III.1 Elementary Properties and Examples

Def Let X be a vector space over $\mathbb{F}$. A <u>seminorm</u> is a map $p: X \to [0,\infty)$ satisfying:

1. $p(x+y) \le p(x) + p(y) \quad \forall x,y \in X$.
2. $p(\alpha x) = |\alpha|\, p(x) \quad \forall \alpha \in \mathbb{F}, x \in X$.

Note that $p(0)=0$. We say p is a <u>norm</u> if it also satisfies

3. $p(x)=0$ iff $x=0$.

- A norm is typically denoted by $\|x\| := p(x)$.
- Note that $d(x,y) = \|x-y\|$ defines a metric on X.

Def A <u>normed space</u> is a pair $(X, \|\cdot\|)$ where X is a vector space and $\|\cdot\|$ is a norm on X. If X is complete with respect to the metric $d(x,y) = \|x-y\|$, then we call it a <u>Banach space</u>.

Ex

0. Inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.
1. Let (X, Ω, μ) be a measure space. Then $L^p(X,\Omega,\mu)$ with $\|\cdot\|_p$ is a Banach space for $1 \le p \le \infty$.
   In particular, $\ell^p(\mathbb{N})$ is a Banach space for $1 \le p \le \infty$.
2. Let X be a locally compact Hausdorff space. Let $C_b(X)$ denote the collection of continuous functions $f: X \to \mathbb{F}$ such that
$$\|f\| := \sup_{x \in X} |f(x)| < \infty$$

pointwise addition and scalar multiplication

Then $C_b(X)$ is a Banach space. Indeed, let $(f_n)_{n\in\mathbb{N}} \subseteq C_b(X)$ be a Cauchy sequence. Then for each $x \in X$
$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|$$
Thus $(f_n(x))_{n\in\mathbb{N}} \subseteq \mathbb{F}$ is a Cauchy sequence, which converges since $\mathbb{F}$ is complete. Define $f: X \to \mathbb{F}$ by
$$f(x) := \lim_{n\to\infty} f_n(x).$$
Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be s.t. $\|f_n - f_m\| < \frac{\varepsilon}{2}$ $\forall n,m \ge N$. Then for any $x \in X$ we have
$$|f(x) - f_n(x)| \le |f(x) - f_m(x)| + |f_m(x) - f_n(x)|$$
This holds for all $m \in \mathbb{N}$, so we can in particular choose $m \ge N$ and large enough so that $|f(x) - f_m(x)| < \frac{\varepsilon}{2}$. We then have
$$|f(x) - f_n(x)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Since $x \in X$ was arbitrary, we obtain $\|f - f_n\| < \varepsilon$. Thus $\|f - f_n\| \to 0$.
It remains to show f is continuous. But as the uniform limit of continuous functions, this follows by the standard $\frac{\varepsilon}{3}$ proof.

3. Let X be a locally compact Hausdorff space. Let $C_0(X) \subset C_b(X)$ be the collection of continuous functions such that $\forall \varepsilon > 0$ $\exists K \subset X$ compact such that
$$\sup_{x \in X \setminus K} |f(x)| < \varepsilon.$$
Equivalently, $\forall \varepsilon > 0$, $\{x : |f(x)| \geq \varepsilon\}$ is compact. Then $C_0(X)$ is a closed subset (Exercise) and hence is a Banach space.

4. Any set I can be made into a locally compact Hausdorff space by endowing it with the discrete topology. In this case, all functions $f: I \to \mathbb{F}$ are continuous. We thus write
$$\ell^\infty(I) := C_b(I) = \{f : I \to \mathbb{F} \text{ with } \sup_{i \in I} |f(i)| < \infty\}$$
$$c_0(I) := C_0(I) = \{f : I \to \mathbb{F} \text{ with } |f|^{-1}([\varepsilon, \infty)) \text{ finite for all } \varepsilon > 0\}.$$
In particular
$$c_0(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{F} \text{ with } \lim_{n \to \infty} f(n) = 0\}.$$
$\square$

$\boxed{\text{Prop}}$ Let X be a normed space

1. $X \times X \ni (x, y) \mapsto x + y \in X$ is continuous
2. $\mathbb{F} \times X \ni (\alpha, x) \mapsto \alpha x \in X$ is continuous.

$\boxed{\text{Proof}}$ (1): $\|(x+y) - (x'+y')\| \leq \|x - x'\| + \|y - y'\|$.

(2) $\|\alpha x - \beta y\| \leq |\alpha - \beta| \|x\| + |\beta| \|x - y\|$. $\square$

$\boxed{\text{Lemma}}$ If p and q are seminorms on a vector space X, then the following are equivalent:

1. $p(x) \leq q(x)$ $\forall x \in X$.
2. $p(x) < 1$ whenever $q(x) < 1$
3. $p(x) \leq 1$ whenever $q(x) \leq 1$
4. $p(x) \leq 1$ whenever $q(x) < 1$.

$\boxed{\text{Proof}}$ $(1) \Rightarrow (2) \Rightarrow (4)$ and $(1) \Rightarrow (3) \Rightarrow (4)$ are clear. So it suffices to show $(4) \Rightarrow (1)$.
For $x \in X$ and $\varepsilon > 0$ we have
$$q\left(\frac{x}{q(x)+\varepsilon}\right) < 1 \Rightarrow p\left(\frac{x}{q(x)+\varepsilon}\right) \leq 1 \rightarrow p(x) \leq q(x) + \varepsilon.$$
Since $\varepsilon > 0$ is arbitrary, we obtain $p(x) \leq q(x)$. $\square$

- We say two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent if they define the same topologies on X. That is, a set is open under the metric $\|x-y\|_1$ iff it is open under the metric $\|x-y\|_2$.

