2.1 Topological Groups

Def A topological group is a group G equipped with a topology such that the operations

$$G \to G \qquad \text{and} \qquad G \times G \to G$$
$$x \mapsto x^{-1} \qquad\qquad\qquad (x,y) \mapsto xy$$

are continuous (where $G \times G$ is given the product topology). □

Ex ① $\mathbb{R}$ with addition and its usual topology is a topological group. Consequently, $\mathbb{Z}, \mathbb{Q} \leq \mathbb{R}$ are topological groups when given the subspace topology.

② $\mathbb{C} \setminus \{0\}$ with multiplication and its usual topology is a topological group. Consequently

$$\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\} \leq \mathbb{C}$$

and

$$\mathbb{T} := \{e^{i\theta} : 0 \leq \theta < 1\} \leq \mathbb{C} \quad \text{torus}$$

are topological groups. □

We will see more examples after we specialize to so-called "locally compact groups."

Exercise Show the continuity of $G \times G \ni (x,y) \mapsto x^{-1}y \in G$ is equivalent to the continuity of both multiplication and inversion. □

Given a subset $A \subseteq G$, we write

$$Ax = \{ax : a \in A\} \qquad xA = \{xa : a \in A\} \qquad A^{-1} = \{a^{-1} : a \in A\},$$

and given another subset $B \subset G$ we write

$$AB = \{ab : a \in A,\ b \in B\}$$

Def We say a subset $A \subset G$ is symmetric if $A^{-1} = A$. □

Let us begin by collecting some basic properties that will be used frequently throughout the course.

Proposition 2.1 Let G be a topological group.

① For subsets $A, B \subset G$, one has $A \cap B = \emptyset$ iff $1 \notin A^{-1}B$.

2. If $U \subset G$ is open then so are U^{-1}, AU, and UA for any subset $A \subset G$. In particular, the topology is invariant under inversion and translation.
3. For any neighborhood N of 1 (i.e. $1 \in N^\circ$) there exists a symmetric neighborhood V of 1 satisfying $VV \subset N$.
4. If $H \leq G$ is a subgroup, then so is $\overline{H}$.
5. If $H \leq G$ is an open subgroup, then H is closed.
6. If $A, B \subset G$ are compact subsets, then so is AB.

Proof 1: $1 \in A^{-1}B$ iff $1 = a^{-1}b$ for some $a \in A, b \in B$ iff $a = b$ for some $a \in A, b \in B$ iff $A \cap B \neq \emptyset$.

2: The continuity of $x \mapsto x^{-1}$ implies U^{-1} is open, and the continuity of $x \mapsto ax$ implies aU is open for all $a \in A$. Hence

$$AU = \bigcup_{a \in A} aU$$

is open. Similarly for UA.

3: The continuity of multiplication implies there exist open neighborhoods W_1, W_2 of 1 satisfying $W_1 \cdot W_2 \subset N$. Set

$$V := W_1 \cap W_2 \cap W_1^{-1} \cap W_2^{-1},$$

and note V is open by 2.

4: Given $x, y \in \overline{H}$, we can find nets $(x_i)_{i \in I}, (y_j)_{j \in J} \subset H$ converging to x and y, respectively. Then

$$xy = \lim_{i \to \infty} \lim_{j \to \infty} x_i y_j \in \overline{H}$$

and

$$x^{-1} = \lim_{i \to \infty} x_i^{-1} \in \overline{H}$$

so that $\overline{H}$ is a subgroup.

5: Let $C \subset G$ be coset representatives for $H \leq G$, with $1 \in C$ the representative of the trivial coset. Then

$$G \setminus H = (C \setminus \{1\}) \cdot H$$

is open by 2, and hence H is closed.

6: Note that $A \times B$ is compact and hence so is its image AB under the continuous map $(x, y) \mapsto xy$. $\square$

Let $H \le G$ be a subgroup of a topological group. Denote by G/H the space of left cosets and let

$$q: G \to G/H$$
$$x \mapsto xH$$

be the quotient map. We equip G/H with the quotient topology: $U \subset G/H$ is open iff $q^{-1}(U)$ is open. Note that for $V \subset G$ open, $q(V)$ is open since

$$q^{-1}(q(V)) = VH$$

is open by Proposition 2.1.2.

Proposition 2.2 Let $H \le G$ be a subgroup of a topological group.

1. If H is closed then G/H is Hausdorff.
2. If G is locally compact then so is G/H.
3. If H is normal then G/H is a topological group.

Proof 1: Let $xH \neq yH$ in G/H. Then in particular $y \notin xH$ so that $1 \notin xHy^{-1}$. Note that xHy^{-1} is also closed since H is closed. Let U be a symmetric neighborhood of 1 satisfying $UU \subset G \setminus (xHy^{-1})$, which exists by Proposition 2.1.3. Then

$$1 \notin (UxH)(UyH)^{-1} = U(xH \cdot Hy^{-1})U^{-1} = U^{-1}(xHy^{-1})U^{-1}$$

since $UU \cap (xHy^{-1}) = \emptyset$. Hence $q(Ux)$ and $q(Uy)$ are disjoint neighborhoods of xH and yH, respectively.

2: Since the quotient map $q: G \to G/H$ is both continuous and open, any compact neighborhood $1 \in K \subset G$ yields a compact neighborhood $q(Kx)$ for xH.

3: For $x, y \in G$, let $U \subset G/H$ be a neighborhood of $(x^{-1}y)H$. We can then find neighborhoods V and W of x and y such that $V^{-1}W \subset q^{-1}(U)$, which then implies $q(V)$ and $q(W)$ are neighborhoods of xH and yH satisfying

$$q(V)^{-1}q(W) = q(V^{-1}W) \subset q(q^{-1}(U)) = U$$

Thus $(xH, yH) \mapsto x^{-1}yH$ is continuous and G/H is a topological group. □

Corollary 2.3 If G is T_1 then it is Hausdorff. If G is not T_1, then $\overline{\{1\}}$ is a closed normal subgroup and $G/\overline{\{1\}}$ is a Hausdorff topological group.

Proof If G is T_1, then $H := \{1\}$ is a closed subgroup and hence $G = G/\{1\}$ is Hausdorff by Proposition 2.2.1.

If G is not T_1, first $\overline{\{1\}}$ is a closed subgroup by Proposition 2.1.4. If $H \le G$ is any other closed subgroup, then $1 \in H$ implies $\overline{\{1\}} \le H$. Hence

$$\overline{\{1\}} = \bigcap_{\substack{H \leq G \\ \text{closed}}} H$$

Since conjugation sends closed subgroups to closed subgroups, it follows that $\overline{\{1\}}$ is normal. Thus $G/\overline{\{1\}}$ is a Hausdorff topological group by Proposition 2.2.1,3. □

In light of the previous corollary, we will only consider Hausdorff topological groups.

Def A locally compact group is a topological group whose topology is locally compact and Hausdorff. □

Ex 1. $\mathbb{R}, \mathbb{Z}, \mathbb{T}$ are all locally compact groups, but $\mathbb{Q}$ with the subspace topology is not. Indeed, let $N \subset \mathbb{Q}$ be a neighborhood of 0 in the subspace topology. Then $[-\varepsilon, \varepsilon] \cap \mathbb{Q} \subset N$ for sufficiently small $\varepsilon > 0$. But this subset contains sequences that converge to irrational numbers; that is, sequences with no convergent subnets. Hence N cannot be compact. Note $\mathbb{Z} \leq \mathbb{R}$ presents no issues because the subspace topology is the discrete topology. Also, we can identify $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ by $t + \mathbb{Z} \mapsto e^{2\pi i t}$.

2. For $n \in \mathbb{N}$, $GL_n(\mathbb{R}) = \{x \in M_n(\mathbb{R}) : x \text{ invertible}\}$ is a locally compact group when $GL_n(\mathbb{R})$ inherits its topology from $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

3. $\mathbb{Z}/2\mathbb{Z}$ with the discrete topology is a compact locally compact group. Hence for any set I (countable or not)

$$\prod_{i \in I} (\mathbb{Z}/2\mathbb{Z})$$

is a compact locally compact group by Tychonoff's theorem. We can identify it, as a set, with the collection of subsets of I.

4. For any prime number p, the p-adic numbers $\mathbb{Q}_p$ forms a locally compact group under addition. We will treat this example in more detail soon. □

1/22

Proposition 2.4 Every locally compact group G admits a subgroup H that is open, closed, and σ-compact.

Proof Let $K \subset G$ be a compact neighborhood of 1. By replacing K with $K \cap K^{-1}$ if necessary, we may assume K is symmetric. Define

$$H := \bigcup_{n=1}^{\infty} \underbrace{K \cdot K \cdots K}_{n \text{ times}}$$

Then $H = \langle K \rangle$ is a subgroup. Since $(K \cdot K \cdots K) \cdot K$ is a neighborhood of $K \cdot K \cdots K$, it follows that H is open, and hence closed by Proposition 2.1.5. Finally, each $K \cdot K \cdots K$ is compact by Proposition 2.1.6 and so H is σ-compact. □

Note that if G in the previous proposition is connected, then necessarily $G = H$. In general, as a topological space, one has

$$G = \bigsqcup_{xH \in G/H} xH \cong H^{|G/H|}$$

If $|G/H|$ is countable, then it follows that G is also σ-compact.

Def Let G be a topological group. For a function $f: G \to \mathbb{C}$ we define its left (resp. right) translation by $y \in G$ as the function

$$(L_y f)(x) := f(y^{-1}x) \qquad (\text{resp. } (R_y f)(x) := f(xy)) \qquad x \in G.$$

We say f is left (resp. right) uniformly continuous if

$$\lim_{y \to 1} \| L_y f - f \|_\infty = 0 \qquad (\text{resp } \lim_{y \to \infty} \| R_y f - f \|_\infty)$$

□

Observe that $L_y L_z = L_{yz}$ and $R_y R_z = R_{yz}$.

Proposition 2.5 Every $f \in C_c(G)$ is left and right uniformly continuous.

Proof Fix $f \in C_c(G)$ and let

$$K := \operatorname{supp} f = \overline{\{x \in G : f(x) \neq 0\}}$$

be the compact support of f. Let $\varepsilon > 0$. For each $x \in K$, the continuity of f yields a neighborhood U_x of 1 such that

$$|f(y^{-1}x) - f(x)| < \frac{\varepsilon}{2} \qquad \forall y \in U_x.$$

Let V_x be a symmetric neighborhood of 1 such that $V_x V_x \subset U_x$. As K is compact we can reduce the cover $\{V_x \cdot x : x \in K\}$ of K to a finite subcover $\{V_{x_1} \cdot x_1, \ldots, V_{x_n} \cdot x_n\}$. Set

$$V := V_{x_1} \cap \cdots \cap V_{x_n},$$

which is also a symmetric neighborhood of 1. We claim $\| L_y f - f \|_\infty < \varepsilon$ for all $y \in V$. Indeed, if $x \in K$ then there exists $1 \le j \le n$ such that $x \in V_{x_j} x_j$, or $x x_j^{-1} \in V_{x_j}$. Then for $y \in V$ we have

$$y^{-1}x = y^{-1} x x_j^{-1} x_j \in V^{-1} V_{x_j} x_j \subseteq V_{x_j} V_{x_j} x_j \subseteq U_{x_j} \cdot x_j$$

and

$$x = x x_j^{-1} \cdot x_j \in V_{x_j} x_j \subseteq U_{x_j} x_j$$

so that

$$|f(y^{-1}x) - f(x)| \le |f(y^{-1}x) - f(x_j)| + |f(x_j) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If $x \notin K$ but $y^{-1}x \in K$ for some $y \in V$, we instead find $1 \le j \le n$ so that $y^{-1}x \in V_{x_j} x_j$. Then

$$y^{-1}x = (y^{-1}x\, x_j^{-1}) \cdot x_j \in V_{x_j} x_j \subseteq U_{x_j} x_j$$

and

$$x = y(y^{-1}x) \cdot x_j^{-1} \cdot x_j \in V \cdot V_{x_j} \cdot x_j \subset V_{x_j} V_{x_j} \cdot x_j \subset U_{x_j} \cdot x_j,$$

so that $|f(y^{-1}x) - f(x)| < \varepsilon$ as well. Finally if $x \notin K$ and $y^{-1}x \notin K$, then

$$|f(y^{-1}x) - f(x)| = |0 - 0| < \varepsilon.$$

Thus $\|L_y f - f\|_\infty < \varepsilon$. The argument for R_y is similar (see the textbook). □

The p-adic numbers

Fix a prime number p. Every $r \in \mathbb{Q} \setminus \{0\}$ can be uniquely written as $r = p^m q$ for some $m \in \mathbb{Z}$ and $q \in \mathbb{Q}$ with numerator and denominator not divisible by p.

Def For $r \in \mathbb{Q} \setminus \{0\}$, its p-adic norm is

$$|r|_p := p^{-m}$$

where $r = p^m q$ is the unique decomposition from above. We set $|0|_p = 0$. □

Observe that for $r_i = p^{m_i} q_i$, $i = 1, 2$,

$$|r_1 \cdot r_2|_p = |p^{m_1 + m_2} \cdot q_1 q_2|_p = p^{-(m_1 + m_2)} = |r_1|_p\, |r_2|_p \qquad *$$

and if $m = \min\{m_1, m_2\}$, then

$$|r_1 + r_2|_p = |p^m(p^{m_1 - m} q_1 + p^{m_2 - m} q_2)|_p \le p^{-m} = \max\{|r_1|_p, |r_2|_p\} \le |r_1|_p + |r_2|_p. \qquad **$$

Consequently,

$$d_p(r_1, r_2) := |r_1 - r_2|_p$$

defines a metric on $\mathbb{Q}$, and the arithmetic operations are continuous with respect to the induced topology by $(*)$ and $(**)$. By $(**)$, one has

$$d_p(r_1, r_2) = |r_1 - r_2| \le \max\{|r_1|_p, |-r_2|_p\} = \max\{|r_1|_p, |r_2|_p\}.$$

Thus if $r_1 \ne r_2$, then they can only be close in this metric if they share a common large power of p.

Def The completion of $\mathbb{Q}$ with respect to d_p is denoted by $\mathbb{Q}_p$, and its elements are called p-adic numbers. □

Note that the arithmetic operations on $\mathbb{Q}$ extend continuously to $\mathbb{Q}_p$ making it a field.

$\boxed{\text{Proposition 2.6}}$ If $m \in \mathbb{Z}$ and $c_j \in \{0, 1, \ldots, p-1\}$ for $j \geq m$, then the series

$$\sum_{j=m}^{\infty} c_j p^j$$

converges in $\mathbb{Q}_p$. Moreover, every p-adic number is the sum of such a series.

