L_x f)\| \leq \|L_x(\psi_u * f - f)\|_1 = \|\psi_u * f - f\|_1 \longrightarrow 0$. So $(\pi(L_x\psi_u))_{u\in\mathcal{U}}$ converges strongly on D to a linear operator

$$\pi_0(x)\left(\sum_{i=1}^n \pi(f_i)\xi_i\right) := \lim_{u\to\infty} \pi(L_x\psi_u) \sum_{i=1}^n \pi(f_i)\xi_i = \sum_{i=1}^n \pi(L_x f_i)\xi_i$$

The above also implies that for all $\xi \in D$ we has

$$\|\pi_0(x)\xi\| \leq \limsup_{u\to\infty} \|\pi(L_x\psi_u)\|\,\|\xi\| \leq \limsup_{u\to\infty} \|L_x\psi_u\|_1 \|\xi\| = \|\xi\|.$$

So $\pi_0(x)$ is bounded on D and hence extends uniquely to an operator $\mathcal{H}$ with norm at most 1, which we continue to denote by $\pi_0(x)$.

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We claim that π_0 is a unitary representation of G. First, $\pi_0(1) = 1$ on D and hence on $\mathcal{H}$. Next, for $x,y \in G$ we have

$$\pi_0(x)\pi_0(y)\pi(f)\xi = \pi_0(x)\pi(L_y f)\xi = \pi(L_x L_y f)\xi = \pi(L_{xy} f)\xi = \pi_0(xy)\pi(f)\xi$$

and hence $\pi_0(x)\pi_0(y) = \pi_0(xy)$ on $\mathcal{H}$. So π_0 is a homomorphism from G into the invertible operators on $\mathcal{H}$. Further, for each $\xi \in \mathcal{H}$ we have

$$\|\xi\| = \|\pi_0(1)\xi\| = \|\pi_0(x^{-1})\pi_0(x)\xi\| \leq \|\pi_0(x)\xi\| \leq \|\xi\|$$

So $\pi_0(x)$ is an invertible isometry and hence a unitary. For the claim it remains to show π_0 is continuous. Suppose $(x_i)_{i\in I} \subset G$ converges to $x \in G$. Then $\|L_{x_i} f - L_x f\|_1 \to 0$ for all $f \in L^1(G,\mu)$, and hence $\pi_0(x_i) \to \pi_0(x)$ strongly on D. Since D is dense and the net $(\pi_0(x_i))_{i\in I}$ is uniformly bounded, we also have strong convergence on $\mathcal{H}$. This completes the claim.

We next show

$$\pi(f) = \int_G f(x)\pi_0(x)\,d\mu(x) =: \pi_0(f) \qquad \forall f \in L^1(G,\mu).$$

First consider $f \in C_c(G)$ and fix $g \in L^1(G,\mu)$ and $\xi,\eta \in \mathcal{H}$. In the proof of Theorem 2.30, we saw that $f * g$ could be approximated in L^1-norm by sums of the form

$$R(x) = \sum_{i=1}^n f(y_i)\mu(E_i)(L_{y_i} g)(x),$$

where $E_1 \sqcup \cdots \sqcup E_n = \operatorname{supp} f$. Hence $\langle \pi(f*g)\xi, \eta\rangle$ can be approximated by

$$\langle \pi(R)\xi,\eta\rangle = \sum_{i=1}^{n} f(y_i)\mu(E_i)\langle \pi(L_{y_i}g)\xi,\eta\rangle = \sum_{i=1}^{n} f(y_i)\mu(E_i)\langle \pi_0(y_i)\pi(g)\xi,\eta\rangle$$

The continuity of $y \mapsto f(y)\langle \pi_0(y)\pi(g)\xi,\eta\rangle$ implies the above also approximates

$$\sum_{i=1}^{n}\int_{E_i} f(y)\langle \pi_0(y)\pi(g)\xi,\eta\rangle\, d\mu(y) = \int_G f(y)\langle \pi_0(y)\pi(g)\xi,\eta\rangle\, d\mu(y) = \langle \pi_0(f)\pi(g)\xi,\eta\rangle$$

Since $\eta\in\mathcal{H}$ was arbitrary, the above implies $\pi(f)\pi(g)\xi = \pi(f*g)\xi = \pi_0(f)\pi(g)\xi$. Thus $\pi(f)=\pi_0(f)$ on $\mathcal{D}$ and hence all of $\mathcal{H}$. Finally, for $f\in L^1(G,\mu)$ we find $(f_n)_{n\in\mathbb{N}}\subset C_c(G)$ with $\|f-f_n\|_1\to 0$ so that

$$\|\pi(f)-\pi_0(f)\| = \|\pi(f)-\pi(f_n)+\pi_0(f_n)-\pi_0(f)\| \le \|\pi(f)-\pi(f_n)\| + \|\pi_0(f_n)-\pi_0(f)\| \le 2\|f-f_n\|_1 \to 0$$

by the first case above.

Finally, suppose $\pi_1: G\to\mathcal{U}(\mathcal{H})$ is another unitary representation satisfying $\pi(f)=\pi_1(f)$ for all $f\in L^1(G,\mu)$. For $\xi,\eta\in\mathcal{H}$ let $U=\{x\in G : |\langle(\pi_0-\pi_1)(x)\xi,\eta\rangle|>0\}$, which is an open set. If $U\neq\emptyset$ then $\int_U f\,d\mu>0$ for some $f\in C_c^+(G)$. Define

$$g := f\cdot \overline{\operatorname{sgn}(\langle(\pi_0-\pi_1)(x)\xi,\eta\rangle)} \in L^1(G,\mu)$$

Then

$$\begin{aligned} 0 &< \int_U f(x)\,|\langle(\pi_0-\pi_1)(x)\xi,\eta\rangle|\, d\mu(x) \\ &= \int_G g(x)\langle \pi_0(x)\xi,\eta\rangle\, d\mu(x) - \int_G g\,\langle \pi_1(x)\xi,\eta\rangle\, d\mu(x) \\ &= \langle \pi_0(g)\xi,\eta\rangle - \langle \pi_1(g)\xi,\eta\rangle = 0, \end{aligned}$$

a contradiction. So we must have $U=\emptyset$ and hence $\langle \pi_0(x)\xi,\eta\rangle = \langle \pi_1(x)\xi,\eta\rangle$. Since $\xi,\eta\in\mathcal{H}$ were arbitrary, this gives $\pi_0(x)=\pi_1(x)$. $\square$

If $\pi: G\to\mathcal{U}(\mathcal{H})$ is a unitary representation of a discrete group, then $\pi(G)\subset\pi(L^1(G,\mu))$ since $\pi(x)=\pi(\delta_x)$ and

$$\overline{\operatorname{span}\,\pi(G)} = \overline{\pi(L^1(G,\mu))} =: C^*(\pi)$$

where the closure is with respect to the operator norm. But in general one can have $\pi(G)\cap\pi(L^1(G,\mu))=\emptyset$ and failure of the above equality. If one instead works with von Neumann algebras then the situation is much nicer. Recall that for a unital $*$-subalgebra $M\subset B(\mathcal{H})$, von Neumann's bicommutant theorem says

$$\overline{M}^{SOT} = \overline{M}^{WOT} = M''$$

strong operator topology closure — weak operator topology closure — double commutant

Moreover, we say M is a von Neumann algebra if and only if it equals one (hence all) of the above sets.