$\boxed{\text{Prop}}$ If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on X, then they are equivalent if and only if $\exists\, c, C > 0$ such that
$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \forall x \in X.$$
$\boxed{\text{Proof}}$ $(\Longrightarrow)$ $0 \in \{x \in X : \|x\|_1 < 1\}$ so $\exists\, c > 0$ such that $\{x \in X : \|x\|_2 < c\} \subset \{x \in X : \|x\|_1 < 1\}$.
That is, $\frac{1}{c}\|x\|_2 < 1$ implies $\|x\|_1 < 1$, so $\|x\|_1 \leq \frac{1}{c}\|x\|_2$ or $c\|x\|_1 \leq \|x\|_2$ $\forall x \in X$.
Using $0 \in \{x \in X : \|x\|_2 < 1\}$ we can obtain $C > 0$

($\Leftarrow$) Fix $x_0 \in X$. Then

$$\{x \in X : \|x - x_0\|_2 < r\} \supset \{x \in X : \|x - x_0\|_1 < \tfrac{r}{C}\}$$
$$\{x \in X : \|x - x_0\|_1 < r\} \supset \{x \in X : \|x - x_0\|_2 < rc\}$$

This shows any open ball in $\|\cdot\|_1$ contains an open ball in $\|\cdot\|_2$ and vice versa. $\square$

$\boxed{\text{Prop}}$ (Reverse Triangle Inequality) If p is a seminorm on X

$$p(x-y) \geq |p(x) - p(y)| \qquad \forall x, y \in X.$$

In particular, for a norm we have $\|x - y\| \geq |\|x\| - \|y\|| \qquad \forall x, y \in X.$

Proof we have

$$p(x) \leq p(x-y) + p(y) \implies p(x) - p(y) \leq p(x-y)$$
$$p(y) \leq p(y-x) + p(x) \implies p(y) - p(x) \leq p(y-x) = p(x-y).$$

Hence $|p(x) - p(y)| \leq p(x-y)$. $\square$

$\boxed{\text{Def}}$ Let X and Y be normed spaces. An isomorphism is a linear bijection $T: X \to Y$ such that T and T^{-1} are continuous (i.e. T is a linear homeomorphism).
An isometric isomorphism is a linear surjective isometry.

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III. 2 Linear Operators on Normed Spaces

- For X and Y normed spaces, let $B(X,Y)$ denote the collection of continuous linear transformations $A: X \to Y$. We write $B(X) := B(X,X)$. The proof of the following proposition is similar to the corresponding proofs in Sections I.3 and II.1, so we leave it as an exercise.

Prop. If X, Y are normed spaces and $A: X \to Y$ is a linear transformation, then the following are equivalent:

1. $A \in B(X,Y)$
2. A is continuous at 0.
3. A is continuous at some point.
4. There exists $C > 0$ such that $\frac{\|Ax\|}{\|x\|} \leq C$ for all $x \in X$.

In this case we have

$$\|A\| := \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\| \leq 1} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| = \inf\{C > 0 : \|Ax\| \leq C\|x\| \ \forall x \in X\}.$$

- As before, $\|A\|$ is called the norm of A. These norms make $B(X,Y)$ into a normed space with pointwise addition and scalar multiplication. If Y is a Banach space, then so is $B(X,Y)$. (The converse is not true, e.g. $B(\{0\}, Y)$.)

Ex 1. For (X, Ω, μ) a σ-finite measure space and $\phi \in L^\infty(X, \Omega, \mu)$, $M_\phi \in B(L^p(X, \Omega, \mu))$ for all $1 \leq p \leq \infty$ with $\|M_\phi\| = \|\phi\|_\infty$.

2. For (X, Ω, μ) a measure space and $k: X \times X \to \mathbb{F}$ satisfying

$$\int_X |k(x,y)| \, d\mu(y) \leq c_1 \quad \mu\text{-a.e. } x \in X \qquad \text{and} \qquad \int_X |k(x,y)| \, d\mu(x) \leq c_2 \quad \mu\text{-a.e. } y \in X$$

Then the corresponding integral operator $K \in B(L^p(X, \Omega, \mu))$ with $\|K\| \leq c_1^{1/q} c_2^{1/p}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

3. Let X, Y be compact Hausdorff spaces with $\phi: Y \to X$ continuous. Then

$$\begin{aligned} C(X) &\longrightarrow C(Y) \\ f &\longmapsto f \circ \phi \end{aligned}$$

defines a bounded operator $A \in B(C(X), C(Y))$ with $\|A\| = 1$. $\square$

$\boxed{\text{III.3}}$ Finite Dimensional Normed Spaces

$\boxed{\text{Thm}}$ If X is a finite dimensional vector space over $\mathbb{F}$, then any two norms on X are equivalent.

Proof Let $\{e_1, \dots, e_d\}$ be a basis for X. For $x = x_1e_1 + \dots + x_de_d$ define

$$\|x\|_\infty := \max_{1\le j\le d} |x_j|$$

This is a norm on X (Exercise verify), so it suffices to show any other norm $\|\cdot\|$ on X is equivalent to $\|\cdot\|_\infty$.

Define $C := \sum_{j=1}^{d} \|e_j\|$, then

$$* \qquad \|x\| \le \sum_{j=1}^{d} |x_j| \, \|e_j\| \le \|x\|_\infty \cdot C$$

Define $f: X \to [0,\infty)$ by $f(x) = \|x\|$. We claim f is continuous with respect to $\|\cdot\|_\infty$. Indeed, fix $x_0 \in X$ and let $\varepsilon > 0$. If

$$\|x - x_0\|_\infty < \frac{\varepsilon}{C}$$

Then by the reverse triangle inequality and $*$ we have:

$$|f(x) - f(x_0)| = |\, \|x\| - \|x_0\| \,| \le \|x - x_0\| \le \|x - x_0\|_\infty \cdot C < \varepsilon.$$

Now, noting that $(X, \|\cdot\|_\infty)$ is isometrically isomorphic to $(\mathbb{F}^d, \|\cdot\|_\infty)$, we obtain that $S := \{x \in X : \|x\|_\infty = 1\}$ is compact. Consequently, f achieves a minimum on S, say

$$f(x_1) = \min_{x\in S} f(x) =: c$$

Note that $c > 0$, since otherwise $0 = f(x_1) = \|x_1\| \Rightarrow x_1 = 0$, contradicting $0 \notin S$. Thus we have $\|x\| \ge c$ $\forall x \in S$. Consequently, $\forall x \in X \setminus \{0\}$ we have

$$c \cdot \|x\|_\infty \le \left\| \frac{x}{\|x\|_\infty} \right\| \cdot \|x\|_\infty = \|x\|. \qquad \square$$

$\boxed{\text{Cor}}$ Any finite dimensional normed space is isometrically isomorphic to $(\mathbb{F}^d, \|\cdot\|)$ for some d and some norm $\|\cdot\|$.