Proof For $M \leq N$ observe

$$d\left(\sum_{j=1}^{M} c_j p^j, \sum_{j=1}^{N} c_j p^j\right) = \left|\sum_{j=M+1}^{N} c_j p^j\right|_p \leq \max\{|c_j p^j|_p : M+1 \leq j \leq N\} = p^{-M+1}$$

Thus the partial sums form a Cauchy sequence and therefore converge. In particular,

$$\sum_{j=m}^{\infty} (p-1) \cdot p^j = \sum_{j=m}^{\infty} (p^{j+1} - p^j) = -p^m \qquad (***)$$

Now, let F denote all such convergent sums in $\mathbb{Q}_p$. By using the analogues of decimal addition, multiplication, and division we see that F is closed under these operations. Also, using $(***)$ we also see that F is closed under subtraction: if $c_m \neq 0$ then

$$-\sum_{j=m}^{\infty} c_j p^j = p^{m+1} - c_m \cdot p^m + -p^{m+1} - \sum_{j=m+1} c_j p^j$$
$$= (p - c_m) p^m + \sum_{j=m+1}^{\infty} (p-1-c_j) p^j$$

Since $1 = 1 \cdot p^0 \in F$, we then have $\mathbb{Q} \subset F$ and it therefore suffices to show F is complete. Suppose

$$x_n = \sum_{j=m(n)}^{\infty} c_j(n) p^j$$

is a Cauchy sequence in F. Note that $|x_n|_p = p^{-m(n)}$, and since Cauchy sequences are bounded we have

$$M := \min_{n \in \mathbb{N}} m(n) > -\infty.$$

For $k \in \mathbb{N}$ let $N \in \mathbb{N}$ be such that

$$|x_{n_1} - x_{n_2}|_p < p^{-k} \qquad \forall n_1, n_2 \geq N.$$

This implies $c_j(n_1) = c_j(n_2)$ for $M \leq j \leq k$. Indeed, if $j \in \mathbb{Z}$ is the smallest integer such that

$$c := c_j(n_1) - c_j(n_2) \neq 0,$$

then by $(**)$ we have

$$p^{-j} = |c \cdot p^j|_p = |c \cdot p^j - (x_{n_1} - x_{n_2}) + (x_{n_1} - x_{n_2})|_p \leq |x_{n_1} - x_{n_2}|_p < p^{-k}.$$

It follows that for each $j \geq M$, the sequence $(c_j(n))_{n\in\mathbb{N}}$ is eventually constant. Set

$$c_j := \lim_{n\to\infty} c_j(n)$$

and

$$x := \sum c_j p^j$$

Then for $k \in \mathbb{N}$, let $N \in \mathbb{N}$ be large enough so that for each $j = 1, \dots, k$, $c_j(n) = c_j$ for all $n \geq N$. Then $|x_n - x|_p < p^{-k}$ for all $n \geq N$. Hence $x_n \to x$ and F is complete. □

1/25

For $r \geq 0$ and $x \in \mathbb{Q}_p$, denote

$$\bar{B}(x,r) := \{y \in \mathbb{Q}_p : |x-y|_p \leq r\},$$

which is a closed set. However, the metric only takes values in the set $\{0\} \cup p^{\mathbb{Z}}$, and so for $r > 0$ $\exists \varepsilon > 0$ so that $\bar{B}(x,r)$ equals the open ball

$$B(x, r+\varepsilon) := \{y \in \mathbb{Q}_p : |x-y|_p < r+\varepsilon\}.$$

Thus $\bar{B}(x,r)$ is clopen for all $x \in \mathbb{Q}_p$ and $r > 0$. This implies $\mathbb{Q}_p$ is totally disconnected but has no isolated points.

By (**) we have

$$\bar{B}(x,r) = \bar{B}(y,r) \qquad \forall y \in \bar{B}(x,r)$$

So every point in $\bar{B}(x,r)$ is the center.

Using (**) again, we see that each $\bar{B}(0,r)$ is an additive subgroup. For $r \leq 1$, it is further a subring by (*).

$\boxed{\text{Def}}$ The set of <u>p-adic integers</u> is the clopen subring of $\mathbb{Q}_p$ given by

$$\mathbb{Z}_p := \bar{B}(0,1)$$

□

Observe that $\mathbb{Z} \subset \mathbb{Z}_p$, and in fact $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the metric d_p. Additionally, $p\mathbb{Z}_p = \bar{B}(0,p^{-1})$ is an additive subgroup of $\mathbb{Z}_p$ with left cosets $c + p\mathbb{Z}_p$, $c = 0, 1, \dots, p-1$. Hence

$$\mathbb{Z}_p / p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}.$$

$\boxed{\text{Theorem 2.7}}$ $\mathbb{Z}_p$ is a compact subgroup. Consequently, $\mathbb{Q}_p$ is a locally compact group under addition, and $\mathbb{Q}_p \setminus \{0\}$ is a locally compact group under multiplication.

<u>Proof</u> The continuity of multiplication and addition follows from (*) and (**), respectively. The continuity of inversion for these operations thus follows from

$$|x^{-1}-y^{-1}|_p = \frac{|y-x|_p}{|x|_p|y|_p} \quad \text{and} \quad |(-x)-(-y)|_p = |x-y|_p$$

so $\mathbb{Q}_p$ and $\mathbb{Q}_p\setminus\{0\}$ are topological groups under their respective operations. To show that they are locally compact, it suffices to show $\mathbb{Z}_p$ is compact since $x+\mathbb{Z}_p$ is a compact neighborhood of $x\in\mathbb{Q}_p$, and $y+p\cdot\mathbb{Z}_p$ is a compact neighborhood of $y\in\mathbb{Q}_p\setminus\{0\}$. Since $\mathbb{Z}_p$ is closed and hence complete, it further suffices to show it is totally bounded. Given $\varepsilon>0$, let $k\in\mathbb{N}$ be large enough so that $p^{-k}<\varepsilon$. Then

$$\mathbb{Z}_p = \bigsqcup_{0\le x<p^k}(x+p^k\cdot\mathbb{Z}_p) = \bigsqcup_{0\le x<p^k} B(x,p^{-k}).$$

□

Semidirect products

Actions of topological groups on other topological groups is another source of examples.

Def Let H, N be topological groups. An <u>action</u> of H on N is a group homomorphism

$$\alpha: H\to \mathrm{Aut}(N)$$

such that the map

$$\begin{aligned} H\times N &\longrightarrow N \\ (x,y) &\longmapsto \alpha_x(y) \end{aligned}$$

is continuous. In this case, we write $H \overset{\alpha}{\curvearrowright} N$. □

Observe that each α_h is a homeomorphism of N, and so we can take $\mathrm{Aut}(N)$ above to mean automorphisms as a topological group: group automorphisms that are topological homeomorphisms.

Exercise For a topological group G, show that

$$G \overset{c}{\curvearrowright} G$$
$$\mathrm{Ad}_y(x) := yxy^{-1}$$

defines an action of G on itself. □

Def Let $H \overset{\alpha}{\curvearrowright} N$ be an action of topological groups. The <u>semidirect product</u> of this action is a topological group denoted $H\ltimes_\alpha N$ consisting of the set $H\times N$ equipped with the product topology and group operations

$$(x,y)(s,t) = (xs, \alpha_s^{-1}(y)t) \qquad (x,y)^{-1} = (x^{-1}, \alpha_x(y^{-1}))$$

□

Exercise Check that $H \rtimes_\alpha N$ is really a topological group.

Note that we can identify H and N with the subgroups $H \times \{1\}$ and $\{1\} \times N$, respectively, and that N then forms a normal subgroup:

$$(x,y)(1,a)(x,y)^{-1} = (x, ya)(x^{-1}, \alpha_x^{-1}(y^{-1})) = (1, \alpha_x(yay^{-1})).$$

In particular,

$$(x,1)(1,a)(x,1) = (1, \alpha_x(a))$$

Lastly, if H and N are locally compact groups, then so is $H \rtimes_\alpha N$.

Remark One can also define $N \rtimes_\alpha H := N \times H$ with group operations

$$(y,x)(t,s) = (y\,\alpha_x(t), xs) \quad \text{and} \quad (y,x)^{-1} = (\alpha_x^{-1}(y^{-1}), x^{-1}).$$

However $N \rtimes_\alpha H \cong H \rtimes_\alpha N$ via $(y,x) \mapsto (x^{-1}, y^{-1})^{-1}$. We will tend to favor $N \rtimes_\alpha H$ for no other reason than the fact that we will focus on left rather than right Haar measures. $\square$

Exercise Show that

$$G \rtimes_{\mathrm{Ad}} G \longrightarrow G$$
$$(x,y) \longmapsto xy$$

is a continuous, surjective group homomorphism with kernel $\{(x, x^{-1}) : x \in G\}$.

3/15

Ex ① If $H \curvearrowright N$ by the trivial action, $\alpha_x(y) = y$ for all $x \in H, y \in N$, then $H \rtimes_\alpha N \cong H \times N$.

② $\mathbb{R}_+ \curvearrowright \mathbb{R}$ by multiplication, and one has

$$\mathbb{R} \rtimes \mathbb{R}_+ \xleftrightarrow{\cong} \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}$$
$$(b,a) \longmapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Indeed, $(b,a)^{-1}(d,c) = (-\frac{1}{a} b, \frac{1}{a})(d,c) = (\frac{d-b}{a}, \frac{c}{a})$ and

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c/a & (d-b)/a \\ 0 & 1 \end{pmatrix}$$

We call this matrix group the "$ax+b$" group.

③ $\mathbb{Q}_p \setminus \{0\} \curvearrowright \mathbb{Q}_p$ also by multiplication, so one can form a p-adic $ax+b$ group.

④ $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ by the usual matrix multiplication, similarly for any subgroup of $GL_n(\mathbb{R})$. $\square$

2.2 Haar Measure

Before we define Haar measures, will prove some more general results about Radon measures.

Radon measures

Def Let X be a locally compact Hausdorff space equipped with its Borel σ-algebra $\mathcal{B}_X$. For a Borel measure μ on X and $E \in \mathcal{B}_X$ we say μ is:

- outer regular on E if
$$\mu(E) = \inf\{\mu(U) : U \supset E \text{ open}\}$$
- inner regular on E if
$$\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$$
- a Radon measure if it is outer regular on all $E \in \mathcal{B}_X$, inner regular on open sets, and finite on all compact sets. $\square$

The following is the most important theorem in the theory of Radon measures, and will be useful in our study of Haar measures. Denote
$$C_c^+(X) := \{f \in C_c(X) : f \geq 0 \text{ and } f \not\equiv 0\}.$$
Note that $C_c(X) = \operatorname{span}_{\mathbb{C}} C_c^+(X)$ since $f = \operatorname{Re}(f)^+ - \operatorname{Re}(f)^- + i\operatorname{Im}(f)^+ - i\operatorname{Im}(f)^-$.

Theorem 2.8 (Riesz Representation Theorem)
Let X be a locally compact Hausdorff space and suppose
$$\varphi : C_c(X) \to \mathbb{C}$$
is a linear functional satisfying $\varphi(C_c^+(X)) \subset [0,\infty)$. Then there exists a unique Radon measure μ on X such that
$$\varphi(f) = \int_X f \, d\mu \qquad \forall f \in C_c(X).$$
Moreover, one has
$$\mu(U) = \sup\{\varphi(f) : f \in C_c(X),\ 0 \leq f \leq 1,\ \operatorname{supp}(f) \subset U\}$$
for all open sets $U \subset X$, and
$$\mu(K) = \inf\{\varphi(f) : f \in C_c(X),\ f \geq 1_K\}$$
for all compact sets $K \subset X$.

See Theorem 7.2 in Folland's Real Analysis textbook for a proof. Another useful lemma in the general theory of Radon measures is the following:

Lemma 2.9 Let μ be a Radon measure on a locally compact Hausdorff space X. Then μ is inner regular on all σ-finite sets.

Proof First suppose $E \in \mathcal{B}_X$ satisfies $\mu(E) < \infty$. Given $\varepsilon > 0$, outer regularity allows us to find $U \supset E$ open such that $\mu(U) < \mu(E) + \varepsilon$. Consequently,

$$\mu(U \setminus E) = \mu(U) - \mu(E) < \varepsilon.$$

Using inner regularity, we can next find $F \subset U$ compact such that

$$\mu(F) > \mu(U) - \varepsilon$$

Invoking outer regularity again, this time on $U \setminus E$, we can find $V \supset U \setminus E$ with $\mu(V) < \varepsilon$. Set $K := F \setminus V$, which is compact as a closed subset of F. Observe that

$$K = F \setminus V \subset U \setminus (U \setminus E) = E.$$

and

$$\mu(K) = \mu(F) - \mu(F \cap V) > \mu(U) - \varepsilon - \mu(V) > \mu(E) - 2\varepsilon.$$

Thus μ is inner regular on E.

Now, suppose $\mu(E) = +\infty$, but E is σ-finite for μ. Then for any $R > 0$ we can find $E_0 \subset E$ satisfying $R < \mu(E_0) < +\infty$. The above argument then yields $K \subset E_0 \subset E$ compact such that

$$\mu(K) > \mu(E_0) - \frac{R}{2} > \frac{R}{2}.$$

Since R was arbitrary, it follows that

$$\sup\{\mu(K) : K \subset E \text{ compact}\} = +\infty = \mu(E),$$

and so μ is inner regular on E. □

Exercise Show that the previous lemma can be upgraded to include $E \in \mathcal{B}_X$ that are μ-semifinite. □

Recall that the monotone convergence theorem (for general measures) is a result about increasing <u>sequences</u> of functions. In particular, it does not apply to increasing <u>nets</u>, at least not in general. However, for Radon measures a fairly general result along these lines does exist if one further restricts the class of functions.

Def Let X be a topological space. We say a function $f : X \to (-\infty, +\infty]$ is <u>lower semicontinuous</u> if $f^{-1}((a, +\infty])$ is open for all $a \in \mathbb{R}$. We say a function

$g: X \to [-\infty, \infty)$ is upper semicontinuous if $g^{-1}([-\infty, b))$ is open for all $b \in \mathbb{R}$. $\square$

Since rays of the form $(a, +\infty]$ and $[-\infty, b)$ both generate the Borel σ-algebra on $\mathbb{R}$, it follows that lower and upper semicontinuous functions are Borel measurable (when X is given its Borel σ-algebra). We record a few other observations that are left as exercises.

Proposition 2.10 Let X be a topological space.

(a) A function f on X is lower semicontinuous if and only if $-f$ is upper semicontinuous.

(b) If f and g are lower (resp. upper) semicontinuous then $f+g$ is lower (resp. upper) semicontinuous

(c) 1_U is lower semicontinuous for all open sets $U \subset X$.

(d) 1_F is upper semicontinuous for all closed sets $F \subset X$.

(e) If $\mathcal{L}$ is a family of lower semicontinuous functions on X, then

$$f(x) := \sup_{g \in \mathcal{L}} g(x)$$

is lower semicontinuous.

(f) If $\mathcal{U}$ is a family of upper semicontinuous functions on X, then

$$f(x) := \inf_{g \in \mathcal{L}} g(x)$$

is upper semicontinuous.