Exercise For a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, show $\operatorname{span}(\pi(G))$ is a unital $*$-algebra. □

Theorem 3.10 Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a locally compact group.

(a) $\pi(G)'' = \pi(L^1(G,\mu))''$.

(b) $C(\pi) = \pi(L^1(G,\mu))'$.

(c) $K \leq \mathcal{H}$ is invariant for π if and only if $\pi(f)K \subset K$ for all $f \in L^1(G,\mu)$.

Proof (a): First let $g \in C_c(G)$ and $\varepsilon > 0$. Given $\xi_1, \ldots, \xi_n \in \mathcal{H}$, let

$$\operatorname{supp} g = E_1 \sqcup \cdots \sqcup E_m$$

be a measurable partition such that whenever $x, y \in E_j$ for some $j = 1, \ldots, m$ one has

$$\max_{1 \leq i \leq n} \| (g(x)\pi(x) - g(y)\pi(y))\xi_i \| < \frac{\varepsilon}{\mu(\operatorname{supp} g)}$$

Fixing a $x_j \in E_j$ for each $j = 1, \ldots, m$, it follows that each $i = 1, \ldots, n$ one has

$$\| \pi(g)\xi_i - \sum_{j=1}^{m} \mu(E_j) g(x_j)\pi(x_j)\xi_i \| \leq \sum_{j=1}^{m} \int_{E_j} \| (g(x)\pi(x) - g(x_j)\pi(x_j))\xi_i \| \, d\mu(x)$$

$$< \sum_{j=1}^{m} \int_{E_j} \frac{\varepsilon}{\mu(\operatorname{supp} g)} \, d\mu(x) = \varepsilon.$$

Since $\sum_{j=1}^{m} \mu(E_j) g(x_j)\pi(x_j) \in \operatorname{span} \pi(G)$, it follows that $\pi(g)$ lies in the strong operator topology closure of $\operatorname{span} \pi(G)$, which is $\pi(G)''$ by the bicommutant theorem. Thus

$$\pi(C_c(G)) \subset \pi(G)'',$$

and since the latter is closed under norms we get $\pi(L^1(G,\mu)) \subset \pi(G)''$ and hence $\pi(L^1(G,\mu))'' \subset \pi(G)''$.

Conversely, let $(\varphi_u)_{u \in \mathcal{U}} \subset L^1(G,\mu)$ be an approximate unit. We saw in the proof of Theorem 3.9 that

$$\pi(x) = \lim_{u \to \infty} \pi(\lambda_x \varphi_u),$$

where the limit is with respect to the strong operator topology. Thus $\pi(G) \subset \pi(L^1(G,\mu))''$ and so $\pi(G)'' = \pi(L^1(G,\mu))''$.

(b): We have $C(\pi) = \pi(G)' = (\pi(G)'')' = (\pi(L^1(G,\mu))'')' = \pi(L^1(G,\mu))'$.

(c): Proposition 3.2 says K is invariant for π if and only if the projection $p \in B(\mathcal{H})$ onto K lies in $C(\pi)$. By (b), this is further equivalent to $p \in \pi(L^1(G,\mu))'$. The analogue of Proposition 3.2 for the $*$-representation of $L^1(G,\mu)$ completes the proof. □

3.3 Functions of Positive Type

Recall that $L^\infty(X,\mu)$ depends only on the measure class of μ. That is, if ν is equivalent to μ, then $L^\infty(X,\mu) = L^\infty(X,\nu)$. Since all left and right Haar measures on G have the same measure class (see Theorem 2.15 and Proposition 2.23), we will write $L^\infty(G)$ for their common L^∞-space.

Def Let G be a locally compact group. A function $\phi \in L^\infty(G)$ is of <u>positive type</u> if

$$\int_G (f^* * f)\,\phi\, d\mu \geq 0 \qquad \forall f \in L^1(G,\mu),$$

where μ is any left Haar measure on G. We denote by $P(G)$ the continuous functions of positive type. □

That is, integration against $\phi \in P(G)$ defines a positive linear functional on the Banach $*$-algebra $L^1(G,\mu)$. Our first goal in this section is to establish a correspondence between elements of $P(G)$ and cyclic unitary representations of G. This correspondence in conjunction with Proposition 3.7 says that the theory of unitary representations can be reduced to the theory of functions of positive type, which has the advantage of being internal to G. We will also show every $\phi \in L^\infty(G)$ of positive type equals a continuous function almost everywhere, so that our definition $P(G)$ does not exclude any almost everywhere equivalence classes.

Let us first make a general observation. For $f \in L^1(G,\mu)$ and $\phi \in L^\infty(G)$ we have

$$\begin{aligned}
\int_G (f^* * f)(x)\,\phi(x)\,d\mu(x) &= \int_G \int_G \Delta(y)^{-1}\, \overline{f(y^{-1})}\, f(y^{-1}x)\, d\mu(y)\, \phi(x)\, d\mu(x) \\
&= \int_G \int_G \overline{f(y)}\, f(yx)\, d\mu(y)\, \phi(x)\, d\mu(x) \\
&= \int_G \int_G \overline{f(y)}\, f(yx)\, \phi(x)\, d\mu(x)\, d\mu(y) \\
&= \int_G \int_G \overline{f(y)}\, f(x)\, \phi(y^{-1}x)\, d\mu(x)\, d\mu(y)
\end{aligned}$$

Thus $\phi \in L^\infty(G)$ is of positive type if and only if

$$\int_G \int_G f(x)\,\overline{f(y)}\,\phi(y^{-1}x)\,d\mu(x)\,d\mu(y) \geq 0 \qquad \forall f \in L^1(G,\mu). \qquad (*)$$

By taking the complex conjugate of $(*)$ we obtain:

Proposition 3.11 If $\phi \in L^\infty(G)$ is of positive type, then so is $\overline{\phi}$.

Proposition 3.12 Let $\pi: G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. For $\xi \in \mathcal{H}$ define
$$\phi(x) := \langle \pi(x)\xi, \xi \rangle$$

Then $\phi \in \mathcal{P}(G)$.