$\boxed{\text{Cor}}$ If X is a normed space and $Y \le X$ is a finite dimensional subspace, then Y is closed.

Proof Y is complete by the previous corollary, hence closed. $\square$

$\boxed{\text{Prop}}$ Let X and Y be normed spaces. If X is finite dimensional then any linear transformation $T: X \to Y$ is bounded.

Proof Define $\|\cdot\|_\infty$ relative to a basis $\{e_1, \dots, e_d\}$ for X. Then for $x = x_1e_1 + \dots + x_de_d$ we have

$$\|T(x)\| \le \sum_{j=1}^{d} |x_j| \, \|T(e_j)\| \le \|x\|_\infty \underbrace{\sum_{j=1}^{d} \|T(e_j)\|}_{=:C}$$

Thus T is bounded — and consequently continuous — with respect to $\|\cdot\|_\infty$. This implies T is continuous with respect to any norm on X since they are all continuous. $\square$

$\boxed{\text{III.4}}$ Quotients and Direct Sums of Normed Spaces

Notation Let X be a normed space. We write $Y \leq X$ when Y is a closed subspace of X.

Quotients

- Let X be a normed space and let Y be a (not necessarily closed) subspace. Recall that

cosets

$$X/Y = \{x+Y : x \in X\}$$

inherits a vector space structure from X. Let $Q: X \to X/Y$ be the quotient map $Qx = x+Y$. Define

$$\|x+Y\| := \inf_{y \in Y} \|x+y\| = \inf_{y\in Y} \|x-y\| = \operatorname{dist}(x,Y).$$

This gives a seminorm on X/Y (Exercise verify), and is a norm iff Y is closed.

Thm For $Y \leq X$, let $Q: X \to X/Y$ and $\|x+Y\|$ be as above.

1. $Q \in B(X, X/Y)$ with $\|Q\| \leq 1$
2. If X is a Banach space then so is X/Y.
3. If $U \subset X$ is open, then $Q(U)$ is open.
4. $W \subset X/Y$ is open iff $Q^{-1}(W)$ is open.

Proof (1): Since $0 \in Y$ we have $\|Q(x)\| = \|x+Y\| \leq \|x+0\| = \|x\|$.

(2): Let $(x_n+Y)_{n\in\mathbb{N}} \subset X/Y$ be a Cauchy sequence. For each $k \in \mathbb{N}$, let $N_k \in \mathbb{N}$ be such that

$$\|(x_n+Y) - (x_m+Y)\| < 2^{-k} \qquad \forall n,m \geq N_k.$$

Set $n_k = N_1 + N_2 + \cdots + N_k$ so that $n_1 < n_2 < \cdots$ and

$$\|(x_{n_k} - x_{n_{k+1}}) + Y\| = \|(x_{n_k}+Y) - (x_{n_{k+1}}+Y)\| < 2^{-k}$$

Thus $\exists\, z_k \in Y$ such that

$$\|x_{n_k} - x_{n_{k+1}} + z_k\| \leq \|(x_{n_k} - x_{n_{k+1}}) + Y\| + 2^{-k} < 2 \cdot 2^{-k}$$

Set $y_1 := 0$ and inductively define $y_{k+1} := y_k - z_k \in Y$. Then we have

$$\|(x_{n_k}+y_k) - (x_{n_{k+1}}+y_{k+1})\| = \|x_{n_k} - x_{n_{k+1}} + z_k\| < 2 \cdot 2^{-k}$$

It follows that for $l > k$

$$\|(x_{n_k}+y_k) - (x_{n_l}+y_l)\| \leq \sum_{j=0}^{l-k-1} \|(x_{n_{k+j}} + y_{k+j}) - (x_{n_{k+j+1}} - y_{k+j+1})\|$$

$$< \sum_{j=0}^{l-k-1} 2 \cdot 2^{-(k+j)} \leq 4 \cdot 2^{-k}$$

Thus $(x_{n_k}+y_k)_{k\in\mathbb{N}} \subset X$ is a Cauchy sequence. So if X is a Banach space then this converges to some $x \in X$. By (1) we have that

$$x_{n_k} + Y = Q(x_{n_k}+y_k) \longrightarrow Q(x).$$

Since the original sequence was Cauchy, we obtain $x_n + Y \to Q(x)$. Thus X/Y is complete.

(3): Let $U \subset X$ be open. Let $x+Y \in Q(U)$ and let $z \in U$ be such that $Q(z) = x+Y$. Then $z - x \in Y$. Let $r > 0$ be such that $B(z,r) \subseteq U$. We claim $Q(B(0,r)) = B(0+Y, r)$. Indeed, if $\|x\| < r$

then $\|x+Y\| \le \|x+0\| \le r$. Conversely, if $\|x+Y\| < r$ then there exists $y \in Y$ such that $\|x+y\| \le \frac{\|x+y\|+r}{2} < r$, and we have $Q(x+y) = x+Y$. So $Q(B(0,r)) = B(0+Y, r)$ and it follows that

$$Q(U) \supseteq Q(B(z,r)) = Q(z + B(0,r)) = x+Y + B(0+Y, r) = B(x+Y, r)$$

Thus $Q(U)$ is open.

(4): If W is open, $Q^{-1}(W)$ is open since Q is continuous. If $Q^{-1}(W)$ is open, then $W = Q(Q^{-1}(W))$ is open by 3. □

- 3 says that Q is an open map. It is not necessarily true that Q sends closed sets to closed sets.