Observe that supremums (resp. infimums) of families of continuous functions are lower (resp. upper) semicontinuous.

Exercise Let X be a locally compact Hausdorff space. For $f: X \to [0, +\infty]$ lower semicontinuous, show that

$$f(x) = \sup\{g(x) : g \in C_c(X),\ 0 \leq g \leq f\}$$

for all $x \in X$. $\square$

Using lower semicontinuous functions, we can obtain a version of the monotone convergence theorem that applies to increasing nets of functions.

Proposition 2.11 Let X be a locally compact Hausdorff space, and let $(f_i)_{i \in I}$ be an increasing net of nonnegative, lower semicontinuous functions on X. Then

$$f(x) := \sup_{i \in I} f_i(x) = \lim_{i \to \infty} f_i(x)$$

satisfies

$$\int_X f \, d\mu = \lim_{i \to \infty} \int_X f_i \, d\mu$$

for all Radon measures on X.

Proof First note that f is lower semicontinuous by Proposition 2.10.e, and hence Borel measurable. The fact that f is the pointwise limit of the f_i follows from $(f_i(x))_{i\in I} \subset [0,+\infty]$ being an increasing net for all $x\in X$.

Now, fix a Radon measure μ on X. For each $n\in\mathbb{N}$ define a simple function ϕ_n on X by

$$\phi_n(x) := \sum_{j=1}^{2^{2n}} \frac{1}{2^n} 1_{U_{n,j}}(x)$$

where $U_{n,j} = f^{-1}((\frac{j}{2^n}, +\infty])$. Since $U_{n,j}$ is open, ϕ_n is lower semicontinuous by Proposition 2.10.b,c. Also, for fixed $x\in X$, if $k = \max\{j : x\in U_{n,j}\}$, then

$$\phi_n(x) = \sum_{j=1}^{k} \frac{1}{2^n} 1_{U_{n,j}}(x) = \frac{k}{2^n} < f(x).$$

Additionally, $\phi_n \leq \phi_{n+1}$ for all $n\in\mathbb{N}$ and $f = \sup_n \phi_n$ (Exercise: check this). So by the monotone convergence theorem, one has

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X \phi_n\, d\mu.$$

2/19

Fix $a < \int_X f\, d\mu$ and let $N\in\mathbb{N}$ be large enough so that $\int_X \phi_N\, d\mu > a$. For each $j = 1,\ldots,2^{2N}$, we can use the inner regularity of μ on $U_{N,j}$ to find $K_{N,j}\subset U_{N,j}$ compact so that

$$\psi := \sum_{j=1}^{2^{2N}} \frac{1}{2^N} 1_{K_{N,j}}$$

satisfies $\int_X \psi\, d\mu > a$. Since $\psi \leq \phi_N < f$, for each $x\in X$ we can find $i_x \in I$ large enough so that $\psi(x) < f_{i_x}(x) \leq f(x)$. Note that $f_{i_x} - \psi$ is lower semicontinuous by Proposition 2.10.a,b,d, and consequently

$$V_x := (f_{i_x} - \psi)^{-1}((0,+\infty]) = \{y\in X : \psi(y) < f_{i_x}(y)\}$$

is an open neighborhood of x. We can reduce the open cover $\{V_x : x\in K_1\cup\cdots\cup K_{2^N}\}$ of the compact set $K_1\cup\cdots\cup K_{2^n}$ to a finite subcover $\{V_{x_1},\ldots,V_{x_d}\}$. Let $i_0\in I$ be s.t. $i_0 \geq i_{x_\ell}$ for $\ell = 1,\ldots,d$. Then for $i \geq i_0$ we have $f_i(x) \geq f_{i_{x_\ell}}(x) > \psi(x)$ for $x\in V_{x_\ell}$. Thus $f_i \geq \psi$ since $\operatorname{supp}\psi \subseteq V_{x_1}\cup\cdots\cup V_{x_d}$, and so

$$\int_X f_i\, d\mu \geq \int_X \psi\, d\mu > a.$$

Since $a < \int_X f\, d\mu$ was arbitrary, we have

$$\lim_{i\to\infty} \int_X f_i\, d\mu = \sup_{i\in I} \int_X f_i\, d\mu \geq \int_X f\, d\mu.$$

The reverse inequality follows from $f_i \leq f$ and the monotonicity of the integral. $\square$

While the previous proposition is interesting in its own right, for the purpose of this course its main utility will be in helping prove the following proposition. This proposition will be useful to us in computing and constructing examples of Haar measures.

Proposition 2.12 Let μ be a Radon measure on a locally compact Hausdorff space, and let $\phi: X \to (0,\infty)$ be a continuous function. Then $d\nu := \phi\, d\mu$ defines a Radon measure and it satisfies

$$\int_X f\, d\nu = \int_X f\phi\, d\mu$$

for all $f \in C_c(X)$.

Proof The Riesz representation theorem (Theorem 2.8) yields a unique Radon measure $\tilde{\nu}$ on X satisfying

$$\int_X f\, d\tilde{\nu} = \int_X f\phi\, d\mu$$

for all $f \in C_c(X)$. We will show $\nu = \tilde{\nu}$, and to do so it suffices to show ν is outer regular and $\nu(U) = \tilde{\nu}(U)$ for all open sets $U \subset X$.

Let $E \subset X$ be a Borel set with $\nu(E) < \infty$ (note that outer regularity is automatic when $\nu(E) = \infty$) and let $\varepsilon > 0$. For $k \in \mathbb{Z}$ define

$$E_k := \{x \in E : 2^k < \phi(x) < 2^{k+2}\} = E \cap \phi^{-1}((2^k, 2^{k+2})),$$

which satisfy

$$E = \bigcup_{k \in \mathbb{Z}} E_k$$

Then $2^k \mu(E_k) < \nu(E_k) \leq \nu(E) < \infty$ implies we can find $U_k \supset E_k$ open such that $\mu(U_k \setminus E_k) < \varepsilon\, 2^{-2|k|-2}$. We may assume $U_k \subset \phi^{-1}((2^k, 2^{k+2}))$ by taking the intersection if necessary. Then we have

$$\nu(U_k \setminus E_k) < 2^{k+2} \mu(U_k \setminus E_k) < \varepsilon\, 2^{-|k|},$$

and so $U := \bigcup_{k \in \mathbb{Z}} U_k$ is an open set containing E and satisfying

$$\nu(U \setminus E) \leq \sum_{k \in \mathbb{Z}} \nu(U_k \setminus E) \leq \sum_{k \in \mathbb{Z}} \nu(U_k \setminus E_k) < 3\varepsilon.$$

Hence ν is outer regular.

Finally, let $V \subset X$ be an arbitrary open set. Consider the set of functions

$$I := \{f \in C_c(X) : 0 \leq f \leq 1_V\}.$$

Since $\max(f,g) \in I$ for all $f, g \in I$, it follows that I is a directed set and we can define an increasing net $(h_f)_{f \in I} \subset C_c(X)$ by $h_f := f$. Then

$$\sup_{f \in I} h_f \phi = 1_V \phi$$

(by the above exercise), and hence by Proposition 2.11 we have

$$\nu(V) = \int_X 1_V \phi \, d\mu = \sup_{f \in \mathcal{I}} \int_X h_f \phi \, d\mu = \sup_{f \in \mathcal{I}} \int_X f \, d\tilde{\nu} = \tilde{\nu}(V)$$

where the last equality holds by the Riesz representation theorem (Theorem 2.8 — in fact, the equality holds even for the a priori smaller supremum of $\{f \in C_c(X) : 0 \le f \le 1 \text{ and } \operatorname{supp} f \subset U\}$. □

Def Let X be a locally compact Hausdorff space. We say Radon measures μ and ν on X are strongly equivalent if there exists a continuous function $\phi : X \to (0,\infty)$ satisfying $d\nu = \phi \, d\mu$. □

Note that if $d\mu = \phi \, d\nu$ and $\phi(x) > 0$ for all $x \in X$, then $\frac{1}{\phi(x)}$ is continuous and $d\mu(x) = \frac{1}{\phi(x)} d\nu(x)$. Thus μ and ν are, in particular, equivalent.

2/19

Haar measures, existence and uniqueness

Def Let G be a locally compact group. A left (resp. right) Haar measure on G is a non-zero Radon measure μ on G that satisfies

$$\mu(xE) = \mu(E) \qquad (\text{resp. } \mu(Ex) = \mu(E))$$

for all Borel sets $E \subset G$ and all $x \in G$. □

Proposition 2.13. Let μ be a non-zero Radon measure on a locally compact group G.

(a) If $\tilde{\mu}(E) := \mu(E^{-1})$, then μ is a left Haar measure if and only if $\tilde{\mu}$ is a right Haar measure

(b) μ is a left Haar measure if and only if

$$\int_G L_y f \, d\mu = \int_G f \, d\mu$$

for all $f \in C_c^+(G)$ and $y \in G$.

Proof (a): This follows from

$$\tilde{\mu}(Ex) = \mu(Ex)^{-1}) = \mu(x^{-1}E^{-1}).$$

(b): Approximating f by simple functions, one has

$$\int_G L_y f \, d\mu = \int_G f \, d\mu_y$$

where $\mu_y(E) = \mu(E)$. Thus if μ is a Haar measure then the claimed equality holds for all $f \in C_c^+(G)$ and $y \in G$. Conversely, if the equality holds for all $f \in C_c^+(G)$ and $y \in G$, then $\mu = \mu_y$ holds by the uniqueness part of the Riesz Representation Theorem, and hence μ is a Haar measure. □

The previous proposition allows to focus our studies on only one left or right

Haar measures, and by convention we will focus on the former.

Our main objective in this section will be to prove the existence and uniqueness (up to scaling) of a left Haar measure an arbitrary locally compact group. For the uniqueness part, we will need the following result:

$\boxed{\text{Proposition 2.14}}$ If μ is a left Haar measure on G, then $\mu(U)>0$ for all nonempty open sets $U \subset G$, and $\int_G f \, d\mu > 0$ for all $f \in C_c^+(G)$.

Proof: Suppose, towards a contradiction, that $\mu(U)=0$ for some nonempty open set. Replacing U with $x^{-1}U$ for some $x \in U$, we may assume U is a neighborhood of $1 \in G$. For any compact set K, the open cover $\{xU : x \in K\}$ of K can be reduced to a finite subcover $\{x_1 U, \ldots, x_n U\}$ and hence

$$\mu(K) \leq \sum_{j=1}^{n} \mu(x_j U) = \sum_{j=1}^{n} \mu(U) = 0$$

So μ is zero on compact sets and hence $\mu(G)=0$ by inner regularity, contradicting μ being non-zero.

For $f \in C_c^+(G)$, $U := \{x \in G : |f(x)| > \frac{1}{2}\|f\|_\infty\}$ is a nonempty open set satisfying $\frac{1}{2}\|f\|_\infty 1_U \leq f$. Hence

$$\int_G f \, d\mu \geq \frac{1}{2}\|f\|_\infty \cdot \mu(U) > 0$$

□

Before beginning the proof of our main objective, let us consider a concrete example to to demonstrate the intuition behind the proof.

$\boxed{\text{Ex}}$ From Math 828 we knew the Lebesgue measure (restricted to $\mathcal{B}_{\mathbb{R}}$) is a (left and right) Haar measure on $\mathbb{R}$. By the Riesz representation theorem, it is determined by

$$C_c^+(\mathbb{R}) \ni f \longmapsto \int_{\mathbb{R}} f \, dm$$

It turns out the ordering on $C_c^+(\mathbb{R})$ determines m (up to a scalar). For $f, g \in C_c^+(\mathbb{R})$ define

$$(f:g) := \inf \left\{ \sum_{j=1}^{n} c_j : f \leq \sum_{j=1}^{n} c_j L_{x_j} g \text{ for some } x_1, \ldots, x_n \in \mathbb{R} \right\}$$

Note that the compact support implies one can always find c_j and x_j as above. For each $n \in \mathbb{N}$, consider $\phi_n \in C_c^+(\mathbb{R})$ satisfying $0 \leq \phi_n \leq 1$, $\phi_n \equiv 1$ on $[0, \frac{1}{n}]$, and $\operatorname{supp} \phi_n \subset [-\frac{1}{n^2}, \frac{1}{n} + \frac{1}{n^2}]$. Then iff $\operatorname{supp} f \subseteq [-N, N]$ one has

$$f \leq \sum_{k=-Nn}^{Nn+1} \underbrace{\left(\max_{\frac{k}{n} \leq t \leq \frac{k+1}{n}} f(t) \right)}_{=: c_k} \cdot L_{\frac{k}{n}} \phi_n$$

So that
$$(f:\phi_n) \le \sum_k c_k \approx \underbrace{\sum_k c_k \cdot \tfrac{1}{n}}_{\text{Riemann sum}} \left(\int_{\mathbb{R}} \phi_n \, dm\right)^{-1} \approx \int_{\mathbb{R}} f \, dm \left(\int_{\mathbb{R}} \phi_n \, dm\right)^{-1}$$

Thus if we fix some $f_0 \in C_c^+(\mathbb{R})$, then
$$\lim_{n\to\infty} \frac{(f:\phi_n)}{(f_0:\phi_n)} = \frac{\int_{\mathbb{R}} f\,dm}{\int_{\mathbb{R}} f_0\,dm} = \int_{\mathbb{R}} f \, d(c\cdot m)$$
where $c = \left(\int_{\mathbb{R}} f_0\,dm\right)^{-1}$. □

Theorem 2.15. Every locally compact group G admits a left Haar measure μ. Moreover, if ν is another left Haar measure then there exists $0<c<\infty$ so that $\mu = c\nu$.