Proof Continuity follows from the strong (and hence weak) operator topology continuity of π, and $\|\phi\|_\infty = \|\xi\|^2$ so that $\phi \in L^\infty(G)$. Next, for $f \in L^1(G,\mu)$ we observe
$$\int_G \int_G f(x)\overline{f(y)}\,\phi(y^{-1}x)\,d\mu(x)d\mu(y) = \int_G \overline{f(y)} \int_G f(x) \langle \pi(x)\xi, \pi(y)\xi \rangle\, d\mu(x)\, d\mu(y)$$
$$= \int_G \overline{f(y)} \langle \pi(f)\xi, \pi(y)\xi \rangle d\mu(y)$$
$$= \|\pi(f)\xi\|^2 \geq 0.$$

Thus ϕ is of positive type by (*) $\square$

Corollary 3.13 For $f \in L^2(G,\mu)$, denote $\tilde{f}(x) := \overline{f(x^{-1})}$. Then $f * \tilde{f} \in \mathcal{P}(G)$.

Proof For the left regular representation $\lambda: G \to \mathcal{U}(L^2(G,\mu))$ we have
$$\langle \lambda(x)f, f\rangle = \int_G f(x^{-1}y)\overline{f(y)} = \overline{\int_G f(y)\tilde{f}(y^{-1}x)\,d\mu(y)} = \overline{f * \tilde{f}(x)}$$

Thus $f*\tilde{f} \in \mathcal{P}(G)$ by Propositions 3.11 and 3.12. $\square$

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We wish to ~~show~~ show every non-zero function of positive type is of the form in Proposition 3.12. For $\phi \in L^\infty(G)\setminus\{0\}$ of positive type, observe that
$$\langle f, g\rangle_\phi := \int_G (g^* * f)\,\phi\, d\mu = \int_G \int_G f(x)\overline{g(y)}\,\phi(y^{-1}x)\,d\mu(x)d\mu(y) \qquad f,g \in L^1(G,\mu)$$

is a positive semidefinite sesquilinear form satisfying:
$$|\langle f, g\rangle_\phi| \leq \|f\|_1 \cdot \|g\|_1 \, \|\phi\|_\infty. \qquad **$$

Denote $N_\phi := \{f \in L^1(G,\mu) : \langle f, f\rangle_\phi = 0\}$. The Cauchy-Schwarz inequality implies $f \in N_\phi$ if and only if $\langle f, g\rangle_\phi = 0$ for all $g \in L^1(G,\mu)$. Consequently, on the quotient space $L^1(G,\mu)/N_\phi$ we obtain an inner product defined by
$$\langle f + N_\phi, g + N_\phi \rangle_\phi := \langle f, g\rangle_\phi,$$

and we let $\mathcal{H}_\phi$ be the Hilbert space formed by the completion of $L^1(G,\mu)/N_\phi$ with respect to this inner product. We denote $\check{f} := f + N_\phi$ and note that by (**)
$$\|\check{f}\|_\phi \leq \|\phi\|_\infty^{1/2}\, \|f\|_1. \qquad ***$$

Now, for $f, g \in L^1(G,\mu)$ and $x \in G$ observe that
$$\langle L_x f, L_x g\rangle_\phi = \int_G \int_G f(x^{-1}y)\,\overline{g(x^{-1}z)}\,\phi(z^{-1}y)\,d\mu(y)d\mu(z)$$
$$= \int_G \int_G f(y)\,\overline{g(z)}\,\phi((xz)^{-1}(xy))\,d\mu(y)\,d\mu(z) = \langle f, g\rangle_\phi.$$

Thus $L_x N_\phi \subset N_\phi$ and the operators L_x induce a unitary representation $\pi_\phi : G \to \mathcal{U}(\mathcal{H}_\phi)$ determined by

$$\pi_\phi(x)\tilde{f} = \widetilde{(L_x f)} \qquad f \in L^1(G,\mu)$$

The continuity of π_ϕ follows from (***). Also, for $f, g, h \in L^1(G,\mu)$ we have

$$\int_G f(x) \langle \pi_\phi(x)\tilde{g}, \tilde{h} \rangle_\phi \, d\mu(x) = \int_G f(x) \int_G \int_G g(x^{-1}y) \overline{h(z)} \phi(z^{-1}y) \, d\mu(y) d\mu(z) d\mu(x)$$

$$= \int_G (f * g)(y) \overline{h(z)} \phi(z^{-1}y) \, d\mu(y) d\mu(z) = \langle \widetilde{f*g}, \tilde{h} \rangle_\phi$$

Thus the non-degenerate $*$-representation of $L^1(G,\mu)$ corresponding to π_ϕ is determined by

$$\pi_\phi(f)\tilde{g} = \widetilde{(f*g)} \qquad g \in L^1(G,\mu).$$

Theorem 3.14 For $\phi \in L^\infty(G)$ of positive type, adopt the above notation. Then there exists a cyclic vector $e \in \mathcal{H}_\phi$ for π_ϕ satisfying $\pi_\phi(f) e = \tilde{f}$ for all $f \in L^1(G,\mu)$. If G is σ-compact (resp. if $\phi \in P(G)$) then one also has

$$\phi(x) = \langle \pi_\phi(x) e, e \rangle$$

almost everywhere (resp. for all $x \in G$).

Proof Let $(\psi_U)_{U \in \mathcal{U}} \subset L^1(G,\mu)$ be an approximate unit. Then $(\psi_U^*)_{U \in \mathcal{U}}$ is almost an approximate unit; it satisfies $\operatorname{supp} \psi_U^* \subset U$, $\psi_U^* \geq 0$, and $\int_G \psi_U^* \, d\mu = 1$ but not $\psi_U^*(x^{-1}) = \psi_U^*(x)$ (unless G is unimodular). Nevertheless, this is suffices to give

$$\lim_{U \to \infty} \|\psi_U^* * f - f\|_1 = 0 \qquad \forall f \in L^1(G,\mu)$$

(what fails is $f * \psi_U \to f$.) Consequently, for all $f \in L^1(G,\mu)$ we have

$$\langle \tilde{f}, \tilde{\psi}_U \rangle_\phi = \int_G (\psi_U^* * f)(x) \phi(x) \, d\mu(x) \longrightarrow \int_G f \phi \, d\mu,$$

Additionally, $\|\tilde{\psi}_U\|_\phi \leq \|\phi\|_\infty^{1/2}$ by (***). Consequently, $(\tilde{\psi}_U)_{U \in \mathcal{U}}$ converges weakly in $\mathcal{H}_\phi$ and we let e denote its weak limit, which satisfies

$$\langle \tilde{f}, e \rangle_\phi = \lim_{U \to \infty} \langle \tilde{f}, \tilde{\psi}_U \rangle_\phi = \int_G f \phi \, d\mu.$$

For $f \in L^1(G,\mu)$ and $y \in G$ we compute

$$\langle \tilde{f}, \pi_\phi(y) e \rangle_\phi = \langle \pi_\phi(y^{-1}) \tilde{f}, e \rangle_\phi$$

$$= \langle \widetilde{L_{y^{-1}} f}, e \rangle_\phi$$

$$= \int_G f(yx) \phi(x) \, d\mu(x) = \int_G f(x) \phi(y^{-1}x) \, d\mu(x).$$

Thus for $g \in L^1(G,\mu)$ we have

$$\langle \tilde{f}, \tilde{g} \rangle_\phi = \int_G \int_G f(x) \overline{g(y)} \phi(y^{-1}x) d\mu(x) d\mu(y)$$

$$= \int_G \langle \tilde{f}, \pi_\phi(y) e \rangle_\phi \overline{g(y)} d\mu(y) = \langle \tilde{f}, \pi_\phi(g) e \rangle_\phi$$

Therefore $\pi_\phi(g) e = \tilde{g}$ for all $g \in L^1(G,\mu)$, as claimed. Also note that if $\langle \tilde{f}, \pi_\phi(y) e \rangle_\phi = 0$ for all $y \in G$, then the above implies $\langle \tilde{f}, \tilde{g} \rangle_\phi = 0$ for all $g \in L^1(G,\mu)$. Thus $(\pi(G)e)^\perp = \{0\}$ so that

$$\overline{\text{span}}\, \pi(G) e = ((\pi(G)e)^\perp)^\perp = (\{0\})^\perp = \mathcal{H}_\phi.$$

That is, e is cyclic for π_ϕ.