Prop Let X be a normed space, $Y \le X$, and $Z \le X$ finite dimensional. Then $Y+Z \le X$.

Proof Let $Q: X \to X/Y$. Note that $\dim Q(Z) \le \dim(Z) < \infty$, so $Q(Z) \le X/Y$. Since Q is continuous, $Y+Z = Q^{-1}(Q(Z))$ is closed. □

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Direct Sums

- Let $\{X_i : i \in I\}$ be a collection of normed spaces. We will define their direct sum just like we did for Hilbert spaces, but will have a choice for each $1 \le p \le \infty$
- Let $\prod_{i \in I} X_i$ denote arbitrary sequences, with coordinate-wise addition and scalar multiplication. For $1 \le p < \infty$ define

$$\bigoplus_{i \in I}^{p} X_i := \left\{ x \in \prod X_i : \|x\|_p := \left(\sum_{i \in I} \|x(i)\|^p \right)^{1/p} < \infty \right\}$$

Define

$$\bigoplus_{i \in I}^{\infty} X_i := \left\{ x \in \prod X_i : \|x\|_\infty := \sup_{i \in I} \|x(i)\| < \infty \right\}$$

When $I = \mathbb{N}$, we also define

$$\bigoplus_{n \in \mathbb{N}}^{0} X_n := \left\{ x \in \prod X_n : \lim_{n \to \infty} \|x(n)\| = 0 \right\} \le \bigoplus_{n \in \mathbb{N}}^{\infty} X_n$$

If $X_i = X$ $\forall i \in I$ we write

$$\ell^p(I, X) := \bigoplus_{i \in I}^{p} X \quad \text{and} \quad c_0(I, X) = \bigoplus_{i \in I}^{0} X.$$

Prop Let $\{X_i : i \in I\}$ be a collection of normed spaces, and let $X := \bigoplus_{i \in I}^{p} X_i$ for some $1 \le p \le \infty$.

1. X is a normed space, and the coordinate projection $P_i : X \to X_i$ satisfies $\|P_i\| \le 1$.
2. X is a Banach space if and only if X_i is a Banach space for all $i \in I$.
3. If $U \subset X$ is open, then $P_i(U) \subset X_i$ is open.

Proof Exercise. □

III.5 Linear Functionals

Def Let X be a vector space over $\mathbb{F}$. A subspace $Y \subseteq X$ is called a hyperplane if $\dim(X/Y) = 1$

Ex Let $f: X \to \mathbb{F}$ be a linear functional. Then $X/\ker(f) \cong \mathbb{F}$, so $\ker(f)$ is a hyperplane.

Prop A subspace $Y \subseteq X$ is a hyperplane iff Y is the kernel of a linear functional. Moreover, two linear functionals have the same kernel iff one is a non-zero scalar multiple of the other.

Proof The previous example shows the first "if" direction. Suppose Y is a hyperplane. Let $T: X/Y \to \mathbb{F}$ be an isomorphism, and let $Q: X \to X/Y$ be the quotient map. Then $T \circ Q: X \to \mathbb{F}$ is a linear functional with $\ker(T \circ Q) = \ker(Q) = Y$.

Next suppose $f, g: X \to \mathbb{F}$ satisfy $\ker f = \ker g$. If $\ker f = X$ then $f = g \equiv 0$. Otherwise, let $x_0 \in X \setminus \ker f$ with $f(x_0) = 1$. Then $g(x_0) \neq 0$. Note that for any $x \in X$

$$f(x - f(x) \cdot x_0) = f(x) - f(x) f(x_0) = 0$$

so $x - f(x) \cdot x_0 \in \ker(f) = \ker(g)$, which implies

$$0 = g(x - f(x) x_0) = g(x) - f(x) \cdot g(x_0)$$

or $g(x) = g(x_0) \cdot f(x)$. That is $g = g(x_0) \cdot f$. The converse is immediate. $\square$

Prop If X is a normed space and $Y \subseteq X$ is a hyperplane, then Y is either closed or dense.

Proof Since $\dim(X/Y) = 1$, the only intermediate subspaces $Y \subseteq Z \subseteq X$ are $Z = Y$ or X. Since $Y \subseteq \overline{Y} \subseteq X$, we have either $\overline{Y} = Y$ (so Y is closed) or $\overline{Y} = X$ (so Y is dense). $\square$

Ex ① Let X be a normed space and let $f: X \to \mathbb{F}$ be a continuous linear functional. Then $\ker f$ is closed.

② Let $X = c_0(\mathbb{N})$. Let $\mathcal{E} = \{e_n : n \in \mathbb{N}\}$ where $e_n(k) = \delta_{k=n}$. Let $x_0 \in c_0(\mathbb{N})$ be $x_0(n) = \frac{1}{n}$. Note that $\{x_0\} \cup \mathcal{E}$ is linearly independent. Extend it to a Hamel basis (maximal linearly independent set):

$$\mathcal{B} = \{x_0\} \cup \mathcal{E} \cup \mathcal{B}_0$$

Define $f: c_0(\mathbb{N}) \to \mathbb{F}$ by

$$f\Big(\alpha_0 x_0 + \sum \alpha_n e_n + \sum_{b \in \mathcal{B}_0} \beta(b) b\Big) = \alpha_0.$$

Then $\mathcal{E} \subseteq \ker(f)$, so the kernel is dense, but $x_0 \notin \ker f$ so it is not closed. $\square$

$\boxed{\text{Thm}}$ If X is a normed space and $f: X \to \mathbb{F}$ is a linear functional, then f is continuous iff $\ker f$ is closed.

Proof ($\Rightarrow$) This is immediate.