Proof (Existence): As in the previous example, for $f, g \in C_c^+(G)$ we define
$$(f:g) := \inf\left\{ \sum_{j=1}^n c_j : f \le \sum_{j=1}^n c_j L_{x_j} g \text{ for some } x_1,\ldots,x_n \in G \right\}$$

This quantity has the following properties:

1. $(L_y f : g) = (f:g)$ for all $y \in G$. This follows from $L_y f \le \sum_{j=1}^n c_j L_{yx_j} g$.
2. $(f_1+f_2 : g) \le (f_1:g) + (f_2:g)$, since $f_k \le \sum c_j^{(k)} L_{x_j^{(k)}} g$ then
$$f_1 + f_2 \le \sum \left(c_j^{(1)} L_{x_j^{(1)}} g + c_j^{(2)} L_{x_j^{(2)}} g\right).$$
3. $(cf:g) = c(f:g)$, since $cf \le c\sum c_j L_{x_j} g$.
4. $(f_1:g) \le (f_2:g)$ whenever $f_1 \le f_2$, since $f_2 \le \sum c_j L_{x_j} g$ implies $f_1 \le \sum c_j L_{x_j} g$.
5. $(f:g) \ge \|f\|_\infty / \|g\|_\infty$, since $f \le \sum c_j L_{x_j} g$ implies
$$\|f\|_\infty \le \left\|\sum c_j L_{x_j} g\right\|_\infty \le \sum c_j \|L_{x_j} g\|_\infty = \sum c_j \|g\|_\infty.$$
6. $(f:g) \le (f:h)(h:g)$, since $f \le \sum c_j L_{x_j} h$ and $h \le \sum b_k L_{y_k} g$ implies
$$f \le \sum c_j b_k L_{x_j y_k} g.$$

Fix some $f_0 \in C_c^+(G)$ and for each $\phi \in C_c^+(G)$ define $I_\phi : C_c^+(G) \to [0,\infty)$ by
$$I_\phi(f) := \frac{(f:\phi)}{(f_0:\phi)} \qquad f \in C_c^+(G)$$

Properties 1 - 4 of $(\,:\,)$ imply I_ϕ is left invariant, subadditive, homogeneous, and monotone. Also, the property 5 implies $(\,:\,)>0$ and property 6 implies
$$(f_0:f)^{-1} \le I_\phi(f) \le (f:f_0). \quad *$$

Claim For $f_1, f_2 \in C_c^+(G)$ and $\varepsilon>0$, there exists a neighborhood V of 1 so that
$$I_\phi(f_1) + I_\phi(f_2) \le I_\phi(f_1+f_2) + \varepsilon$$
whenever $\operatorname{supp}\phi \subset V$.

Indeed, fix $g \in C_c^+(G)$ such that $g \equiv 1$ on $\operatorname{supp}(f_1+f_2)$ and some $\delta>0$ to be determined later set $h := f_1 + f_2 + \delta g$ and for $i=1,2$

$$h_i(x) = \begin{cases} 0 & \text{if } f_i(x)=0 \\ \frac{f_i(x)}{h(x)} & \text{otherwise} \end{cases}.$$

Then $h_i \in C_c^+(G)$, and so by Proposition 2.5 there exists a neighborhood V of 1 so that

$$\| R_y h_i - h_i \|_\infty < \delta$$

for all $y \in V$. Equivalently, $|h_i(x) - h_i(y)| < \delta$ whenever $y^{-1}x \in V$. Suppose $\phi \in C_c^+(G)$ satisfies $\operatorname{supp}\phi \subset U$. If $h \leq \sum c_j L_{x_j}\phi$, then

$$f_i(x) = h(x) h_i(x) \leq \sum c_j \phi(x_j^{-1}x) h_i(x) \leq \sum c_j \phi(x_j^{-1}x)[h_i(x_j)+\delta]$$

because $|h_i(x) - h_i(x_j)| < \delta$ whenever $x_j^{-1}x \in \operatorname{supp}\phi \subset V$. Since $h_1 + h_2 \leq 1$, this gives

$$(f_1 : \phi) + (f_2 : \phi) \leq \sum c_j [h_1(x_j)+\delta] + \sum c_j [h_2(x_j)+\delta] \leq \sum c_j [1+2\delta].$$

Taking the infimum over all such $\sum c_j$ and dividing by $(f_0 : \phi)$ we get

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(h)[1+2\delta] \leq (I_\phi(f_1) + I_\phi(f_2) + \delta I_\phi(g))[1+2\delta],$$

where the last inequality follows from properties 2 and 3 of $(\cdot:\cdot)$. By taking δ sufficiently small (note f_1, f_2, and g are all fixed), we obtain the claimed inequality $\square$

With the claim in hand, we now finish the proof of the existence of a left Haar measure. Define

$$X := \prod_{f \in C_c^+(G)} [(f : f_0)^{-1}, (f : f_0)]$$

Equipping X with the product topology makes it into a compact space by Tychonoff's theorem. We will identify X with the set of all maps $C_c^+(G) \to (0,\infty)$ whose value at f lies in the interval $[(f:f_0)^{-1}, (f:f_0)]$, and under this identification the topology becomes that of pointwise convergence. Thus $I_\phi \in X$ for all $\phi \in C_c^+(G)$ by (*). For each neighborhood V of 1 in G, define

$$K(V) := \overline{\{ I_\phi : \operatorname{supp}(\phi) \subset V \}} \subset X.$$

That is, $K(V)$ is the set of pointwise limits of I_ϕ with $\operatorname{supp}(\phi) \subset V$. It is also a compact set as a closed subset of X. We claim $\{K(V) : 1 \in V^\circ \subset G\}$ has the finite intersection property. Indeed, if $V_1, \ldots, V_n \subset G$ are all neighborhoods of 1, then

$$\bigcap_{j=1}^n K(V_j) \supset K\Big(\bigcap_{j=1}^n V_j\Big) \neq \emptyset$$

by definition of $K(\cdot)$. The compactness of X then yields the existence of an

$$I \in \bigcap_{1 \in V^\circ \subset G} K(V)$$

Since I is a pointwise limit of I_ϕ's, 1 and 3 imply I is invariant under left translation and commutes with scalar multiplication. Moreover, since the support of the ϕ's can be made arbitrarily small, the claim implies I also commutes with addition. This allows us to extend I linearly to $C_c(G,\mathbb{R})$ by setting

$$I(f-g) := I(f) - I(g) \qquad \forall f, g \in C_c^+(G).$$

Note that if $f-g = f_1 - g_1$, then $f+g_1 = f_1+g$ so that

$$I(f)+I(g_1) = I(f+g_1) = I(f_1+g) = I(f_1)+I(g),$$

and hence

$$I(f)-I(g) = I(f_1) - I(g_1).$$

We can then further extend I to $C_c(G)$ by

$$I(f) := I(\mathrm{Re}(f)) + i\, I(\mathrm{Im}(f)).$$

Thus I is a linear functional on $C_c(G)$ satisfying $I(C_c^+(G))>0$. The Riesz representation theorem thus yields a Radon measure μ on G satisfying

$$I(f) = \int_G f \, d\mu \qquad \forall f \in C_c(X).$$

The left invariance of implies, by Proposition 2.13.b, that μ is left translation invariant. Finally, note that $\mu \neq 0$ since I was first identified as a function $C_c^+(G) \to (0,\infty)$. This completes the proof of existence.

2/1

(Uniqueness): Let μ and ν both be left Haar measures on G. Recall from Proposition 2.14 that $\int_G f\,d\mu, \int_G f\,d\nu > 0$ for all $f \in C_c^+(G)$. To show $\mu = c\nu$, it therefore suffices to show

$$C_c^+(G) \ni f \longmapsto \frac{\int_G f\,d\nu}{\int_G f\,d\mu}$$

is constant. Fix $f, g \in C_c^+(G)$ and a symmetric compact neighborhood K of 1. Set

$$A := (\mathrm{supp} f)K \cup K(\mathrm{supp} f) \qquad \text{and} \qquad B := (\mathrm{supp}\, g)K \cup K(\mathrm{supp}\, g).$$

Then A, B are compact and for $y \in K$ one has $f(\cdot y) - f(y\,\cdot)$ and $g(\cdot y) - g(y\,\cdot)$ are supported in A and B, respectively. Recalling that f and g are left and right uniformly continuous by Proposition 2.5, for $\varepsilon > 0$ we can find a symmetric neighborhood $V \subset K$ of 1 such that

$$\| f(\cdot y) - f(y\,\cdot)\|_\infty, \ \| g(\cdot y) - g(y\,\cdot)\|_\infty < \varepsilon \qquad \forall y \in V.$$

Let $h \in C_c^+(G)$ satisfy $h(x) = h(x^{-1})$ (take $h = h_0 + h_0(\cdot^{-1})$ for example) and $\mathrm{supp}(h) \subset V$. Then

$$\int_G h\,d\mu \int_G f\,d\nu = \iint_{G\,G} h(y) f(x)\, d\nu(x)\, d\mu(y) = \iint_{G\,G} h(y) f(yx)\, d\nu(x)\, d\mu(y)$$

and

$$\int_G h\,d\nu \int_G f\,d\mu = \iint_G h(x) f(y)\,d\nu(x)\,d\mu(y)$$
$$= \int_G \int_G h(y^{-1}x) f(y)\,d\nu(x)\,d\mu(y)$$
$$= \int_G \int_G h(x^{-1}y) f(y)\,d\nu(x)\,d\mu(y) = \int_G \int_G h(y) f(xy)\,d\nu(x)\,d\mu(y).$$

Thus

$$\left| \int_G h\,d\mu \int_G f\,d\nu - \int_G h\,d\nu \int_G f\,d\mu \right| = \left| \int_G \int_G h(y) \left(f(yx) - f(xy) \right) d\nu(x)\,d\mu(y) \right|$$
$$\leq \varepsilon\, \nu(A) \cdot \int_G h\,d\mu.$$

Dividing by $\int_G h\,d\mu \int_G f\,d\mu$ then yields

$$\left| \frac{\int_G f\,d\nu}{\int_G f\,d\mu} - \frac{\int_G h\,d\nu}{\int_G h\,d\mu} \right| \leq \varepsilon \cdot \frac{\nu(A)}{\int_G f\,d\mu}$$

Similarly,

$$\left| \frac{\int_G g\,d\nu}{\int_G g\,d\mu} - \frac{\int_G h\,d\nu}{\int_G h\,d\mu} \right| \leq \varepsilon \cdot \frac{\nu(B)}{\int_G g\,d\mu}$$

and combining these estimates gives

$$\left| \frac{\int_G f\,d\nu}{\int_G f\,d\mu} - \frac{\int_G g\,d\nu}{\int_G g\,d\mu} \right| \leq \varepsilon \left[\frac{\nu(A)}{\int_G f\,d\mu} + \frac{\nu(B)}{\int_G g\,d\mu} \right]$$

Letting $\varepsilon \to 0$ completes the proof. $\square$

Examples from affine actions on $\mathbb{R}^n$

Let X be a Banach space. Recall that we say $A: X \to X$ is affine if there exists an invertible $T \in B(X)$ satisfying

$$A(x) - A(y) = T(x-y)$$

for all $x, y \in X$. Equivalently,

$$A(x) = T(x) + A(0)$$

for all $x \in X$. In particular, T is uniquely determined by A. Denote the set of affine maps on X by $\mathrm{Aff}(X)$.

Each $A \in \mathrm{Aff}(X)$ is Lipschitz continuous since

$$\|A(x) - A(y)\| = \|T(x-y)\| \leq \|T\| \cdot \|x-y\|,$$

and invertible with

$$A^{-1}(v) = T^{-1}(v - A(0)) = T^{-1}(v) - A^{-1}(0).$$

$Aff(X)$ is also closed under composition:

$$A_1 \circ A_2(x) = T_1(T_2(x) + A_2(0)) + A_1(0) = T_1 \circ T_2(x) + A_1 \circ A_2(0)$$

Hence $Aff(X)$ is a group.

Exercise For affine maps $A_i(x) = T_i(x) + A_i(0)$, $i=1,2$, on a Banach space X, show that

$$d(A_1, A_2) := \|T_1 - T_2\| + \|A_1(0) - A_2(0)\|$$

defines a metric making $Aff(X)$ a topological group. When is it locally compact? □

Def Let G be a locally compact group. A (continuous) affine action of G on a Banach space X is a continuous group homomorphism

$$A : G \to Aff(X).$$
$$x \mapsto A_x$$

In this case we write T_x for the associated invertible linear transformation. □

Proposition 2.16 Let G be a locally compact group and let m be the Lebesgue measure on $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Suppose there exists a homeomorphism $\pi : G \to \pi(G)$ onto a Borel subset $\pi(G) \subset \mathbb{R}^n$ with $m(\pi(G)) > 0$ and a continuous affine action $A: G \to \mathbb{R}^n$ such that

$$\pi(xy) = A_x(\pi(y)) = T_x(\pi(y)) + A_x(0)$$

Then the left Haar measure on G is given by

$$d\mu = |\det(T)|^{-1}\, d(\pi^* m),$$

where $\pi^* m(E) = m(\pi(E))$ is the pullback of m by π.

Proof First note that, since m is inner and outer regular on all Borel sets, the restriction of m to $\pi(G)$ (with the subspace topology) is a non-trivial Radon measure (Exercise check this). Then $\pi^* m$ is also a non-trivial Radon measure on G since π is a homeomorphism. In fact $\pi^* m$ is both inner and outer regular on all Borel sets. Since

$$G \ni x \mapsto |\det T_x|^{-1} \in (0, \infty)$$

is continuous, μ defined as above is a Radon measure by Proposition 2.12. By Proposition 2.13.b, it therefore suffices to show:

$$\int_G L_y f\, d\mu = \int_G f\, d\mu \qquad \forall\, y \in G,\ f \in C_c(G)$$

Fix $f \in C_c(G)$. Then $f \circ \pi^{-1}(t)\, |\det(T_{\pi^{-1}(t)})|^{-1} \in L^1(\mathbb{R}^n, m)$ (it may fail to be continuous if $\pi(G)$ is not open). For $y \in G$, note that $A_{y^{-1}}: \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism with

$$D_t A_{y^{-1}} = T_{y^{-1}} \qquad \forall t \in \mathbb{R}^n.$$

Thus the Lebesgue change of variables formula (see Theorem 2.47 in Folland's Real Analysis book) gives us:

$$\int_G L_y f\, d\mu = \int_G f(y^{-1}x)\, |\det(T_x)|^{-1}\, d(\pi^* m)(x)$$

$$= \int_{\mathbb{R}^n} f(y^{-1}\pi^{-1}(t))\, |\det(T_{\pi^{-1}(t)})|^{-1}\, dm(t)$$

$$= \int_{\mathbb{R}^n} f \circ \pi^{-1}(A_{y^{-1}}(t))\, |\det(T_{y^{-1}\pi^{-1}(t)})|^{-1}\, |\det(T_{y^{-1}})|\, dm(t)$$

$$= \int_{\mathbb{R}^n} f \circ \pi^{-1}(A_{y^{-1}}(t))\, |\det(T_{\pi^{-1}(A_{y^{-1}}(t))})|^{-1}\, |\det(T_{y^{-1}})|\, dm(t)$$

$$= \int_{\mathbb{R}^n} f \circ \pi^{-1}(s)\, |\det(T_{\pi^{-1}(s)})|^{-1}\, dm(s) = \int_G f\, d\mu \qquad \square$$

2/5

The previous proposition allows us to give fairly explicit descriptions of Haar measures. Note that similar argument shows

$$d\nu = |\det(S)|^{-1}\, d(\pi^* m)$$

is a right Haar measure whenever $\pi(xy) = B_y(\pi(x)) = S_y(\pi(x)) + B_y(0)$ is another affine action.

Ex ① For $\mathbb{R}_+$ with multiplication, if $\iota: \mathbb{R}_+ \to \mathbb{R}$ is the canonical embedding then

$$\iota(xy) = xy = A_x(\iota(y))$$

where $A_x = T_x \in B(\mathbb{R})$ is multiplication by x. Thus $|x|^{-1} dm(x)$ is the (left and right) Haar measure.