Finally, for $f \in L^1(G,\mu)$ we have

$$\int_G f(x) \langle \pi_\phi(x) e, e \rangle_\phi d\mu(x) = \langle \pi_\phi(f) e, e \rangle_\phi = \langle \tilde{f}, e \rangle_\phi = \int_G f \phi \, d\mu$$

So $\langle \pi_\phi(x) e, e \rangle$ and $\phi(x)$ induce the same element of $L^1(G,\mu)^*$. Thus when G is σ-compact (so that μ is σ-finite), the isometric isomorphism $L^1(G,\mu)^* \cong L^\infty(G)$ gives that $\langle \pi_\phi(x) e, e \rangle = \phi(x)$ μ-almost everywhere. Alternatively, if $\phi \in \mathcal{P}(G)$ then continuity of ϕ and $\langle \pi_\phi(x) e, e \rangle$ implies each can be approximated by the above integrals for $f = 1_U$ with U a compact neighborhood of x. Thus $\phi(x) = \langle \pi_\phi(x) e, e \rangle_\phi$ for all $x \in G$. □

Remark Without the σ-compactness assumption on G, one may not have $L^1(G,\mu)^* \cong L^\infty(G)$, but one can still say $\langle \pi_\phi(x) e, e \rangle = \phi(x)$ locally almost everywhere. This means

$$E := \{x \in G : \langle \pi_\phi(x) e, e \rangle \neq \phi(x)\}$$

satisfies that whenever $F \in \mathcal{B}_0(G)$ has $\mu(F) < \infty$ then $E \cap F \in \mathcal{B}(G)$ with $\mu(E \cap F) = 0$. □

Corollary 3.15 For σ-compact G, every $\phi \in L^\infty(G,\mu)$ of positive type agrees almost everywhere with an element of $\mathcal{P}(G)$

Corollary 3.16 Each $\phi \in \mathcal{P}(G)$ satisfies $\|\phi\|_\infty = \phi(1)$ and $\phi(x^{-1}) = \overline{\phi(x)}$.

Proof We have $\phi(x) = \langle \pi_\phi(x) e, e \rangle_\phi$ for all $x \in G$, where e is the cyclic vector from Theorem 3.14. Thus

$$\|\phi\|_\infty = \sup_{x \in G} |\phi(x)| \leq \|e\|^2 = \phi(1) \leq \|\phi\|_\infty.$$

Also $\phi(x^{-1}) = \langle \pi_\phi(x^{-1}) e, e \rangle_\phi = \langle e, \pi_\phi(x) e \rangle_\phi = \overline{\phi(x)}$. □

For $\phi \in \mathcal{P}(G)$ the following proposition will imply as an immediate corollary that π_ϕ is the unique cyclic representation giving ϕ as in Proposition 3.12. (The representation π in Proposition 3.12 was not assumed to be cyclic, but $\langle \pi(x)\xi,\xi\rangle$ depends only only the cyclic subrepresentation on $\overline{\text{span}}\, \pi(G)\xi$.) This will therefore complete the one-to-one correspondence between $\mathcal{P}(G)$ and cyclic representations.

$\boxed{\textbf{Proposition 3.17}}$ Suppose π and ρ are cyclic representations of G with cyclic vectors ξ and η, respectively. If

$$\langle \pi(x)\xi,\xi\rangle = \langle \rho(x)\eta,\eta\rangle$$

for all $x\in G$, then there exists a unitary $U\in C(\pi,\rho)$ satisfying $U\xi=\eta$.

Proof For $x,y\in G$ observe

$$\langle \pi(x)\xi,\pi(y)\xi\rangle = \langle \pi(y^{-1}x)\xi,\xi\rangle = \langle \rho(y^{-1}x)\eta,\eta\rangle = \langle \rho(x)\eta,\rho(y)\eta\rangle$$

Thus $U(\sum_{i=1}^{n}\pi(x_i)\xi) = \sum_{i=1}^{n}\rho(x_i)\eta$ is a well-defined linear map that extends to an isometry on $\overline{\text{span}}\,\pi(G)\xi$. Its image is $\overline{\text{span}}\,\rho(G)\xi$, hence it is unitary. Moreover

$$\rho(y)\,U\pi(x)\xi = \rho(y)\rho(x)\xi = \rho(yx)\xi = U\pi(yx)\xi = U\pi(y)\pi(x)\xi$$

implies $\rho(y)U = U\pi(y)$ so that $U\in C(\pi,\rho)$. $\square$

$\boxed{\textbf{Corollary 3.18}}$ Let π be a cyclic representation with cyclic vector ξ. For

$$\phi(x) := \langle \pi(x)\xi,\xi\rangle \in \mathcal{P}(G),$$

π is unitarily equivalent to π_ϕ.

Our last goal in this section (and chapter) is to prove that there are sufficiently many irreducible representations for G. More precisely, we will show the irreducible representations separate points: for $x,y\in G$ distinct there exists an irreducible representation π satisfying $\pi(x)\neq\pi(y)$. We will need to collect a few more results along the way.

Let us denote

$$P_1(G) := \{\phi\in\mathcal{P}(G) : \|\phi\|_\infty = 1\} = \{\phi\in\mathcal{P}(G) : \phi(1)=1\}$$
$$P_0(G) := \{\phi\in\mathcal{P}(G) : \|\phi\|_\infty \leq 1\} = \{\phi\in\mathcal{P}(G) : \phi(1)\leq 1\},$$

where the equalities follow from Corollary 3.16. Note that both sets are bounded convex subsets of $L^\infty(G)$, and we will denote their extreme points by $\text{ext}(P_j(G))$ for $j=0,1$.

$\boxed{\textbf{Theorem 3.19}}$ For $\phi\in P_1(G)$, one has $\phi\in\text{ext}(P_1(G))$ if and only if π_ϕ is irreducible.