($\Leftarrow$) Let $Q: X \to X/\ker f$. Since $\ker f \leq X$, Q is continuous by a theorem in Section III.4. Let $T: X/\ker f \to \mathbb{F}$ be an isomorphism. Since $X/\ker f$ is finite dimensional, T is automatically continuous. Define $g := T \circ Q : X \to \mathbb{F}$, which is continuous. Then

$$\ker g = \ker(Q) = \ker f.$$

Thus we have $f = \alpha g$ for some scalar α, and consequently f is continuous. $\square$

- If X is a normed space, then a linear functional $f: X \to \mathbb{F}$ is in particular a linear transformation. Hence all of the equivalent conditions for f being continuous hold, and

$$\|f\| = \sup_{x \in X \setminus \{0\}} \frac{|f(x)|}{\|x\|} = \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|x\| = 1} |f(x)| = \inf\{c > 0 : |f(x)| \leq c\|x\| \ \forall x \in X\}.$$

$\boxed{\text{Def}}$ Let X be a normed space. A bounded linear functional $f: X \to \mathbb{F}$ is a continuous linear functional. The collection of bounded linear functionals on X is called the dual space of X and is denoted X^*.

- Note that $X^* = \mathcal{B}(X, \mathbb{F})$.

$\boxed{\text{Prop}}$ If X is a normal space, then X^* is a Banach space.

Proof Since $\mathbb{F}$ is a Banach space, $X^* = \mathcal{B}(X, \mathbb{F})$ is a Banach space. $\square$

$\boxed{\text{Ex}}$ 0 For $\mathcal{H}$ a Hilbert space, $\mathcal{H}^*$ is isometrically isomorphic to $\mathcal{H}$ by the Reisz rep. Thm.

1 Let (X, Ω, μ) be a σ-finite measure space. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, let $g \in L^q(X, \Omega, \mu)$ and define $F_g \in L^p(X, \Omega, \mu)^*$

$$F_g(f) := \int_X fg \, d\mu.$$

Then

$$\begin{aligned} L^q(X, \Omega, \mu) &\to L^p(X, \Omega, \mu)^* \\ g &\mapsto F_g \end{aligned}$$

is an isometric isomorphism.

2 Let (X, Ω, μ) be a σ-finite measure space. For $g \in L^\infty(X, \Omega, \mu)$, define $F_g \in L^1(X, \Omega, \mu)^*$ by

$$F_g(f) = \int_X fg \, d\mu$$

Then

$$\begin{aligned} L^\infty(X, \Omega, \mu) &\to L^1(X, \Omega, \mu)^* \\ g &\mapsto F_g \end{aligned}$$

is an isometric isomorphism. $\square$

Let X be a set and let Ω be a σ-algebra. For μ a (complex-valued) measure on (X,Ω),
$$|\mu|(E) := \sup\left\{ \sum_{j=1}^{\infty} |\mu(E_j)| : \{E_j\}_{j=1}^{\infty} \text{ is a } \Omega\text{-measurable partitition of } E \right\}$$
defines a positive measure on (X,Ω). The total variation of μ is
$$\|\mu\| = |\mu|(X)$$

Def Let X be a locally compact Hausdorff space. Let Ω be the σ-algebra generated by the open subsets of X. A positive measure μ on (X,Ω) is a regular Borel measure if

1. $\mu(K) < \infty$ for all compact $K \subset X$.
2. $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ for all $E \in \Omega$
3. $\mu(E) = \inf\{\mu(U) : U \supset E \text{ open}\}$ for all $E \in \Omega$.

If μ is complex-valued, we say μ is regular Borel if $|\mu|$ is regular Borel.

Notation Let X be a locally compact Hausdorff space. Denote by $M(X)$ the space of $\mathbb{F}$-valued regular Borel measures on X, which is a normed space when equipped with the total variation norm.

Thm (Riesz Representation Theorem)
If X is a locally compact Hausdorff space and $\mu \in M(X)$, define $F_\mu : C_0(X) \to \mathbb{F}$ by
$$F_\mu(f) = \int_X f \, d\mu.$$
Then $F_\mu \in C_0(X)^*$ and the map
$$M(X) \to C_0(X)^*$$
$$\mu \mapsto F_\mu$$
is an isometric isomorphism.

Proof See Theorem 7.17 of Folland's Real Analysis. $\square$

Ex ① $C_0(\mathbb{N}) = c_0(\mathbb{N})$, while $\mu \in M(\mathbb{N})$ satisfies
$$|\mu|(\mathbb{N}) = \sum_{n=1}^{\infty} |\mu(\{n\})|$$
That is, $(\mu(\{n\}))_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$. In fact $M(\mathbb{N})$ is isometrically isomorphic to $\ell^1(\mathbb{N})$ by this map. Thus the theorem implies
$$c_0(\mathbb{N})^* \cong \ell^1(\mathbb{N})$$

② As a special case of the examples on the previous page, we obtain
$$\ell^1(\mathbb{N})^* \cong \ell^\infty(\mathbb{N})$$ $\square$

- Consider
$$L^\infty(M(X)) := \{ F : M(X) \to L^\infty(X,\mu) \ : \ F(\mu) = F(\nu) \ \mu\text{-a.e. if } \mu \ll \nu \}$$
This will turn out to be the dual of $M(X)$.

Lemma For $F \in L^\infty(M(X))$,
$$\|F\| := \sup_{\mu \in M(X)} \|F(\mu)\|_\infty < \infty$$

Proof If $\|F\| = \infty$, then $\exists (\mu_n)_{n\in\mathbb{N}} \subseteq M(X)$ s.t. $\|F(\mu_n)\|_\infty \geq n$ $\forall n \in \mathbb{N}$. Consider
$$\mu := \sum_{n=1}^\infty \frac{2^{-n}}{\|\mu_n\|} \cdot |\mu_n| \in M(X)$$

Then $\mu_n \ll \mu$ $\forall n \in \mathbb{N}$ so that $F(\mu) = F(\mu_n)$ μ_n-a.e. Therefore
$$\|F(\mu)\|_\infty \geq \|F(\mu_n)\|_\infty \geq n \qquad \forall n \in \mathbb{N}$$
a contradiction. $\square$

Thm Let X be a locally compact Hausdorff space and $F \in L^\infty(M(X))$.
Define $\Phi_F : M(X) \to \mathbb{F}$ by
$$\Phi_F(\mu) = \int F(\mu)\, d\mu.$$
Then $\Phi_F \in M(X)^*$ and the map
$$L^\infty(M(X)) \longrightarrow M(X)^*$$
$$F \longmapsto \Phi_F$$
is an isometric isomorphism.