This can also be verified by noting that $\exp: \mathbb{R} \to \mathbb{R}_+$ is an isomorphism as topological groups and hence the pushforward $\exp_* m$ must be the Haar measure on $\mathbb{R}_+$. For $f \in C_c(\mathbb{R}_+)$ we have

$$\int_{\mathbb{R}_+} f\, d(\exp_* m) = \int_{\mathbb{R}} f(e^t)\, dm(t) = \int_{\mathbb{R}} f(s) \tfrac{1}{s}\, dm(s)$$

(2) For $\mathbb{C}\setminus\{0\}$ with multiplication, if $\pi : \mathbb{C}\setminus\{0\} \to \mathbb{R}^2$ is given by $\pi(z) = (\mathrm{Re}(z), \mathrm{Im}(z))$ then

$$\pi(zw) = \begin{pmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix} (\mathrm{Re}(w), \mathrm{Im}(w))$$

Thus $|\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2|^{-1} \, dm(\pi(z)) = |x^2+y^2| \, dm(x,y)$ is the (left and right Haar measure).

(3) For $x \in GL_n(\mathbb{R})$, let $x_1, \ldots, x_n \in \mathbb{R}^n$ denote the columns of x. Define $\pi : GL_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ by

$$\pi(x) = \pi(x_1 | \cdots | x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Note $m(\pi(GL_n(\mathbb{R}))) > 0$ since

$$\mathbb{R}^{n^2} \setminus \pi(GL_n(\mathbb{R})) = \pi(M_n(\mathbb{R}) \setminus GL_n(\mathbb{R})) = \pi(\det^{-1}(\{0\}))$$

is a n^2-1 dimensional submanifold and hence m-null. Also

$$\pi(xy) = \pi(xy_1 | \cdots | x y_n)$$
$$= \begin{pmatrix} xy_1 \\ \vdots \\ xy_n \end{pmatrix} = \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

so that

$$A_x = T_x = \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix}$$

and hence $\det T_x = \det(x)^n$. Thus $\det(x)^{-n} \, dm(\pi(x))$ is a left Haar measure for $GL_n(\mathbb{R})$. To see that this is also a right Haar measure, note that

$$\pi(x^T) = S \cdot \pi(x)$$

for a fixed permutation matrix $S \in M_{n^2}(\mathbb{R})$. Thus

$$\det(x^T)^{-n} \, dm(\pi(x^T)) = \det(x)^{-n} \det(S)^{-1} dm(\pi(x)) = \det(x)^{-n} dm(\pi(x)) = d\mu(x)$$

So $\mu(x) = \mu(x^T)$ and the latter is a right Haar measure.

(4) Consider the Heisenberg group

$$H_n(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in M_n(\mathbb{R}) \right\}.$$

It is a subgroup of $GL_n(\mathbb{R})$, but a null set for its Haar measure. So we cannot use the previous example to immediately compute its Haar measure, but we can do so indirectly. Let $\pi : GL_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ be as in the previous example and define $\pi' : H_n(\mathbb{R}) \to \mathbb{R}^{n(n-1)/2}$ by

$$\pi'(x) := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} \qquad \text{where} \qquad x = \begin{pmatrix} 1 & x_1 & & & & \\ 0 & 1 & & x_j & & \\ & 0 & & & & \\ \vdots & \vdots & \cdots & 1 & \cdots & x_{n-1} \\ & & & 0 & & \\ & & & \vdots & & \\ 0 & 0 & & 0 & & 1 \end{pmatrix}$$

Note that $\pi'(I_n) = 0$. Let $V : \mathbb{R}^{n(n-1)/2} \to \mathbb{R}^{n^2}$ be the affine map uniquely determined by

$$V\pi'(x) = \pi(x) \qquad \forall x \in H_n(\mathbb{R}).$$

Then

$$V\pi'(x) = W\pi'(x) + V(0)$$

where

$$W = \begin{array}{cc} \begin{pmatrix} 0 & & \cdots & & 0 \\ \begin{smallmatrix} I_1 \\ 0 \end{smallmatrix} & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ & & \begin{smallmatrix} I_j \\ 0 \end{smallmatrix} & & \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & \begin{smallmatrix} I_{n-1} \\ 0 \end{smallmatrix} \end{pmatrix} & \begin{matrix} I_n \\ I_n \\ \vdots \\ I_n \\ \\ I_n \end{matrix} \\ \begin{matrix} 1 & \quad j & \quad\quad n-1 \end{matrix} & \end{array} \qquad \text{and} \qquad V(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \hline \vdots \\ \hline 0 \\ \vdots \\ 1 \end{pmatrix} = \pi(I_n)$$

Using the computation in the previous example we have

$$V\pi'(xy) = \pi(xy) = \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix}\pi(y) = \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} V\pi'(y)$$

so that

$$\pi'(xy) = V^{-1}\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} V\pi'(y)$$

Note that $A_x := V^{-1}\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} V$ satisfies

$$A_x(0) = V^{-1}\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix}\pi(I_n) = V^{-1}\pi(x) = \pi'(x)$$

and

$$A_x\pi'(y) = W^T\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} W\pi'(y) + \pi'(x)$$

Thus

$$T_x = W^T\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} W = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_{n-1} \end{pmatrix} \qquad *$$

where m_j is the jth minor of x for each $j = 1, \ldots, n-1$. Noting $\det(m_j) = 1$ (since m_j is upper triangular with 1's on the diagonal), we see that

$$dm(\pi'(x)) = \prod_{i<j} dx_{ij}$$

is the left Haar measure for $H_n(\mathbb{R})$.

Heuristically, in the product xy note that y_j only interacts with m_j for each $j=1,\dots,n$, which justifies (✱). Similarly, if we write

$$y = \begin{pmatrix} 1 & \tilde{y}_1 & & \\ & 1 & \tilde{y}_2 & \\ & & \ddots & \\ 0 & & 1 & \tilde{y}_{n-1} \\ & & & 1 \end{pmatrix},$$

Then in the product $yx = (x^T y^T)^T$, $\tilde{y}_j$ only interacts with $\tilde{m}_j$ for each $j=1,\dots,n$ where $\tilde{m}_j$ is the bottom right minor of X^T. Thus if

$$\pi''(y) := \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_{n-1} \end{pmatrix} \in \mathbb{R}^{\frac{n(n-1)}{2}},$$

it follows that

$$\pi''(xy) = \begin{pmatrix} \tilde{m}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{m}_{n-1} \end{pmatrix} \pi''(y) + \pi''(x)$$

and hence

$$\left|\det\begin{pmatrix} \tilde{m}_1 & & 0 \\ 0 & \ddots & \tilde{m}_{n-1} \end{pmatrix}\right|^{-1} d\mathrm{m}(\pi''(x)) = \prod_{i<j} dx_{ij}$$

so that the left Haar measure is also a right Haar measure.

5 The "$ax+b$ group" is the subgroup of $Gl_2(\mathbb{R})$ given by

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a>0,\ b\in\mathbb{R} \right\}$$

For $\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, one has

$$\pi\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} ac \\ ad+b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

so that for $x = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ one has $T_x = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $\det(T_x) = a^2$. So $a^{-2}\,da\,db$ is a left Haar measure on G.

On the other hand

$$\pi\left(\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} ca \\ cb+d \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

so that for $y = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ one has $\det(S_y) = a$, and hence $a^{-1}\,da\,db$ is a right Haar measure on G.

6 Let us compute the Haar measure μ for $\mathbb{Q}_p$. Recall that $\mathbb{Z}_p$ is both compact and open. So $0 < \mu(\mathbb{Z}_p) < \infty$ and hence we assume $\mu(\mathbb{Z}_p) = 1$, rescaling if necessary. We claim that

$$\mu(\overline{B}(x, p^m)) = p^m$$

for all $x \in \mathbb{Q}_p$ and $m \in \mathbb{Z}$. By translation invariance, it suffices to consider $x=0$.

Recall from the proof of Theorem 27 that for each $k \in \mathbb{N}$ one has

$$\mathbb{Z}_p = \bigsqcup_{0 \le x < p^k} (x + p^k \mathbb{Z}_p) = \bigsqcup_{0 \le x < p^k} \bar{B}(x, p^{-k}) \quad {**}$$

Thus

$$p^k \cdot \mu(\bar{B}(0, p^{-k})) = \sum_{0 \le x < p^k} \mu(\bar{B}(x, p^{-k})) = \mu(\mathbb{Z}_p) = 1$$

so that the claim holds for $m = -k$. Scaling $(**)$ by p^{-k} (recall $(|p^{-k}x|_p = |p^{-k}|_p |x|_p = p^k |x|_p)$ we obtain

$$\bar{B}(0, p^k) = p^{-k} \mathbb{Z}_p = \bigsqcup_{0 \le x < p^k} \bar{B}(x, 1)$$

and hence

$$\mu(\bar{B}(0, p^k)) = \sum_{0 \le x < p^k} \mu(\bar{B}(x, 1)) = p^k \mu(\mathbb{Z}_p) = p^k.$$

So the claim holds.

Now, given an open set $U \subset \mathbb{Q}_p$, we can write

$$U = \bigcup_{x \in U} \bar{B}(x, p^{m_x})$$

for some function $U \ni x \mapsto \mathbb{Z}$. Recalling that $\bar{B}(x, p^{m_x}) = \bar{B}(y, p^{m_x})$ for any $y \in \bar{B}(x, p^{m_x})$, we can reduce the above union to a countable disjoint union by picking a representative from $\mathbb{Q} \cap \bar{B}(x, p^{m_x})$ for each ball. Thus

$$U = \bigsqcup_{i \in I} \bar{B}(x_i, p^{m_i})$$

for some countable set I, and

$$\mu(U) = \sum_{i \in I} p^{m_i}$$

Using the outer regularity of μ we then have

$$\mu(E) = \inf \left\{ \sum_{i \in I} p^{m_i} : E \subset \bigcup_{i \in I} \bar{B}(x_i, p^{m_i}),\ I \text{ countable} \right\}$$

for any Borel set $E \subset \mathbb{Q}_p$. Compare the above to the formula for the Lebesgue measure on $\mathbb{R}$.

2/8

7 Recall that $|xy|_p = |x|_p |y|_p$. So if $|y|_p = p^k$ for $k \in \mathbb{Z}$, then the claim in previous example implies

$$\mu(y \cdot \bar{B}(x, p^m)) = \mu(\bar{B}(yx, p^{m+k})) = p^{m+k} = |y|_p\, p^m$$

It then follows that $\mu(y \cdot E) = |y|_p \mu(E)$ for all Borel sets E and $y \in \mathbb{Q}_p \setminus \{0\}$. Thus if $\mu_y(E) := \mu(y \cdot E)$ then $d\mu_y / d\mu = |y|_p$, and we claim

$$d\nu := |x|_p^{-1}\, d\mu(x)$$

is the left Haar measure for $\mathbb{Q}_p\setminus\{0\}$ with multiplication. That ν is a Radon measure follows Proposition 2.12 and denoting $\nu_y(E) := \nu(y\cdot E)$ we have

$$\frac{d\nu_y}{d\nu}(x) = \frac{d\nu_y}{d\mu_y}(x)\cdot\frac{d\mu_y}{d\mu}(x)\cdot\frac{d\mu}{d\nu}(x) = |yx|_p^{-1}\cdot|y|_p\cdot|x|_p = 1$$

so that $\nu_y = \nu$ and hence ν is a left (and right) Haar measure. $\square$

Haar measures on product groups

If $G_1, \ldots, G_n$ are locally compact groups, then their product

$$G := G_1\times\cdots\times G_n$$

is a locally compact group with entrywise operations and the product topology. If μ_j is a left Haar measure on G_j for $j=1,\ldots,n$, then

$$\varphi\colon C_c(G)\ni f\longmapsto \int_{G_n}\cdots\int_{G_1} f(x_1,\ldots,x_n)\,d\mu_1(x_1)\cdots d\mu_n(x_n)$$

is a non-trivial, left translation invariant functional satisfying $\varphi(C_c^+(G))\geq 0$. Thus is μ is the unique Radon measure associated to φ by the Riesz representation theorem (Theorem 2.8), then μ is a left Haar measure for G. This measure is also known as the Radon product of $\mu_1,\ldots,\mu_n$ and is denoted

$$\mu_1\hat{\times}\cdots\hat{\times}\mu_n := \mu.$$

How does this compare to the usual product measure? If the G_j's are σ-compact, then it is an extension of $\mu_1\times\cdots\times\mu_n$ from $\mathcal{B}_{G_1}\otimes\cdots\otimes\mathcal{B}_{G_n}$ to $\mathcal{B}_G$. If the G_j's are second countable, then these σ-algebras agree and hence so do $\mu_1\times\cdots\times\mu_n$ and $\mu_1\hat{\times}\cdots\hat{\times}\mu_n$. In general, however, they can be quite different as the following example illustrates.

$\boxed{\text{Ex}}$ Let $\mathbb{R}$ and $\mathbb{R}_d$ denote the real numbers with the usual and discrete topologies, respectively, and the additive group structure. Note that $\mathbb{R}$ is second countable, but $\mathbb{R}_d$ is not. Their Haar measures are the Lebesgue measure m and the counting measure $\#$. For the product measure on $\mathbb{R}\times\mathbb{R}_d$, we have

$$m\times\#(\{0\}\times\mathbb{R}_d) = m(\{0\})\,\#(\mathbb{R}_d) = 0\cdot\infty = 0$$

under the usual convention $(0\cdot\infty=0)$ taken for product measures. On the other hand

$$m\hat{\times}\#(\{0\}\times\mathbb{R}_d) = \infty.$$

Indeed, by outer regularity this will follow from $m \hat{\times} \#(\mathcal{U}) = \infty$ for all open sets $\mathcal{U} \supset \{0\} \times \mathbb{R}_d$. Fix one such open set $\mathcal{U}$, and note that for each $x \in \mathbb{R}_d$ $\exists\, \varepsilon_x > 0$ so that $(-\varepsilon_x, \varepsilon_x) \times \{x\} \subset \mathcal{U}$. Since $\mathbb{R}_d$ is uncountable, $\exists\, n \in \mathbb{N}$ so that

$$\{x \in \mathbb{R}_d : \varepsilon_x \geq \tfrac{1}{n}\}$$

is an infinite set. Let $\{x_k : k \in \mathbb{N}\}$ be a countable subset of the above. Then

$$m \hat{\times} \#(\mathcal{U}) \geq m \hat{\times} \#\Big(\bigcup_{k=1}^{\infty} (-\tfrac{1}{n}, \tfrac{1}{n}) \times \{x_k\}\Big) = \sum_{k=1}^{\infty} \frac{2}{n} \cdot 1 = \infty.$$

The same argument shows $m \times \#$ and $m \hat{\times} \#$ disagree on any set of the form $I \times J \subset \mathbb{R} \times \mathbb{R}_d$ where $I \neq \emptyset$ is m-null and J is uncountable. Also that $m \hat{\times} \#(K) = 0$ for all compact $K \subset I \times J$ since $K_x = \{y \in \mathbb{R}_d : (x,y) \in K \text{ for some } x \in \mathbb{R}\}$ is necessarily finite and $K^y \subseteq I$ is m-null. $\square$

One can also make sense of infinite products of locally compact groups provided they admit a compact open subgroup to "anchor" a left Haar measure at 1.