Proof $(\Rightarrow)$ We proceed by contrapositive and suppose π_ϕ is reducible, say with $\mathcal{K}\leq\mathcal{H}_\phi$ a non-trivial invariant subspace. Since $e\in\mathcal{H}_\phi$ is a cyclic vector, we cannot have $e\in\mathcal{K}$ or $e\in\mathcal{K}^\perp$, and hence $Pe\neq 0$

and $(1-p)e \neq 0$ where $p \in B(\mathcal{H}_\phi)$ is the projection onto K. Define

$$\xi_1 := \frac{1}{\|pe\|} pe \qquad \text{and} \qquad \xi_2 := \frac{1}{\|(1-p)e\|}(1-p)e$$

and

$$\psi_1(x) := \langle \pi_\phi(x)\xi_1, \xi_1\rangle_\phi \qquad \text{and} \qquad \psi_2(x) := \langle \pi_\phi(x)\xi_2, \xi_2\rangle_\phi$$

Then $\psi_1(1) = \|\xi_1\|^2 = 1$ and $\psi_2(1) = \|\xi_2\|^2 = 1$ so that $\psi_1, \psi_2 \in P_1(G)$. We also have

$$\langle \pi_\phi(x)e, \|pe\|\xi_1 - \|(1-p)e\|\xi_2\rangle_\phi = \psi_1(x) - \psi_2(x)$$

Note that the above is not identically zero since $\|pe\|\xi_1 - \|(1-p)e\|\xi_2 \neq 0$ and since e is a cyclic vector for π_ϕ. Thus $\psi_1 \neq \psi_2$, but

$$\phi(x) = \langle \pi_\phi(x)e, e\rangle_\phi = \|pe\|^2 \cdot \langle \pi_\phi(x)\xi_1, \xi_1\rangle_\phi + \|(1-p)e\|^2 \langle \pi_\phi(x)\xi_2, \xi_2\rangle = \|pe\|^2\psi_1(x) + \|(1-p)e\|^2\psi_2(x)$$

and

$$\|pe\|^2 + \|(1-p)e\|^2 = \|e\|^2 = \phi(1) = 1.$$

Hence $\phi \notin \mathrm{ext}(P_1(G))$.

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($\Longleftarrow$) Assume π_ϕ is irreducible and suppose $\phi = \psi + \psi'$ for $\psi, \psi' \in P(G)$. For $f \in L^1(G,\mu)$ we have

$$\|f\|_\psi^2 = \int_G (f^* * f)\psi\, d\mu = \int_G (f^* * f)(\phi - \psi')\, d\mu \leq \int_G (f^* * f)\phi\, d\mu = \|f\|_\phi^2$$

Thus for $f, g \in L^1(G,\mu)$ we have

$$|\langle f, g\rangle_\psi| \leq \|f\|_\psi \|g\|_\psi \leq \|f\|_\phi \cdot \|g\|_\phi.$$

Hence $(\tilde{f}, \tilde{g}) \mapsto \langle f, g\rangle_\psi$ is a well-defined, bounded sesquilinear form on $\mathcal{H}_\phi$, and therefore there exists $T \in B(\mathcal{H}_\phi)$ self-adjoint with $\|T\| \leq 1$ satisfying $\langle T\tilde{f}, \tilde{g}\rangle_\phi = \langle f, g\rangle_\psi$. Then for $x \in G$ we have

$$\langle T\pi_\phi(x)\tilde{f}, \tilde{g}\rangle_\phi = \langle T(\widetilde{L_x f}), \tilde{g}\rangle_\phi = \langle L_x f, g\rangle_\psi = \langle f, L_{x^{-1}} g\rangle_\psi$$

$$= \langle T\tilde{f}, (\widetilde{L_{x^{-1}} g})\rangle_\phi = \langle T\tilde{f}, \pi_\phi(x^{-1})\tilde{g}\rangle_\phi = \langle \pi_\phi(x) T\tilde{f}, \tilde{g}\rangle_\phi.$$

Thus $T \in C(\pi_\phi)$, but by Schur's Lemma (Theorem 3.3) this implies $T = \alpha \cdot 1$ for some $\alpha \in \mathbb{C}$. Thus

$$\int_G (g^* * f)\psi\, d\mu = \langle f, g\rangle_\psi = \langle T\tilde{f}, \tilde{g}\rangle_\phi = \alpha\langle f, g\rangle_\phi = \alpha \int_G (g^* * f)\phi\, d\mu.$$

By considering $g^* * f$ with small compact support near $x \in G$, the above implies $\psi(x) = \alpha\phi(x)$. Hence $\psi' = \phi - \psi = (1-\alpha)\phi$ and therefore ϕ is extreme. □

Suppose G is σ-compact so that $L^\infty(G) \cong L^1(G,\mu)^*$. Then $P(G)$ is a weak* closed subset since the condition

$$\int_G (f^* * f)\,\phi\, d\mu \geq 0 \qquad f \in L^1(G,\mu)$$

is preserved under weak* limits. $P_0(G)$ is then weak* compact by Banach–Alaoglu because it is bounded. Furthermore, its convexity and the Krein–Milman theorem imply

$$P_0(G) = \overline{\mathrm{conv}}(\mathrm{ext}(P_0(G))) \qquad (****)$$

where the closure is with respect to the weak* topology. $P_1(G)$, on the other hand, is <u>not</u> weak* closed (unless G is discrete and hence $\phi \mapsto \phi(1)$ is a bounded linear functional on $L^\infty(G)$). Nevertheless, the analogue of $(****)$ for $P_1(G)$ still holds. To see this, we first require a lemma.

Lemma 3.20 $\mathrm{ext}(P_0(G)) = \mathrm{ext}(P_1(G)) \cup \{0\}$.

Proof Suppose $t\phi_1 + (1-t)\phi_2 = 0$ for $\phi_1, \phi_2 \in P_0(G)$. Then $t\phi_1(1) + (1-t)\phi_2(1) = 0$ implies $\|\phi_i\|_\infty = \phi_i(1) = 0$ for each $i = 1,2$. Hence $0 \in \mathrm{ext}(P_0(G))$. Next if $t\phi_1 + (1-t)\phi_2 = \phi \in \mathrm{ext}(P_1(G))$ for $\phi_1, \phi_2 \in P_0(G)$, then $t\phi_1(1) + (1-t)\phi_2(1) = 1$ implies $\phi_1(1) = \phi_2(1) = 1$. Thus $\phi_1, \phi_2 \in P_1(G)$ and therefore $\phi_1 = \phi_2 = \phi$ since ϕ is extreme in $P_1(G)$. Hence $\mathrm{ext}(P_1(G)) \subset \mathrm{ext}(P_0(G))$. To complete the proof, it suffices to show $P_0(G) \setminus (P_1(G) \cup \{0\})$ contains no extreme points. For $\phi \in P_0(G) \setminus (P_1(G) \cup \{0\})$ we have $0 < \phi(1) < 1$ and thus

$$\phi = \phi(1)\frac{1}{\phi(1)}\phi + (1-\phi(1)) \cdot 0$$

is a non-trivial convex combination. Therefore ϕ is not extreme in $P_0(G)$. □

Theorem 3.21 For a σ-compact group G, one has

$$P_1(G) = \overline{\mathrm{conv}}(\mathrm{ext}(P_1(G))),$$

where the closure is with respect to the weak* topology on $L^\infty(G)$.