Proof We first show Φ_F is linear. For $\mu, \nu \in M(X)$ and $\alpha \in \mathbb{F}\setminus\{0\}$
$$\mu + \alpha\nu \ll |\mu| + |\alpha|\,|\nu| \gg \mu, \nu$$
Consequently
$$\Phi_F(\mu + \alpha\nu) = \int F(\mu+\alpha\nu)\, d(\mu+\alpha\nu) = \int F(|\mu| + |\alpha|\,|\nu|)\, d(\mu+\alpha\nu)$$
$$= \int F(|\mu| + |\alpha|\,|\nu|)\, d\mu + \alpha \int F(|\mu| + |\alpha|\,|\nu|)\, d\nu$$
$$= \int F(\mu)\, d\mu + \alpha \int F(\nu)\, d\nu = \Phi_F(\mu) + \alpha \Phi_F(\nu).$$
Also
$$|\Phi_F(\mu)| \leq \int |F(\mu)|\, d|\mu| \leq \|F(\mu)\|_\infty \cdot \|\mu\| \leq \|F\| \cdot \|\mu\|.$$
Thus $\Phi_F \in M(X)^*$ with $\|\Phi_F\| \leq \|F\|$.

Next, fix $\Phi \in M(X)^*$. Note that for $\mu \in M(X)$, the Radon-Nikodym theorem implies
$$L^1(X, |\mu|) \xleftrightarrow{1-1} \{\nu \in M(X) : \nu \ll |\mu|\}$$
$$f \longmapsto f\mu$$
In fact, this is an isometric isomorphism (Exercise). Therefore $f \mapsto \Phi(f\mu)$ defines a linear functional on $L^1(X, |\mu|)$. Since
$$|\Phi(f\mu)| \leq \|\Phi\|\,\|f\|,$$
we have $(f \mapsto \Phi(f\mu)) \in L^1(X, |\mu|)^* \cong L^\infty(X, |\mu|)$. So $\exists F(\mu) \in L^\infty(X, |\mu|)$ satisfying
$$\Phi(f\mu) = \int f F\, d\mu \qquad \forall f \in L^1(X, |\mu|).$$

Consequently,

$$* \quad \|F(\mu)\|_\infty = \sup_{\substack{f\in L^1(X,|\mu|)\\ \|f\|_1=1}} |\Phi(f\mu)| \le \|\Phi\|$$

Observe that $\Phi(\mu) = \int 1\, F(\mu)\, d\mu = \int F(\mu)\, d\mu$. So if we can show $F \in L^\infty(M(X))$, then $\Phi = \Phi_F$ and by $(*)$

$$\|\Phi_F\| = \|\Phi\| \ge \sup_{\mu\in M(X)} \|F(\mu)\|_\infty = \|F\|.$$

Let $\mu, \nu \in M(X)$ satisfy $\nu \ll \mu$. Then $\exists\, f \in L^1(X,|\mu|)$ s.t. $\nu = f\mu$. For $g \in L^1(X,|\nu|)$ we have $gf \in L^1(X,|\mu|)$ with $\int gf\, d\mu = \int g\, d\nu$. That is, $g\nu = gf\mu$. Consequently

$$\int g\, F(\nu)\, d\nu = \Phi(g\nu) = \Phi(gf\mu) = \int gf\, F(\mu)\, d\mu = \int g\, F(\mu)\, d\nu$$

Since $g \in L^1(X,|\nu|)$ was arbitrary, we have $F(\nu) = F(\mu)$ ν-a.e. $\square$

III.6 The Hahn-Banach Theorem

- The Hahn-Banach theorem is one of the most important theorems in functional analysis. It is used so frequently throughout the subject that it and its numerous corollaries are typically silently invoked.

Def If X is a vector space, a sublinear functional is a function $q: X \to \mathbb{R}$ satisfying
1. $q(x+y) \le q(x) + q(y)$ $\forall x, y \in X$.
2. $q(\alpha x) = \alpha q(x)$ for $\alpha \ge 0$ and $x \in X$.

- Any seminorm (and consequently any norm) is a sublinear function. But since 2 only applies to $\alpha \ge 0$, the converse is not true.

Ex Define $q: \mathbb{R} \to \mathbb{R}$ by

$$q(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{otherwise} \end{cases}$$

Then one can check q is a sublinear functional on $\mathbb{R}$, but for $\alpha < 0$, $q(\alpha x) \ne |\alpha| q(x)$ so it is not a seminorm.

For any linear $f: X \to \mathbb{R}$, $q \circ f$ is a sublinear functional. $\square$

Thm (The Hahn-Banach Theorem)

Let X be a vector space over $\mathbb{R}$, and let $q: X \to \mathbb{R}$ be a sublinear functional. If $Y \le X$ is a subspace and $f: Y \to \mathbb{R}$ is a linear functional satisfying

$$f(y) \le q(y) \quad \forall y \in Y,$$

then there is a linear functional $F: X \to \mathbb{R}$ such that $F|_Y = f$ and $F(x) \le q(x)$ $\forall x \in X$.

- The important part of the theorem is not that an extension exists, but that we can find one that is still dominated by q. Indeed, extending f to X is trivial: pick a Hamel basis for Y and extend it to a Hamel basis for X, then define F arbitrarily on this extension.
- Just important is the theorem itself are its various corollaries, which people also sometimes broadly call "The Hahn-Banach Theorem". We explore these corollaries before proving the above.

Lemma Let X be a vector space over $\mathbb{C}$.
1. If $f: X \to \mathbb{R}$ is $\mathbb{R}$-linear, then $\tilde{f}(x) := f(x) - i f(ix)$ is $\mathbb{C}$-linear with $f = \mathrm{Re}\, \tilde{f}$.
2. If $g: X \to \mathbb{C}$ is $\mathbb{C}$-linear and $f := \mathrm{Re}\, g$, then $\tilde{f}$ as above equals g.