$\boxed{\text{Def}}$ Let $\{G_i : i \in I\}$ be a family of locally compact groups each admitting a compact open subgroup $K_i \leq G_i$. Their restricted direct product is the group

$$\prod_{i \in I} (G_i, K_i) := \varinjlim \Big(\prod_{i \in F} G_i \times \prod_{i \notin F} K_i \Big),$$

where the direct limit is over finite subsets $F \subset I$, with pointwise group operations. $\square$

Note that each $\prod_{i \notin F} K_i$ is compact by Tychonoff's theorem so that

$$H_F := \prod_{i \in F} G_i \times \prod_{i \notin F} K_i$$

is a locally compact group. Also, if $E \subset F$ then H_E is an open subgroup of H_F since $K_i \leq G_i$ is open for each $i \in F \setminus E$. Thus the restricted direct product is a locally compact group. As a group, it can be identified with

$$\bigcup_{F \subset\subset I} \Big(\prod_{i \in F} G_i \times \prod_{i \notin F} K_i \Big) \leq \prod_{i \in I} G_i,$$

but the subspace topology does not give the right topology on $\prod (G_i, K_i)$ (unless I is finite) since H_F is not open in $\prod G_i$.

Suppose μ_i is a left Haar measure on G_i satisfying $\mu_i(K_i) = 1$ for all $i \in I$. For each finite set $F \subset I$, observe

$$C_c\Big(\prod_{i \in F} G_i\Big) \otimes \mathbb{C} \leq C_c(H_F) \leq C_c\Big(\prod_{i \in I} (G_i, K_i)\Big)$$

since $1 \in C(\prod K_i) = C_c(\prod K_i)$. Also

$$\bigcup_{F \subset\subset I} C_c\Big(\prod_{i \in F} G_i\Big) \otimes \mathbb{C} \subset C_c\Big(\prod_{i \in I} (G_i, K_i)\Big)$$

is dense by the Stone-Weierstrass theorem. Thus the following linear functional extends uniquely to continuous compactly supported functions on the restricted direct product: for $f \otimes 1 \in C_c(G_{i_1} \times \cdots \times G_{i_n}) \otimes \mathbb{C}$

$$\varphi(f \otimes 1) = \int f \, d(\mu_{i_1} \hat{\times} \cdots \hat{\times} \mu_{i_n}).$$

The measure associated to this linear functional via the Riesz representation theorem (Theorem 2.8) is a left Haar measure for $\prod_i (G_i, K_i)$ and we denote it by

$$\widehat{\prod_{i \in I}} \mu_i .$$

Note that if $\{G_i : i \in I\}$ is a family of compact groups, then we can apply the above construction to $K_i := G_i$ and we obtain

$$\prod_{i \in I} (G_i, G_i) = \prod_{i \in I} G_i,$$

which is also a compact group.

$\boxed{\text{Ex}}$ Denote

$$(\mathbb{Z}/2\mathbb{Z})^\omega := \prod_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$$

and

$$\mu_\omega := \widehat{\prod_{n \in \mathbb{N}}} \left(\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_1\right)$$

Then $\pi : (\mathbb{Z}/2\mathbb{Z})^\omega \to [0,1]$ defined by

$$\pi((x_n)_{n \in \mathbb{N}}) := \sum_{n=1}^\infty x_n 2^{-n}$$

is continuous and surjective. It is not injective since

$$2^{-m} = \sum_{n=m+1}^\infty 2^{-n},$$

but it does satisfy $\pi_* \mu = m$ (see the textbook for details). □

2/15 (continued)

Haar measure on semidirect products

Suppose $H \overset{\alpha}{\curvearrowright} N$ is an action of locally compact groups with Haar measures μ_H and μ_N, respectively. We claim that $\mu_H \hat{\times} \mu_N$ is a left Haar measure for $H \ltimes_\alpha N$. Indeed, for $f \in C_c(H)$ and $g \in C_c(N)$ we have

$$\int_N \int_H L_{(s,t)}(f\otimes g)(x,y)\, d\mu_H(x)\, d\mu_N(y) = \int_N \int_H f(sx)\, g(\alpha_x(t)\, y)\, d\mu_H(x)\, d\mu_N(y)$$
$$= \int_H (L_s f)(x) \int_N (L_{\alpha_x(t)}\, g)(y)\, d\mu_N(y)\, d\mu_H(x)$$
$$= \int_H f(x) \int_N g(y)\, d\mu_N(y)\, d\mu_H(x).$$

Since functions of the form $f\otimes g$ densely span $C_c(H \ltimes N)$, this shows $\mu_H \hat{\times} \mu_N$ is left invariant.

Now suppose ν_H and ν_N are right Haar measures of H and N, respectively. In light of the above, it would be natural for $\nu_H \hat{\times} \nu_N$ to be a right Haar measure of $H \ltimes_\alpha N$, but this turns out to not be true in general. Observe that for each $x\in H$, $y\in N$, and $E\subset N$ Borel one has

$$\nu_N(\alpha_x(E\cdot y)) = \nu_N(\alpha_x(E)\cdot \alpha_x(y)) = \nu_N(\alpha_x(E)).$$

Thus $\nu_N\circ\alpha_x$ is a right Haar measure for N and therefore

$$\nu_N\circ\alpha_x = c_x\, \nu_N$$

for some constant $c_x>0$ by the uniqueness of Haar measures. Also observe that $c_1 = 1$ and

$$c_{xy} = \frac{d\nu_N\circ\alpha_{xy}}{d\nu_N} = \frac{d\nu_N\circ\alpha_x\circ\alpha_y}{d\nu_N\circ\alpha_x}\cdot\frac{d\nu_N\circ\alpha_x}{d\nu_N} = c_y\, c_x.$$

Thus $H\ni x\mapsto c_x\in\mathbb{R}_\times$ is a group homomorphism. In fact it is continuous (see Exercises).

Proposition 2.17 Let $H \overset{\alpha}{\curvearrowright} N$ be an action of locally compact groups. Then $\mu_H\hat{\times}\mu_N$ is a left Haar measure for $H\ltimes_\alpha N$, where μ_H and μ_N are left Haar measures on H and N, respectively, and

$$d\nu(x,y) := \frac{d(\nu_N\circ\alpha_x)}{d\nu_N}(y)\, d(\nu_H\hat{\times}\nu_N)(x,y)$$

is a right Haar measure for $H\ltimes_\alpha N$, where ν_H and ν_N are right Haar measures on H and N, respectively.

Proof We have already seen that $\mu_H\hat{\times}\mu_N$ is a left Haar measure. Denote

$$c_x := \frac{d(\nu_N\circ\alpha_x)}{d\nu_N}(y),$$

and recall that $x\mapsto c_x$ is a group homomorphism. For $f\in C_c(H\ltimes_\alpha N)$ observe

$$[R(x,y) f](s,t) = f((s,t)\cdot(x,y)) = f(sx, \alpha_x^{-1}(t)\, y)$$

so that

$$\int_{H\ltimes N} R_{(x,y)} f \, d\upsilon = \int_N \int_H f(sx, \alpha_x^{-1}(t)y) \; c_s \; d\nu_H(s) \, d\nu_N(t)$$

$$= \int_N \int_H f(s, \alpha_x^{-1}(t)) \; c_s c_x^{-1} \, d\nu_H(s) \, d\nu_N(t)$$

$$= \int_N \int_H f(s,t) \; c_s \, d\nu_H(s) \;\; c_x^{-1} \, d(\nu_N \circ \alpha_x)(t)$$

$$= \int_N \int_H f(s,t) \; c_s \, d\nu_H(s) \, d\nu_N(t) = \int_{H\ltimes N} f \, d\upsilon. \qquad \square$$

By a symmetric argument, one also has the following

Corollary 2.18 Let $H \overset{\alpha}{\curvearrowright} N$ be an action of locally compact groups. Then $\nu_N \hat{\times} \nu_H$ is a right Haar measure for $N \rtimes_\alpha H$, where ν_N and ν_H are right Haar measures on N and H, respectively, and

$$d\mu(x,y) := \frac{d\mu_N}{d(\mu_N \circ \alpha_y)}(x) \; d(\mu_N \hat{\times} \mu_H)(x,y)$$

is a left Haar measure for $N \rtimes_\alpha H$, where μ_N and μ_H are left Haar measures on N and H, respectively.

Note that even if μ_H and μ_N are left and right Haar measures, then $\mu_H \hat{\times} \mu_N$ (resp. $\mu_N \hat{\times} \mu_H$) can fail to be a right (resp. left) Haar measure on $H \,_\alpha\!\ltimes N$ (resp. $N \rtimes_\alpha H$). This is illustrated in the following example

Ex Recall that $d\mu(x) = \frac{1}{x} dm(x)$ gave a left and right Haar measure on $\mathbb{R}_+$. Thus

$$d(m \hat{\times} \mu)(b,a) = dm(b) \, d\mu(a) = \frac{1}{a} dm(a) \, dm(b)$$

is a right Haar measure on $\mathbb{R} \rtimes \mathbb{R}_+$ by Corollary 2.15. Using $m(a \cdot E) = a\, m(E)$ for $a \in \mathbb{R}_+$, the same corollary implies

$$\frac{1}{a} d(m \hat{\times} \mu)(b,a) = \frac{1}{a^2} dm(a) \, dm(b)$$

is a left Haar measure for $\mathbb{R} \rtimes \mathbb{R}_+$. These match our computations for the left and right Haar measures of the $ax+b$ group, which we showed could be identified with $\mathbb{R} \rtimes \mathbb{R}_+$ in a Section 2.1 example. $\square$

Exercise Let $\mathbb{Q}_p \setminus \{0\}$ act on $\mathbb{Q}_p$ by multiplication. Compute the Haar measures for $(\mathbb{Q}_p \setminus \{0\}) \ltimes \mathbb{Q}_p$ and $\mathbb{Q}_p \rtimes (\mathbb{Q}_p \setminus \{0\})$. $\square$

2.4 The Modular Fuction

Let G be a locally compact group with left Haar measure μ. In this section we will quantify how far μ is from being a right Haar measure. We begin with a few observations. For $x \in G$,

$$\mathcal{B}_G \ni E \mapsto \mu(Ex)$$

defines a left Haar measure since $y(Ex) = y(Ex)$ for $y \in G$. Thus the uniqueness of Haar measures yield a constant $\Delta(x) \in (0,\infty)$ so that $\mu(Ex) = \Delta(x)\mu(E)$ for all Borel sets E. Note that this same uniqueness implies Δ does not depend on μ.

Def $\Delta : G \to (0,\infty)$ defined as above is called the modular function of G. □

Proposition 2.19 The modular function is a continuous homomorphism $\Delta : G \to \mathbb{R}_+$. Moreover, for any $f \in L^1(G,\mu)$ one has

$$\int_G R_y f \, d\mu = \Delta(y^{-1}) \int_G f \, d\mu \qquad *$$

for all $y \in G$. In particular, $d\mu(xy) = \Delta(y)\, d\mu(x)$.

Proof For $x, y \in G$ and $E \in \mathcal{B}_G$ we have

$$\Delta(xy)\mu(E) = \mu(E(xy)) = \mu((Ex)y) = \Delta(y)\mu(Ex) = \Delta(x)\Delta(y)\mu(E),$$

so that Δ is a group homomorphism. Also, using $R_y 1_E(x) = 1_E(xy) = 1_{Ey^{-1}}(x)$ we have

$$\int_G R_y 1_E \, d\mu = \int_G 1_{Ey^{-1}} d\mu = \mu(Ey^{-1}) = \Delta(y^{-1})\mu(E) = \Delta(y^{-1}) \int_G 1_E \, d\mu.$$

So $(*)$ holds by approximating $f \in L^1(G,\mu)$ by simple functions. Note that it implies

$$\Delta(y) \int_G f \, d\mu = \int_G f(xy^{-1}) \, d\mu(x) = \int_G f(x) \, d\mu(xy),$$

so that $\Delta(y) d\mu(x) = d\mu(xy)$ as claimed. Lastly, to check the continuity of Δ, fix $f_0 \in C_c^+(G)$ and note that $(*)$ implies

$$\Delta(y) = \frac{\int_G f_0 \, d\mu}{\int_G R_y \, d\mu}$$

so the continuity follows from the exercise below. □

Exercise Let G be a locally compact group with a left Haar measure μ.

(a) For all $K \subset G$ compact show there exists $\tilde{K} \subset G$ compact with $K \subset \tilde{K}^\circ$.

(b) Let U be a neighborhood of a compact set $K \subset G$. Show there exists a symmetric neighborhood V of $1 \in G$ so that $yK, Ky \in G$ for all $y \in V$.

(c) Show that for all $f \in C_c(G)$,
$$y \mapsto \int_G L_y f \, d\mu \qquad \text{and} \qquad y \mapsto \int_G R_y f \, d\mu$$
are continuous. □

Def We say G is <u>unimodular</u> if its modular function is trivial: $\Delta \equiv 1$. That is, if its left Haar measures are also right Haar measures. □

abelian

Many of the concrete examples we have seen are unimodular: $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{C}\setminus\{0\}$, $\mathbb{T}$, $\mathbb{Q}_p$, $\mathbb{Q}_p\setminus\{0\}$, $GL_n(\mathbb{R})$, and $H_n(\mathbb{R})$. In addition to abelian groups and discrete groups, the following results give additional classes of unimodular groups.

Proposition 2.20 Let G be a locally compact group with modular function $\Delta: G \to \mathbb{R}_+$. If $K \subset G$ is a compact subgroup, then $\Delta|_K \equiv 1$.

Proof One has that $\Delta(K)$ is a compact subgroup of $\mathbb{R}_+$; that is, $\Delta(K) = \{1\}$. □

Corollary 2.21 Compact groups are unimodular.

Def Let G be a topological group. The <u>commutator subgroup</u> of G, denoted $[G,G]$, is the subgroup generated by elements of the form $[x,y] := xyx^{-1}y^{-1}$ for $x,y \in G$. □

Exercise Show $[G,G]$ is a closed and normal subgroup of G.

Proposition 2.22 Let G be a locally compact group. If $G/[G,G]$ is compact, then G is unimodular.