Proof Fix $\phi_0 \in P_1(G)$. By $(****)$ and Lemma 3.20, ϕ_0 is a weak* limit of a net $(\phi_i)_{i \in I}$ of functions of the form

$$\phi_i = t_1\psi_1 + \cdots + t_n\psi_n + t_{n+1} \cdot 0,$$

where $\psi_1, \ldots, \psi_n \in \mathrm{ext}(P_1(G))$ and $t_1, \ldots, t_{n+1} \geq 0$ with $t_1 + \cdots + t_{n+1} = 1$. Note that the set

$$\{f \in L^\infty(G) : \|f\|_\infty \leq 1 - \varepsilon\}$$

is weak* closed for all $\varepsilon > 0$. Thus $\{f \in L^\infty(G) : \|f\|_\infty > 1 - \varepsilon\}$ is a weak* neighborhood of ϕ_0 (since $\|\phi_0\|_\infty = 1$), and consequently

$$\liminf_{i\to\infty} \phi_i(1) = \liminf_{i\to\infty} \|\phi_i\|_\infty > 1-\varepsilon$$

for all $\varepsilon>0$. Since we also have $\|\phi_i\|_\infty \le 1$, this implies $\lim_{i\to\infty}\phi_i(1)=1$. Set $\phi_i' := \frac{1}{\phi_i(1)}\phi_i$ and observe

$$1 = \phi_i'(1) = \frac{1}{\phi_i(1)}(t_1 + \cdots + t_n + 0).$$

Thus $\phi_i' \in \mathrm{conv}(\mathrm{ext}(P_1(G)))$ and $\phi_i' \to \phi_0$ weak*. □

We will next show that the weak* topology on $P_1(G)$ coincides with another natural topology:

Def Let X be a locally compact Hausdorff space and let $C_b(X)$ denote the set of bounded continuous functions $f: X\to\mathbb{C}$. We say a net $(f_i)_{i\in I}\subset C_b(X)$ converges to $f\in C_b(X)$ uniformly on compact subsets if for all $K\subset X$ compact one has

$$\lim_{i\to\infty} \sup_{x\in K} |f_i(x) - f(x)| = 0.$$

We call the associated topology on $C_b(X)$ the topology of compact convergence. □

Exercise For $f_0\in C_b(X)$, show that a neighborhood base for f_0 in the topology of compact convergence is given by

$$N(f_0;\varepsilon,K) := \{f : \sup_{x\in K} |f(x) - f_0(x)| < \varepsilon\}$$

where $\varepsilon>0$ and $K\subset X$ is compact. □

Lemma 3.22 Let X be a Banach space and let $B\subset X^*$ be a bounded subset. The weak* topology on B coincides with the topology of compact convergence on X.

Proof Let $(\phi_i)_{i\in I}\subset B$ and $\phi\in B$. If $\phi_i\to\phi$ uniformly on compact subsets, then in particular it converges pointwise on X and hence in the weak* top. Conversely, suppose $\phi_i\to\phi$ weak*. Let $\varepsilon>0$, $K\subset X$ compact, and denote $R := \sup\{\|\phi\| : \phi\in B\}$. For $\delta := \varepsilon/3R$, we can find $x_1,\ldots,x_n\in K$ such that

$$B(x_1,\delta)\cup\cdots\cup B(x_n,\delta)\supset K.$$

Let $i_0\in I$ be large enough so that for all $i\ge i_0$ one has $|(\phi_i-\phi)(x_j)| < \frac{\varepsilon}{3}$ for each $j=1,\ldots,n$. Then for $i\ge i_0$ and any $x\in K$ we have $x\in B(x_j,\delta)$ for some j and hence

$$|\phi_i(x) - \phi(x)| \le |\phi_i(x-x_j)| + |(\phi_i-\phi)(x_j)| + |\phi(x-x_j)| < R\cdot\delta + \frac{\varepsilon}{3} + R\cdot\delta = \varepsilon$$

Therefore $\phi_i\in N(\phi;\varepsilon,K)$ for all $i\ge i_0$, and the net converges in the topology of compact convergence. □

Lemma 3.23 Let G be a σ-compact group, let $\phi_0\in P_1(G)$, and let $f\in L^1(G,\mu)$. For all $\varepsilon>0$ and

$K \subset G$ compact there exists a weak* neighborhood Φ of ϕ_0 in $\mathcal{P}_1(G)$ satisfying

$$\sup_{x \in K} |f * \phi(x) - f * \phi_0(x)| < \varepsilon$$

for all $\phi \in \Phi$.

Proof Recall $\phi \in \mathcal{P}(G)$ satisfies $\phi(y^{-1}) = \overline{\phi(y)}$ by Corollary 3.16. Thus

$$f * \phi(x) = \int_G f(xy)\,\phi(y^{-1})\,d\mu(y) = \int_G (L_{x^{-1}} f)(y)\,\overline{\phi(y)} = \overline{\int_G (L_{x^{-1}}\overline{f})(y)\,\phi(y)\,d\mu(y)}$$

By Proposition 2.27, $G \ni x \mapsto L_{x^{-1}}\overline{f} \in L^1(G,\mu)$ is continuous and therefore

$$F := \{ L_{x^{-1}}\overline{f} : x \in K \}$$

is a compact subset of $L^1(G,\mu)$. Thus Lemma 3.22 implies $N(\phi_0; \varepsilon, F)$ is a weak* neighborhood of ϕ_0 and

$$\varepsilon > \sup_{g \in F} \left| \int g\,\phi\,d\mu - \int g\,\phi_0\,d\mu \right| = \sup_{x \in K} | f * \phi(x) - f * \phi_0(x)|$$

for all $\phi \in N(\phi_0; \varepsilon, F) =: \Phi$. $\square$

Lemma 3.24 For $\phi \in \mathcal{P}_1(G)$, $|\phi(x) - \phi(y)|^2 \leq 2 - 2\,\mathrm{Re}\,\phi(yx^{-1})$.

Proof We have $\phi(x) = \langle \pi_\phi(x) e, e \rangle_\phi$ by Theorem 3.14. Consequently,

$$|\phi(x) - \phi(y)|^2 = |\langle (\pi_\phi(x) - \pi_\phi(y)) e, e \rangle_\phi|^2 = |\langle e, (\pi_\phi(x^{-1}) - \pi_\phi(y^{-1})) e \rangle_\phi|^2$$

$$\leq \| (\pi_\phi(x^{-1}) - \pi_\phi(y^{-1})) e \|_\phi^2 = 2 - 2\,\mathrm{Re}\,\langle \pi_\phi(x^{-1}) e, \pi_\phi(y^{-1}) e \rangle_\phi = 2 - 2\,\mathrm{Re}\,\phi(yx^{-1}).$$

as claimed. $\square$

Theorem 3.25 Let G be a σ-compact group. On $\mathcal{P}_1(G)$, the weak* topology coincides with the topology of compact convergence on G.