(3) If p is a seminorm on X and f and $\tilde{f}$ as above, then
$$|f(x)| \le p(x) \quad \forall x \in X \iff |\tilde{f}(x)| \le p(x) \quad \forall x \in X$$

(4) If X is a normed space and f and $\tilde{f}$ are as above, then $\|f\| = \|\tilde{f}\|$.

Proof (1), (2): Exercises.

(3): Suppose $|f(x)| \le p(x)$ $\forall x \in X$. Fix $x \in X$ and let θ be s.t. $\tilde{f}(x) = e^{i\theta}|\tilde{f}(x)|$. Then
$$|\tilde{f}(x)| = e^{-i\theta}\tilde{f}(x) = \tilde{f}(e^{-i\theta}x) = \operatorname{Re}\tilde{f}(e^{-i\theta}x) = f(e^{-i\theta}x) \le p(e^{-i\theta}x) = p(x)$$

Conversely, suppose $|\tilde{f}(x)| \le p(x)$ $\forall x \in X$. Note that for all $x \in X$
$$\left.\begin{aligned} f(x) &= \operatorname{Re}\tilde{f}(x) \le |\tilde{f}(x)| \le p(x) \\ -f(x) &= \operatorname{Re}\tilde{f}(-x) \le |\tilde{f}(-x)| \le p(-x) = p(x) \end{aligned}\right\} \implies |f(x)| \le p(x).$$

(4): First observe that
$$|f(x)| = |\operatorname{Re}\tilde{f}(x)| \le |\tilde{f}(x)|$$
so $\|f\| \le \|\tilde{f}\|$. Conversely, fix $x \in X$ and let θ be s.t. $\tilde{f}(x) = e^{i\theta}|\tilde{f}(x)|$. Then by the same computation as in (3) we have $|\tilde{f}(x)| = f(e^{-i\theta}x)$. Consequently
$$|\tilde{f}(x)| = f(e^{-i\theta}x) \le \|f\| \cdot \|e^{-i\theta}x\| = \|f\| \cdot \|x\|.$$
Hence $\|\tilde{f}\| = \|f\|$. $\square$

Cor (Complex Hahn-Banach Theorem)
Let X be a vector space, let $Y \le X$ be a subspace, and let p be a seminorm on X. If $f: Y \to \mathbb{F}$ is a linear functional such that
$$|f(y)| \le p(y) \quad \forall y \in Y,$$
then there is a linear functional $F: X \to \mathbb{F}$ such that $F|_Y = f$ and $|F(x)| \le p(x)$ for all $x \in X$.

Proof If $\mathbb{F} = \mathbb{R}$, then observe
$$f(y) \le |f(y)| \le p(y) \quad \forall y \in Y.$$
So the Hahn-Banach theorem implies $\exists F: X \to \mathbb{R}$ such that $F|_Y = f$ and $F(x) \le p(x)$ $\forall x \in X$. But then
$$-F(x) = F(-x) \le p(-x) = p(x) \quad \forall x \in X$$
so that $|F(x)| \le p(x)$.

If $\mathbb{F} = \mathbb{C}$, let $f_1 := \operatorname{Re} f$ so that $|f_1(y)| \le p(y)$ $\forall y \in Y$ by the lemma. The previous case implies $\exists F_1: X \to \mathbb{R}$ such that $F_1|_Y = f_1$ and $|F_1(x)| \le p(x)$. Define $F(x) := F_1(x) - iF_1(ix)$. Then $F|_Y = f$ and the lemma implies $|F(x)| \le p(x)$ $\forall x \in X$. $\square$

Cor If X is a normed space, $Y \le X$ is a subspace, and $f: Y \to \mathbb{F}$ is a bounded linear functional, then there exists $F \in X^*$ such that $F|_Y = f$ and $\|F\| = \|f\|$.

Proof Define a seminorm $p(x) := \|f\|\,\|x\|$. Then $|f(y)| \le p(y)$ and the previous corollary yields $F: X \to \mathbb{F}$ s.t. $F|_Y = f$ and
$$|F(x)| \le \|f\| \cdot \|x\| \quad \forall x \in X.$$
Thus $\|F\| \le \|f\|$. But $F|_Y = f$ implies $\|F\| = \|f\|$. $\square$

Cor If X is a normed space, $\{x_1, x_2, \dots, x_d\} \subseteq X$ is linearly independent, and $\alpha_1, \dots, \alpha_d \in \mathbb{F}$, then there is an $f \in X^*$ such that $f(x_j) = \alpha_j$ for $j = 1, \dots, d$.

Proof Let $Y = \operatorname{span}\{x_1, \dots, x_d\}$ and define $g : Y \to \mathbb{F}$ by
$$g\left(\sum \beta_j x_j\right) = \sum \beta_j \alpha_j$$
Then g is bounded (since Y is finite-dimensional) and so the previous corollary yields the continuous extension. $\square$

Cor If X is a normed space, then for $x \in X$
$$\|x\| = \sup\{|f(x)| : f \in X^* \text{ with } \|f\| \le 1\}.$$
Moreover, there exists $f_x \in X^*$ such that $|f_x(x)| = \|x\|$ and $\|f_x\| = 1$.

Proof Fix $x \in X$ and let α be the supremum. We immediately have $\alpha \le \|f\|$. Conversely, consider $Y := \operatorname{span}\{x\}$, and define $g : Y \to \mathbb{F}$ by
$$g(\beta x) = \beta \|x\|.$$
Then g is a bounded linear functional with $\|g\| = 1$. Let $f \in X^*$ be its continuous extension, then
$$f(x) = g(x) = \|x\|$$
So that $\alpha = \|x\| = f(x) = |f(x)|$. $\square$

Cor If X is a normed space, $Y \le X$, $x_0 \in X \setminus Y$ with $d := \operatorname{dist}(x_0, M)$, then there is $f \in X^*$ such that $f(x_0) = 1$, $f|_Y \equiv 0$, and $\|f\| = 1/d$.