Proof Note that
$$\Delta([x,y]) = \Delta(x)\Delta(y)\Delta(x)^{-1}\Delta(y)^{-1} = 1$$
since $\mathbb{R}_+$ is abelian. So $[G,G] \subset \ker(\Delta)$ and therefore Δ factors through the quotient $G/[G,G]$:

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "q"'] & \mathbb{R}_+ \\ G/[G,G] \arrow[ur, dashed, "\exists \pi"'] & \end{tikzcd}$$

But then $\Delta(G) = \pi(G/[G,G])$ is a compact subgroup of $\mathbb{R}_+$ and hence is $\{1\}$. □

2/22

The $ax+b$ group is an example of a non-unimodular group, since we previously saw its left and right measures differed by more than a constant. In fact, the following proposition implies the modular function of the $ax+b$ group is given by
$$\Delta\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \frac{1}{a}.$$

Proposition 2.23 Let G be a locally compact group with left Haar measure μ and right Haar measure $d\nu(x) := d\mu(x^{-1})$. Then μ and ν are strongly equivalent with

$$d\nu(x) = \Delta(x^{-1})\, d\mu(x) \qquad **$$

Proof By (*) in Proposition 2.19, for $f \in C_c(G)$ we have

$$\int_G [R_y f](x)\, \Delta(x^{-1})\, d\mu(x) = \Delta(y) \int_G f(xy)\, \Delta((xy)^{-1})\, d\mu(x) = \int_G f(x)\, \Delta(x^{-1})\, d\mu(x).$$

So $\Delta(x^{-1})\, d\mu(x)$ is a right translation invariant. It is also a Radon measure by Proposition 2.12. Thus it is right Haar measure and therefore $\Delta(x^{-1})d\mu(x) = c\, d\nu(x)$ for some constant $0 < c < \infty$. It remains to show $c = 1$.

Suppose, towards a contradiction, that $c \neq 1$. Then there exists a compact symmetric neighborhood V of $1 \in G$ such that

$$|\Delta(x^{-1}) - 1| \leq \tfrac{1}{2}|c - 1| \qquad \forall\, x \in V.$$

Since $\mu(V) = \mu(V^{-1}) = \nu(V)$, it follows that

$$|c-1| \cdot \mu(V) = |c\nu(V) - \mu(V)| = \left|\int_V [\Delta(x^{-1}) - 1]\, d\mu(x)\right| \leq \tfrac{1}{2}|c-1| \cdot \mu(V),$$

a contradiction. □

Note that $d\nu(x) = d\mu(x^{-1})$ and (**) further imply:

$$d\mu(x^{-1}) = \Delta(x^{-1})\, d\mu(x) \quad \text{and} \quad d\nu(x^{-1}) = \Delta(x)\, d\nu(x)$$

Exercise For $\alpha \in \mathrm{Aut}(G)$ show that $\Delta \circ \alpha = \Delta$. □

Since Δ is unbounded as soon as it is non-trivial, it follows that $L^p(G,\mu) \neq L^p(G,\nu)$ for $1 \leq p < \infty$. (But $L^\infty(G,\mu) = L^\infty(G,\nu)$ since μ and ν are equivalent by the previous proposition.) However, both

$$f \mapsto \check{f}(x) := f(x^{-1}) \qquad \text{and} \qquad f \mapsto \Delta^{1/p} f$$

define isometric isomorphisms from $L^p(G,\nu)$ to $L^p(G,\mu)$ (Exercise check this). Consequently the mapping

$$f \mapsto \Delta^{-1/p}\check{f}$$

gives an isometric isomorphism of $L^p(G,\mu)$ onto itself.

Ex ① Let $H \overset{\alpha}{\curvearrowright} N$ be an action of locally compact groups, which have left Haar measures μ_H and μ_N, respectively. Recall that $\mu_H \times \mu_N$ is a left Haar measure for $H \ltimes_\alpha N$ and let Δ denote its modular function. For $x \in H$, $y \in N$ and compact sets $A \subset H$ and $B \subset N$ we have

$$
\begin{aligned}
\Delta(x,y)\, \mu_H \hat{\times} \mu_N(A\times B) &= \mu_H \hat{\times} \mu_N((A\times B)\cdot(x,y)) \\
&= \mu_H \hat{\times} \mu_N((A\cdot x) \times (\alpha_x^{-1}(B)\cdot y)) \\
&= \mu_H(Ax)\, \mu_N(\alpha_x^{-1}(B) y) \\
&= \Delta_H(x)\mu_H(A)\, \Delta_N(y)\, \mu_N(\alpha_x^{-1}(B)) \\
&= \Delta_H(x)\Delta_N(y) \frac{d(\mu_N\circ\alpha_x^{-1})}{d\mu_N}\, \mu_H(A)\mu_N(B) \\
&= \Delta_H(x)\,\Delta_N(y) \frac{d\mu_N}{d(\mu_N\circ\alpha_x)}\, \mu_H \hat{\times} \mu_N(A\times B)
\end{aligned}
$$

So that $\Delta(x,y) = \Delta_H(x)\Delta_N(y) \frac{d\mu_N}{d(\mu_N\circ\alpha_x)}$. Recalling that $(y,x)\mapsto (x^{-1},y^{-1})^{-1}$ defined an isomorphism from $N \rtimes_\alpha H$ to $H \ltimes N$, we see that the modular function of $N\rtimes_\alpha H$ is given by

$$(y,x) \mapsto \Delta((x^{-1},y^{-1})^{-1}) = \Delta(x,y)$$

② Let G be a locally compact group with modular function $\Delta : G \to \mathbb{R}_+$. Define a continuous action $G \overset{\alpha}{\curvearrowright} \mathbb{R}$ by

$$\alpha_x(t) = \Delta(x)\, t.$$

Then for the Lebesgue measure m on $\mathbb{R}$ one has

$$\frac{dm}{d(m\circ\alpha_x)} = \Delta(x)^{-1}$$

The previous example implies the modular function $\tilde{\Delta}$ of $G \ltimes \mathbb{R}$ (and $\mathbb{R}\rtimes_\alpha G$) is given by

$$\tilde{\Delta}(x,y) = \Delta(x)\cdot 1 \cdot \Delta(x)^{-1} = 1$$

That is, $G \,{}_\alpha\!\ltimes \mathbb{R}$ is unimodular. $\square$

2.5 Convolutions

For a locally compact Hausdorff space X, let $M(X)$ denote the space of complex Radon measures, equipped with the total variation norm $\|\mu\| = |\mu|(X)$. Recall the following corollary to the Riesz representation theorem (Theorem 2.8). The key point is that $\|\mu\| < \infty$ implies the linear functional

$$C_c(X) \ni f \mapsto \int_X f \, d\mu$$

has a unique extension to $C_0(X) = \{f \in C(X) : |f|^{-1}([\varepsilon, +\infty)) \text{ is compact } \forall \varepsilon > 0\} = \overline{C_c(X)}^{\|\cdot\|_\infty}$

Theorem 2.24 (Riesz representation theorem for $C_0(X)$)
Let X be a locally compact Hausdorff space. Then

$$\begin{aligned} M(X) &\longrightarrow C_0(X)^* \\ \mu &\longmapsto \left(f \mapsto \int_X f \, d\mu\right) \end{aligned}$$

is an isometric isomorphism.

That is, every bounded linear functional on $C_0(X)$ is given by integration against a unique Radon measure.

Def For a locally compact group G, let $M(G)$ denote the space of complex Radon measures (i.e. μ such that $|\mu|$ is Radon). The <u>convolution</u> of $\mu, \nu \in M(G)$ is the unique Radon measure $\mu * \nu$ satisfying

$$\int f \, d(\mu * \nu) = \int_G \int_G f(xy) \, d\mu(x) \, d\nu(y)$$

$\square$

Note that the existence of $\mu * \nu$ follows from Theorem 2.24 and the bound

$$\left| \int_G \int_G f(xy) \, d\mu(x) \, d\nu(y) \right| \leq \|f\|_\infty \cdot \|\mu\| \, \|\nu\|$$

In particular, $\|\mu * \nu\| \leq \|\mu\| \, \|\nu\|$. We also observe that convolution is associative:

$$\begin{aligned} \int f \, d[\mu * (\nu * \sigma)] &= \iint f(xy) \, d\mu(x) \, d(\nu * \sigma)(y) \\ &= \iiint f(xyz) \, d\mu(x) \, d\nu(y) \, d\sigma(z) \\ &= \iint f(xz) \, d(\mu * \nu)(x) \, d\sigma(z) = \int f \, d[(\mu * \nu) * \sigma] \end{aligned}$$

Convolution is commutative if and only if G is abelian. Indeed, if G is abelian then $f(xy) = f(yx)$ and Fubini's theorem (note all measures here are finite) implies $\mu * \nu = \nu * \mu$. Conversely, suppose convolution is commutative. Then for $x, y \in G$, the point masses $\delta_x, \delta_y \in M(G)$ satisfy

$$\int f\, d(\delta_x * \delta_y) = \iint f(st)\, d\delta_x(s)\, d\delta_y(t) = f(xy) = \int f\, d\delta_{xy}.$$

so $\delta_{xy} = \delta_x * \delta_y = \delta_y * \delta_x = \delta_{yx}$ and therefore $xy = yx$.

The point mass $\delta := \delta_1 \in M(G)$ is a multiplicative identity:

$$\int f\, d(\delta * \mu) = \int f(xy)\, d\delta(x)\, d\mu(y) = \int f(y)\, d\mu(y) = \int f\, d\mu$$

so that $\delta * \mu = \mu$, and similarly $\mu * \delta = \mu$.

Lastly, observe that if we define $\mu^*(E) := \overline{\mu(E^{-1})}$ for $\mu \in M(G)$, then $\mu \mapsto \mu^*$ is an involution. Indeed

$$(\mu^*)^*(E) = \overline{\mu^*(E^{-1})} = \mu(E)$$

and

$$\begin{aligned}\int f\, d(\mu * \nu)^* &= \int f(x^{-1})\, d\overline{(\mu * \nu)}(x) \\ &= \iint f((xy)^{-1})\, d\overline{\mu}(x)\, d\overline{\nu}(y) \\ &= \iint f(y^{-1}x^{-1})\, d\overline{\nu}(y)\, d\overline{\mu}(x) \\ &= \iint f(yx)\, d\nu^*(y)\, d\mu^*(x) = \int f\, d(\nu^* * \mu^*)\end{aligned}$$

so that $(\mu * \nu)^* = \nu^* * \mu^*$.

The above observations imply $M(G)$ is a unital Banach $*$-algebra.

$\boxed{\text{Def}}$ For a locally compact group G, $M(G)$ is called the measure algebra of G. $\square$

Let μ be a left Haar measure and observe that we can define a linear isometry

$$\begin{aligned} L^1(G,\mu) &\longrightarrow M(G) \\ f &\longmapsto f\,d\mu. \end{aligned}$$

Thus we can identify $L^1(G,\mu)$ with a subspace of $M(G)$ and convolution on this subspace is given by

$$f * g(x) = \int_G f(y)\, g(y^{-1}x)\, d\mu(y).$$

Indeed, for $h \in C_c(G)$ we have

$$\int_G\int_G h(yx) f(y) g(x)\, d\mu(x)\, d\mu(y) = \int_G\int_G h(x) f(y) g(y^{-1}x)\, d\mu(x)\, d\mu(y) = \int_G h(x) (f*g)(x)\, d\mu(x).$$

Also note that Fubini's theorem implies $\|f*g\|_1 \le \|f\|_1 \|g\|_1$, so $f*g \in L^1(G,\mu)$. Also, the involution is given by

$$f^*(x) = \overline{f(x^{-1})}\,\Delta(x^{-1})$$

Since

$$\begin{aligned}(f\,d\mu)^*(E) &= \overline{\int 1_{E^{-1}}(x)\, f(x)\,d\mu(x)} \\ &= \int 1_E(x^{-1})\,\overline{f(x)}\,d\mu(x) \\ &= \int 1_E(x)\,\overline{f(x^{-1})}\,d\mu(x^{-1}) \\ &= \int 1_E(x)\,\overline{f(x^{-1})}\,\Delta(x^{-1})\,d\mu(x) = (f^* d\mu)(E)\end{aligned}$$

Thus $L^1(G,\mu)$ is Banach $*$-subalgebra of $M(G)$.

Spring Break 3/4

$\boxed{\text{Proposition 2.25}}$ Suppose $1 \le p \le \infty$ and $f \in L^1(G,\mu)$ and $g \in L^p(G,\mu)$.

(a) $f * g \in L^p(G,\mu)$ with

$$\|f*g\|_p \le \|f\|_1\,\|g\|_p$$

If $p=\infty$, then $f*g$ is continuous.

(b) Suppose either G is unimodular or f has compact support. Then $g*f \in L^p(G,\mu)$ with

$$\|g*f\|_p \le \sup_{x\in \mathrm{supp}(f)} |\Delta(x)|^{\frac{1}{p}-1}\,\|g\|_p\,\|f\|_1$$

If further $p=\infty$, then $g*f$ is continuous.

Proof (a): First observe that

$$f*g(x) = \int_G f(y)\,g(y^{-1}x)\,d\mu(y) = \int_G f(y)\,(L_y g)(x)\,d\mu(y).$$

Applying Minkowski's integral inequality we have

$$\begin{aligned}\|f*g\|_p &= \left(\int_G \left|\int_G f(y)(L_y g)(x)\,d\mu(y)\right|^p d\mu(x)\right)^{1/p} \\ &\le \int_G \left(\int_G |f(y)\,(L_y g)(x)|^p\,d\mu(x)\right)^{1/p} d\mu(y) \\ &= \int |f(y)|\cdot\|L_y g\|_p\,d\mu(y) = \|f\|_1\,\|g\|_p\end{aligned}$$

where the last equality follows from the left invariance of μ.

Now suppose $p=\infty$ and note

$$f*g(x) = \int f(y)\, g(y^{-1}x)\, d\mu(y) = \int_G f(xy)\, g(y^{-1})\, d\mu(y) = \int_G (R_x f)(y)\, g(y^{-1})\, d\mu(y)$$

If $f \in C_c(G)$, then the right uniform continuity of f (Proposition 2.5) and the above implies $f*g$ is continuous: for $K \supset K^\circ \supset \operatorname{supp} f$ compact

$$\limsup_{x \to x_0} |f*g(x) - f*g(x_0)| \leq \lim_{x \to x_0} \|R_x f - R_{x_0} f\|_\infty \cdot \|g\|_\infty \cdot \mu(K) = 0$$

For $f \in L^1(G,\mu)$, we can find $(f_n)_{n \in \mathbb{N}} \subset C_c(G)$ satisfying $\|f_n - f\|_1 \to 0$ (this is a general fact for Radon measures). Then

$$\|f_n * g - f*g\|_\infty \leq \|f_n - f\|_1 \cdot \|g\|_\infty \longrightarrow 0$$

by the first part of the proof, and hence $f*g$ is continuous as a uniform limit of continuous functions.

b: For $\check{f}(x) := f(x^{-1})$ and $\check{g}(x) := g(x^{-1})$ we have

$$g*f(x) = \int_G g(xy)\, f(y^{-1})\, d\mu(y) = \int \check{f}(y)\, \check{g}(y^{-1}x^{-1})\, d\mu(y) = (\check{f} * \check{g})^\vee (x).$$

Recall that for the right Haar measure $d\nu(x) := d\mu(x^{-1}) = \Delta(x^{-1})\, d\mu(x)$, $f \mapsto \check{f}$ defines and isometric isomorphism from $L^p(G,\nu)$ to $L^p(G,\mu)$. Thus if G is unimodular, so that $\nu = \mu$, the above computation and a implies

$$\|g*f\|_p = \|(\check{f}*\check{g})^\vee\|_p = \|\check{f}*\check{g}\|_p \leq \|\check{f}\|_1 \cdot \|\check{g}\|_p = \|g\|_p \|f\|_1.$$

Additionally, for $p=\infty$ $\check{f}*\check{g}$ is continuous by a and thus so is $g*f = (\check{f}*\check{g})^\vee$.