Proof Let $(\phi_i)_{i \in I} \subset \mathcal{P}_1(G)$ be a net and $\phi_0 \in \mathcal{P}_1(G)$. First suppose $\phi_i \to \phi_0$ in the topology of compact convergence. Given $f \in L^1(G,\mu)$ and $\varepsilon > 0$, we can find $K \subset G$ compact such that

$$\int_{G \setminus K} |f|\,d\mu < \frac{\varepsilon}{4}.$$

Let $i_0 \in I$ be such that $\phi_i \in N(\phi_0; \varepsilon/2\|f\|_1, K)$ for all $i \geq i_0$. Then for $i \geq i_0$ we have

$$\left| \int_G f(\phi_i - \phi_0)\,d\mu \right| \leq \int_K |f|\,|\phi_i - \phi_0|\,d\mu + \int_{G \setminus K} |f|\,|\phi_i - \phi_0|\,d\mu < \|f\|_1 \cdot \frac{\varepsilon}{2\|f\|_1} + \frac{\varepsilon}{4} \cdot 2 = \varepsilon$$

Thus $\phi_i \to \phi_0$ in the weak* topology.

Conversely, suppose $\phi_i \to \phi_0$ in the weak* topology. Fix $\varepsilon > 0$ and $K \subset G$ compact. Let $\delta > 0$ to be chosen later, and let V be a compact neighborhood of $1 \in G$ such that

$$|\phi_0(x) - 1| < \delta \qquad x \in V.$$

Since $1_V \in L^1(G,\mu)$, we can find $i_1 \in I$ so that $i \geq i_1$ implies

$$\left|\int_V \phi_i - \phi_0 \, d\mu\right| = \left|\int_V 1_V(\phi_i-\phi_0)\, d\mu\right| < \delta\cdot\mu(V).$$

Observe that for $i_1 \geq i$ we also have

$$\left|\int_V 1 - \mathrm{Re}\,\phi_i\, d\mu\right| \le \left|\int_V 1-\phi_i\, d\mu\right| \le \left|\int_V 1-\phi_0\, d\mu\right| + \left|\int_V \phi_0-\phi_i\, d\mu\right| < 2\delta\mu(V),$$

Thus for $x\in G$ we have

$$|1_V * \phi_i(x) - \mu(V)\phi_i(x)| = \left|\int_V \phi_i(y^{-1}x) - \phi_i(x)\, d\mu(y)\right|$$

$$\le \int_V |\phi_i(y^{-1}x) - \phi_i(x)|\, d\mu(y)$$

Lemma 3.24

$$\le \int_V [2-2\mathrm{Re}\,\phi_i(y)]^{1/2}\, d\mu(y)$$

Cauchy-Schwarz

$$\le \left(\int_V 2-2\mathrm{Re}\,\phi_i\, d\mu\right)^{1/2} \mu(V)^{1/2} < 2\,\delta^{1/2}\mu(V)$$

Similarly for ϕ_0. Now, by Lemma 3.23 there exists $i_2\in I$ so that for all $i\geq i_2$ we have

$$\sup_{x\in K} |1_V*\phi_i(x) - 1_V*\phi_0(x)| < \delta\mu(V)$$

Let $i_0 \geq i_1, i_2$. Then for $i\geq i_0$ the above estimates imply for $x\in K$ that

$$|\phi_i(x)-\phi_0(x)| \le \frac{1}{\mu(V)}\Big(|\mu(V)\phi_i(x) - 1_V*\phi_i(x)| + |1_V*\phi_i(x) - 1_V*\phi_i(x)| + |1_V*\phi_0(x) - \mu(V)\phi_0(x)|\Big)$$

$$\le \mu(V)\Big(2\cdot\delta^{1/2}\mu(V) + \delta\mu(V) + 2\,\delta^{1/2}\mu(V)\Big) = \delta + 4\delta^{1/2}.$$

Thus choosing δ such that $\delta + 4\delta^{1/2} < \varepsilon$ gives $\phi_i \in N(\phi_0; \varepsilon, K)$ for all $i\geq i_0$. That is, $\phi_i \to \phi_0$ in the topology of compact convergence. □

Remark The conclusion of the previous theorem does <u>not</u> hold for $P_0(G)$ in general. For example, $\phi_\xi(x) := e^{i\xi x} \in P_1(\mathbb{R})$ for each $\xi\in\mathbb{R}$ and $\phi_\xi \to 0$ as $\xi\to\infty$ in the weak* topology by the Riemann–Lebesgue lemma from Fourier analysis. However, $\phi_\xi \not\to 0$ in the topology of compact convergence on $\mathbb{R}$ (consider $K=[0,1]$). Since $0\in P_0(\mathbb{R})\setminus P_1(\mathbb{R})$ this does not contradict Theorem 3.25. □

We need one more result before proving our last goal in this chapter. This proposition will also be needed in the following chapter.

Proposition 3.26 For a locally compact group G one has

$$\{f*g : f,g\in C_c(G)\} \subset \mathrm{span}(C_c(G)\cap P(G)).$$

Moreover, the above span is dense in $C_c(G)$ under $\|\cdot\|_\infty$ and in $L^p(G,\mu)$ under $\|\cdot\|_p$ for $1\le p<\infty$.

Proof For $f\in C_c(G)$, write $\tilde{f}(x) := \overline{f(x^{-1})}$. Then Corollary 3.13 implies $f*\tilde{f}\in C_c(G)\cap P(G)$

for all $f \in C_c(G)$. Using the polarization identity, for $f, h \in C_c(G)$ we have

$$f * \tilde{h} = \frac{1}{4} \sum_{k=1}^{4} i^k (f + i^k h) * \widetilde{(f + i^k h)} \in \operatorname{span}\big(C_c(G) \cap P(G)\big).$$

For $g \in C_c(G)$, by setting $h = \tilde{g}$ the above shows $f * g$ is in the span. To see density, we simply consider $g = \psi_U$ for some approximate unit $(\psi_U)_{U \in \mathcal{U}} \subset C_c(G)$. This shows the $\|\cdot\|_\infty$ or $\|\cdot\|_p$ closure of the span contains $C_c(G)$, and in the latter case this implies it contains $L^p(G, \mu)$ by the $\|\cdot\|_p$-density of $C_c(G)$. □

Exercise For a locally compact group G show that $L^1(G, \mu)$ admits an approximate unit consisting of continuous compactly supported functions. □

Theorem 3.27 (The Gelfand–Raikov Theorem) Let G be a σ-compact group. Then for all distinct $x, y \in G$ there exists an irreducible unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ satisfying $\pi(x) \neq \pi(y)$.

Proof Fix distinct $x, y \in G$. Then we can find $f \in C_c(G)$ satisfying $f(x) \neq f(y)$. Proposition 3.26 implies f can be approximated uniformly by

$$\sum_{k=1}^{n} \alpha_k f_k$$

with $f_k \in C_c(G) \cap P(G)$ and $\alpha_k \in \mathbb{C}$. Letting the approximation be finer than $|f(x) - f(y)|$, we may assume $f = \sum \alpha_k f_k$. Using Theorems 3.21 and 3.25 we can find a linear combination g of extreme points in $P_1(G)$ that approximate f on the compact set $\{x, y\}$ well enough so that $g(x) \neq g(y)$. Consequently, there must exist $\phi \in \operatorname{ext}(P_1(G))$ satisfying $\phi(x) \neq \phi(y)$. By Theorem 3.19, π_ϕ is irreducible and it satisfies

$$\langle \pi_\phi(x) e, e \rangle_\phi = \phi(x) \neq \phi(y) = \langle \pi_\phi(y) e, e \rangle_\phi$$

by Theorem 3.14. Hence $\pi_\phi(x) \neq \pi_\phi(y)$. □

We conclude this section by comparing the notion of positive type to a more general notion:

Def Let G be a group. We say a function $\phi : G \to \mathbb{C}$ is positive definite if for all $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in G$ one has

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \phi(x_j^{-1} x_i) \geq 0$$

□

Observe that if $A^{(n)} = (\phi(x_j^{-1} x_i))_{i,j} \in M_n(\mathbb{C})$ and $v = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{C}^n$, then

$$\langle A^{(n)} v, v \rangle = \sum_{i,j=1}^{n} \phi(x_j^{-1} x_i) c_i \bar{c}_j \geq 0.$$

So $A^{(n)}$ is positive semi-definite for all $n \in \mathbb{N}$ if and only if ϕ is positive definite. For $n = 2$, $x_1 = x$, and $x_2 = 1$ we have

$$A^{(2)} = \begin{pmatrix} \phi(1) & \phi(x) \\ \phi(x^{-1}) & \phi(1) \end{pmatrix}$$

Since $A^{(2)}$ is self-adjoint, it follows that $\phi(1) \in \mathbb{R}$ and $\phi(x^{-1}) = \overline{\phi(x)}$. Since $A^{(2)}$ has non-negative eigenvalues,

$$0 \leq \det(A^{(2)}) = \phi(1)^2 - \phi(x)\phi(x^{-1}).$$

Thus

$$|\phi(x)|^2 = \phi(x)\phi(x^{-1}) \leq \phi(1)^2,$$

which implies ϕ is bounded.

If G is discrete, then being of positive type implies being positive definite: for $f := c_1\delta_{x_1} + \cdots + c_n\delta_{x_n}$ we have

$$0 \leq \int_G \int_G f(x)\overline{f(y)}\, \phi(y^{-1}x)\, d\mu(x)\, d\mu(y) = \sum_{i,j=1}^n c_i \overline{c_j}\, \phi(x_j^{-1}x_i).$$

The converse is also true by the density of $\mathrm{span}\{\delta_x : x \in G\}$ in $\ell^1(G)$ (Exercise: prove the converse.) However, in general the two notions are distinct because positive definite functions need not be measurable: $e^{i\psi}$ is positive definite on $\mathbb{R}$ for any $\mathbb{Q}$-linear bijection $\psi: \mathbb{R} \to \mathbb{R}$. Nevertheless, the notions coincide in the continuous case:

Proposition 3.28 For $\phi \in C_b(G)$, the following are equivalent:

(i) ϕ is of positive type;

(ii) ϕ is positive definite;

(iii) for all $f \in C_c(G)$

$$\int_G (f^* * f)\,\phi\, d\mu \geq 0$$

Proof (i) $\Rightarrow$ (ii): Let $(\psi_U)_{U \in \mathcal{U}} \subset L^1(G,\mu)$ be an approximate identity. Given $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in G$, define

$$f_U := \sum_{j=1}^n c_j\, L_{x_j}\psi_U \in L^1(G,\mu).$$

Then

$$0 \leq \int_G \int_G f_U(y)\overline{f_U(z)}\, \phi(z^{-1}y)\, d\mu(y)\, d\mu(z) = \sum_{i,j=1}^n c_i\overline{c_j} \int_G \int_G \psi_U(x_i^{-1}y)\, \psi_U(x_j^{-1}z)\, \phi(z^{-1}y)\, d\mu(y)\, d\mu(z)$$

The continuity of ϕ and the fact that $\mathrm{supp}\,\psi_U \subset U$ implies the last expression converges to

$$\sum_{i,j=1}^n c_i\overline{c_j}\, \phi(x_j^{-1}x_i),$$

as $U \to \infty$ and hence the above is non-negative.

(ii) $\Rightarrow$ (iii): For $f \in C_c(G)$ one has

$$F(x,y) := f(x)\overline{f(y)}\, \phi(y^{-1}x) \in C_c(G \times G)$$

In particular, this function is uniformly continuous. Denote $K := \mathrm{supp}\, f$ so that $\mathrm{supp}\, F \subset K \times K$. Given $\varepsilon > 0$, we can find a measurable partition $K = E_1 \sqcup \cdots \sqcup E_n$ and $x_j \in E_j$, $1 \leq j \leq n$, such that

$$|F(x,y) - F(x_i, x_j)| < \varepsilon \qquad (x,y) \in E_i \times E_j.$$

Then for $c_j := f(x_j)\mu(E_j)$, $1 \le j \le n$ we have

$$\left| \int_G (f^* * f)\phi \, d\mu - \sum_{i,j=1}^{n} c_i \overline{c_j} \, \phi(x_j^{-1} x_i) \right| \le \sum_{i,j=1}^{n} \int_{E_i} \int_{E_j} |F(x,y) - F(x_i, x_j)| \, d\mu(x) d\mu(y)$$

$$< \sum_{i,j=1}^{n} \mu(E_i)\mu(E_j)\,\varepsilon = \mu(K)^2 \cdot \varepsilon.$$

Since $\sum c_i \overline{c_j} \phi(x_j^{-1} x_i) \ge 0$ and $\varepsilon > 0$, we must have $\int_G (f^* * f)\phi \, d\mu \ge 0$.

(iii) $\Rightarrow$ (i): For $f \in L^1(G,\mu)$ let $(f_n)_{n \in \mathbb{N}} \subset C_c(G)$ approximate f in $\|\cdot\|_1$. Then

$$\|f^* * f - f_n^* * f_n\|_1 \le \|f^*\|_1 \|f - f_n\|_1 + \|f^* - f_n^*\| \cdot \|f_n\|_1 \longrightarrow 0$$

Consequently,

$$\int_G (f^* * f)\phi \, d\mu = \lim_{n \to \infty} \int_G (f_n^* * f_n)\phi \, d\mu \ge 0$$

so that ϕ is of positive type. $\square$