Now suppose instead that $K := \operatorname{supp} f$ is compact. Then $C := \|\Delta|_K\|_\infty < \infty$ and Minkowski's integral inequality gives:

$$\|g*f\|_p = \left(\int_G \left| \int_G g(xy)\, f(y^{-1})\, d\mu(y) \right|^p d\mu(x) \right)^{1/p}$$

$$= \left(\int_G \left| \int_G g(xy^{-1})\, f(y)\, \Delta(y^{-1})\, d\mu(y) \right|^p d\mu(x) \right)^{1/p}$$

$$\leq \int_G \left(\int_G |g(xy^{-1})\, f(y)\, \Delta(y^{-1})|^p\, d\mu(x) \right)^{1/p} d\mu(y)$$

$$= \int_G \left(\int_G |g(x)|^p\, \Delta(y)\, d\mu(x) \right)^{1/p} |f(y)\, \Delta(y^{-1})|\, d\mu(y)$$

$$= \int_G \|g\|_p\, |f(y)| \cdot |\Delta(y)|^{1/p - 1}\, d\mu(y) \leq \|g\|_p \|f\|_1\, C^{\frac{1}{p}-1}$$

Finally, assume $p=\infty$. Then

$$(g*f)(x) = \int_G g(y)\, f(y^{-1}x)\, d\mu(y) = \int_G g(y)\, (R_x f)(y^{-1})\, d\mu(x)$$

and the right uniform continuity of f implies the above is continuous. □

Proposition 2.26 Suppose G is unimodular and $1<p,q<\infty$ satisfy $\frac{1}{p}+\frac{1}{q}=1$. Then for $f\in L^p(G,\mu)$ and $g\in L^q(G,\mu)$ one has $f*g\in C_0(G)$ with

$$\|f*g\|_\infty \le \|f\|_p\,\|g\|_q.$$

Proof First note that by Hölder's inequality we have

$$|f*g(x)| \le \|f\|_p\,\|L_{x^{-1}}\check{g}\|_q = \|f\|_p\,\|g\|_q,$$

where the second equality uses the unimodularity of G. Now, if $f,g\in C_c(G)$, then

$$f*g(x) = \int_{\operatorname{supp} f} f(y)\, g(y^{-1}x)\, d\mu(y)$$

is zero unless $y^{-1}x\in \operatorname{supp} g$ for some $y\in\operatorname{supp} f$. That is $\operatorname{supp}(f*g)\subseteq \operatorname{supp} f\cdot \operatorname{supp} g$, which is compact by Proposition 2.1.6. Using Proposition 2.25 we then have $f*g\in C_c(G)\subset C_0(G)$. Now, for $f\in L^p(G,\mu)$ and $g\in L^q(G,\mu)$, we can find $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}\subset C_c(G)$ satisfying $\|f_n-f\|_p, \|g_n-g\|_q\to 0$. Then the above inequality implies

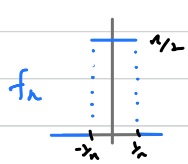
$$\|f_n*g_n - f*g\|_\infty \le \|f_n-f\|_p\,\|g_n\|_q + \|f\|_p\,\|g_n-g\|_q \longrightarrow 0$$

Thus $f*g\in \overline{C_c(G)}^{\|\cdot\|_\infty} = C_0(G)$. □

We saw earlier in the section that $\delta\in M(G)$ is an identity for the measure algebra. If G is discrete, then $\delta\in L^1(G,\mu)$ as the function

$$\delta(x) = \begin{cases} 1 & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$$

However, if G is not discrete then $L^1(G,\mu)$ will not admit any identity under convolution (Exercise: prove this). However, $L^1(G,\mu)$ does admit an "approximate identity". On $\mathbb{R}$, for example, this is given by the sequence



Before establishing their existence in full generality, we require the following.

Proposition 2.27 For $f\in L^p(G,\mu)$ with $1\le p<\infty$, one has

$$\lim_{y\to 1}\|L_y f - f\|_p = 0 \qquad \text{and} \qquad \lim_{y\to 1}\|R_y f - f\|_p = 0$$

Proof Fix a compact neighborhood V of $1\in G$. For $g\in C_c(G)$, consider the compact set

$$K := (\operatorname{supp} g)V^{-1} \cup V(\operatorname{supp} g).$$

Then for $y \in V$ one has $\operatorname{supp}(L_y g), \operatorname{supp}(R_y g) \subset K$ so that

$$\| L_y g - g\|_p \le \mu(K)^{1/p} \| L_y g - g\|_\infty \longrightarrow 0$$

and

$$\|R_y g - g\|_p \le \mu(K)^{1/p} \| R_y g - g\|_\infty \longrightarrow 0$$

by Proposition 2.5.

Now, for $f \in L^p(G,\mu)$ let $\varepsilon > 0$ and let $g \in C_c(G)$ be such that $\|f-g\|_p < \varepsilon$. Then

$$\| L_y f - f\|_p \le \| L_y (f-g)\|_p + \| L_y g - g\|_p + \| g - f\|_p \le 2\varepsilon + \| L_y g - g\|_p \longrightarrow 2\varepsilon$$

Next, let $C := \| \Delta|_V \|_\infty < \infty$. Then

$$\| R_y f - f\|_p \le \| R_y (f-g)\|_p + \| R_y g - g\|_p + \|g - f\|_p \le C(C+1)\varepsilon + \|R_y g - g\|_p \longrightarrow C(C+1)\cdot\varepsilon.$$

□

Proposition 2.28 Let $\mathcal{U}$ be a neighborhood base for $1 \in G$, which we order by reverse inclusion. For each $U \in \mathcal{U}$, let ψ_U be a compactly supported function satisfying: $\operatorname{supp} \psi_U \subset U$; $\psi_U \ge 0$; $\psi_U(x^{-1}) = \psi_U(x)$; and

$$\int_G \psi_U \, d\mu = 1.$$

Then the net $(\psi_U)_{U \in \mathcal{U}}$ satisfies the following.

(a) For $f \in L^p(X,\mu)$ with $1 \le p < \infty$

$$\lim_{U \to \infty} \| f * \psi_U - f\|_p = 0 \quad \text{and} \quad \lim_{U\to\infty} \| \psi_U * f - f\|_p = 0$$

(b) For $f \in L^\infty(X,\mu)$ which is right (resp. left) uniformly continuous

$$\lim_{U\to\infty} \|f * \psi_U - f\|_\infty = 0 \quad \left(\text{resp. } \lim_{U\to\infty} \| \psi_U * f - f\|_\infty = 0\right)$$

Proof (a) Using $\int \psi_U \, d\mu = 1$ and $\psi_U(y^{-1}) = \psi_U(y)$ we have:

$$\begin{aligned} f * \psi_U(x) - f(x) &= \int_G f(xy)\psi_U(y^{-1})\, d\mu(y) - f(x)\int_G \psi_U(y)\, d\mu(y) \\ &= \int_G [R_y f - f](x)\, \psi_U(y)\, d\mu(y) \end{aligned}$$

Thus Minkowski's integral inequality gives

$$\| f * \psi_U - f\|_p \le \int \| R_y f - f\|_p \, \psi_U(y)\, d\mu(y) \le \sup_{y \in U} \| R_y f - f\|_p$$

Given $\varepsilon > 0$, Proposition 2.27 yields $U_0 \in \mathcal{U}$ such that $\|R_y f - f\|_p < \varepsilon$ for all $y \in U_0$. So the above gives

$$\|f * \psi_U - f\|_p \le \varepsilon$$

whenever $U \subset U_0$. Thus the limit as $U \to \infty$ is zero as claimed.

Similarly,

$$\psi_U * f(x) - f(x) = \int_G \psi_U(y)\,[L_y f - f](x)\, d\mu(y)$$

so that

$$\|\psi_U * f - f\|_p \le \sup_{y \in U} \|L_y f - f\|_p \qquad **$$

So the same argument as above shows this tends to zero as $U \to \infty$.

(b): We cannot appeal to Proposition 2.27 in this case, but (*) and (**) are still valid and so the right (resp. left) uniform continuity of f completes the proof. $\square$

$\boxed{\text{Def}}$ A net $(\psi_U)_{U \in \mathcal{U}}$ satisfying the conditions in Proposition 2.28 is called an <u>approximate identity</u> of $L^1(G,\mu)$. $\square$

Note that we can let $\mathcal{U}$ be the collection of compact symmetric neighborhoods of $1 \in G$ and then set $\psi_U := \mu(U)^{-1} 1_U$.

3/7

Observe that

$$(L_z(f*g))(x) = f*g(z^{-1}x) = \int_G f(z^{-1}xy)\, g(y^{-1})\, d\mu(y) = \int_G (L_z f)(xy)\, g(y^{-1})\, d\mu(y) = (L_z f)*g(x)$$

and similarly

$$(R_z(f*g))(x) = f*g(xz) = \int_G f(y)\, g(y^{-1}xz)\, d\mu(y) = \int_G f(y)\, (R_z g)(y^{-1}x)\, d\mu(y) = f*(R_z g)(x).$$

Thus

$$L_z(f*g) = (L_z f)*g \qquad \text{and} \qquad R_z(f*g) = f*(R_z g). \qquad ***$$

Combining this with Proposition 2.27 gives:

$\boxed{\text{Proposition 2.29}}$ If $f \in L^1(G,\mu)$ and $g \in L^\infty(G,\mu)$, then $f*g$ is left uniformly continuous and $g*f$ is right uniformly continuous.

<u>Proof</u> Using (***) and Hölder's inequality we have

$$\|L_z(f*g) - f*g\|_\infty = \|(L_z f - f)*g\|_\infty \le \|L_z f - f\|_1 \cdot \|g\|_\infty$$

$$\|R_z(g*f) - g*f\|_\infty = \|g*(R_z f - f)\|_\infty \le \|g\|_\infty \|R_z f - f\|_1 .$$

These tend to zero as $z \to 1$ by Proposition 2.27. $\square$

As an application of approximate identities, we have the following.

$\boxed{\text{Theorem 2.30}}$ Let $I \subseteq L^1(G,\mu)$ be a closed subspace. Then I is a left (resp. right) ideal if and only if $L_y I \subset I$ (resp. $R_y I \subset I$) for all $y \in G$.

Proof Suppose I is a left ideal. For $f \in I$ and $x \in G$, we must show $L_x f \in I$. Let $\{\psi_u\}_{u \in \mathcal{U}}$ be an approximate identity. Then using (***) we have

$$L_x(\psi_u * f) = (L_x \psi_u) * f \in I$$

for all $u \in \mathcal{U}$. Consequently

$$L_x f = \lim_{u \to \infty} L_x(\psi_u * f) \in I$$

Conversely, suppose I is closed under left translations. For $f \in I$ and $g \in L^1(G,\mu)$, we have

$$g * f(x) = \int_G g(y) f(y^{-1}x)\, d\mu(y) = \int_G g(y)(L_y f)(x)\, d\mu(y).$$

Suppose $g \in C_c(G)$ so that $y \mapsto g(y)(L_y f) \in L^1(G,\mu)$ is a compactly supported $L^1(G,\mu)$-valued continuous function (we are using Proposition 2.27 here). Given $\varepsilon > 0$, we can partition

$$\operatorname{supp} g = E_1 \sqcup \cdots \sqcup E_{N(\varepsilon)}$$

into measurable subsets so that for each $j = 1, \ldots, N(\varepsilon)$

$$\| g(x)(L_x f) - g(y)(L_y f)\|_1 < \varepsilon \qquad \forall x, y \in E_j.$$

Fix $y_j \in E_j$ for each $j = 1, \ldots, N(\varepsilon)$ and consider

$$R(x) := \sum_{j=1}^{N(\varepsilon)} g(y_j)(L_{y_j} f)(x) \cdot \mu(E_j) \in I$$

Then

$$\begin{aligned}
\| g*f - R\|_1 &= \int_G | g*f(x) - R(x)|\, d\mu(x) \\
&= \int_G \Big| \sum_{j=1}^{N(\varepsilon)} \int_{E_j} g(y)(L_y f)(x) - g(y_j)(L_{y_j} f)(x)\, d\mu(y) \Big|\, d\mu(x) \\
&\leq \sum_{j=1}^{N(\varepsilon)} \int_{E_j} \| g(y)(L_y f) - g(y_j)(L_{y_j} f)\|_1 \cdot d\mu(y) \\
&\leq \sum_{j=1}^{N(\varepsilon)} \varepsilon \cdot \mu(E_j) = \varepsilon \cdot \mu(\operatorname{supp} g)
\end{aligned}$$

Since ε was arbitrary, we have $g * f \in I$. Finally, let $g \in L^1(X,\mu)$ and let $(g_n)_{n \in \mathbb{N}} \subset C_c(G)$ be such that $\|g_n - g\|_1 \to 0$. Then

$$\|g_n * f - g * f\|_1 \leq \|g_n - g\|_1 \|f\| \to 0$$

and hence $g * f \in I$. The argument for right ideals is similar. □

Remark For $\nu \in M(G)$ and $f \in L^1(G,\mu)$, define

$$\nu * f(x) := \int f(y^{-1}x)\, d\nu(y)$$
$$f * \nu(x) := \int f(xy^{-1})\, \Delta(y)^{-1}\, d\nu(y)$$

Then

$$\|\nu * f\|_1 \leq \|\nu\|\, \|f\|_1$$
$$\|f * \nu\|_1 \leq \|f\|_1\, \|\nu\|$$

and

$$(\nu * f)\, d\mu = \nu * (f\, d\mu)$$
$$(f * \nu)\, d\mu = (f\, d\mu) * \nu$$

This shows $L^1(G,\mu)$ is a closed two-sided ideal in $M(G)$. More generally, for $f \in L^p(X,\mu)$ one can show $\nu * f \in L^p(G,\mu)$ with

$$\|\nu * f\|_p \leq \|\nu\| \cdot \|f\|_p$$

and, if G is unimodular, $f * \nu \in L^p(G,\mu)$ with

$$\|f * \nu\|_p \leq \|f\|_p\, \|\nu\|.$$

This can be proven using the same argument as in Proposition 2.25. □