Proof Let $Q : X \to X/Y$ be the quotient map. Since $\|Q(x_0)\| = d$, the previous corollary implies $\exists\, g \in (X/Y)^*$ with $g(x_0 + Y) = d$ and $\|g\| = 1$. Let $f := \frac{1}{d} g \circ Q \in X^*$. We have
$$f(x_0) = \tfrac{1}{d} g(x_0 + Y) = 1$$
$$f(y) = \tfrac{1}{d} g(0 + Y) = 0 \qquad \forall y \in Y$$
and $\|f\| \le \frac{1}{d}\|g\| \cdot \|Q\| = \frac{1}{d}$. On the other hand, let $(x_n)_{n\in\omega} \subset X$ be such that $\|Q(x_n)\| \le 1$ and
$$\lim_{n\to\infty} |g(x_n + Y)| = \|g\| = 1.$$
Let $y_n \in Y$ be s.t. $\|x_n + y_n\| \le 1 + \frac{1}{n}$. Then
$$\|f\| \ge \limsup_{n\to\infty} \frac{|f(x_n + y_n)|}{\|x_n + y_n\|} \ge \limsup_{n\to\infty} \frac{1}{d}\frac{|g(x_n + Y)|}{1 + \frac{1}{n}} = \frac{1}{d}$$
So $\|f\| = 1/d$. $\square$

Thm If X is a normed space and $Y \le X$ is a subspace, then
$$\overline{Y} = \bigcap \{\ker f : f \in X^* \text{ with } Y \subseteq \ker f\}$$

Proof The inclusion $\subseteq$ is immediate by the continuity of each $f \in X^*$. Now, for any $x_0 \notin \overline{Y}$, let $d := \operatorname{dist}(x_0, Y) > 0$. Then by the previous corollary $\exists f \in X^*$ with $f(x_0) = 1$ and $f|_Y \equiv 0$. Thus $x_0 \notin \bigcap\{\ker f : f \in X^* \text{ with } Y \subseteq \ker f\}$. This implies the inclusion $\supseteq$. $\square$

Cor In a normed space X, a subspace $Y \le X$ is dense if and only if $Y \subseteq \ker f$ for f^* implies $f \equiv 0$.

- As a lemma to the Hahn-Banach theorem, we first show the result holds for hyperplanes.

Lemma Let X, Y, q, and f be as in the Hahn-Banach theorem. If $\dim(X/Y)=1$, then there exists $F: X \to \mathbb{R}$ such that $F|_Y = f$ and $F(x) \le q(x)$ $\forall x \in X$.

Proof Fix $x_0 \in X \setminus Y$, so that $\forall x \in X$, $x = tx_0 + y$ for some $t \in \mathbb{R}$ and $y \in Y$. Note that t is uniquely determined by the quotient map $Q(x) = tx_0 + Y$, and consequently $y := x - tx_0$ is also unique. Thus to define $F: X \to \mathbb{R}$, it suffices to define $F(x_0)$. We claim that if $\alpha_0 \in \mathbb{R}$ satisfies

$$* \qquad \sup_{y \in Y} \left(f(y) - q(y - x_0)\right) \le \alpha_0 \le \inf_{y \in Y} \left(-f(y) + q(y + x_0)\right)$$

then $F(x_0) := \alpha_0$ works. We will need to show that $(*)$ can be satisfied, but if it is then we have for $t > 0$

$$F(tx_0 + y) = t\alpha_0 + f(y) = t(\alpha_0 + f(\tfrac{1}{t}y)) \le t\, q(\tfrac{1}{t}y + x_0) = q(tx_0 + y)$$

and

$$F(-tx_0 + y) = -t\alpha_0 + f(y) = -t(\alpha_0 - f(\tfrac{1}{t}y)) \le -t(-q(\tfrac{1}{t}y - x_0)) = q(-tx_0 + y)$$

Thus $F(x) \le q(x)$ $\forall x \in X$. So it remains to show the supremum in $(*)$ is dominated by the infimum. For $y_1, y_2 \in Y$, we have:

$$f(y_1) + f(y_2) = f(y_1 + y_2) \le q(y_1 + y_2) = q(y_1 - x_0 + y_2 + x_0) \le q(y_1 - x_0) + q(y_2 + x_0)$$

This implies

$$f(y_1) - q(y_1 - x_0) \le -f(y_2) + q(y_2 + x_0)$$

Thus taking a supremum over $y_1 \in Y$ followed by an infimum over $y_2 \in Y$, we see that $(*)$ can be satisfied. $\square$

Proof of the Hahn-Banach Theorem

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We will use Zorn's lemma to find the desired extension. Let $\mathcal{Z}$ be the collection of pairs (Z, g) consisting of intermediate subspaces $Y \le Z \le X$ and linear functionals $g: Z \to \mathbb{R}$ satisfying $g|_Y = f$ and $g(z) \le q(z)$ $\forall z \in Z$. For $(Z_1, g_1), (Z_2, g_2) \in \mathcal{Z}$, write $(Z_1, g_1) \preceq (Z_2, g_2)$ if $Z_1 \le Z_2$ and $g_2|_{Z_1} = g_1$. Then $(\mathcal{Z}, \preceq)$ is a partially ordered set. Suppose $\mathcal{C} = \{(Z_i, g_i) : i \in I\}$ is a chain in $\mathcal{Z}$. Define

$$Z := \bigcup_{i \in I} Z_i$$

which is a subspace as a union of subspaces. Also $Y \le Z_i \le X$ $\forall i \in I \Rightarrow Y \le Z \le X$. Define $g: Z \to \mathbb{R}$ by $g(z) = g_i(z)$ if $z \in Z_i$. Then one easily checks that g is well-defined, linear, and satisfies $g(z) \le q(z)$ $\forall z \in Z$. Hence $(Z, g) \in \mathcal{Z}$ and so is an upper bound for $\mathcal{C}$. By Zorn's Lemma we obtain a maximal element $(M, F) \in \mathcal{Z}$. If $M \ne X$, then using the lemma we can contradict the maximality of (M, F). Thus $M = X$ and F is the desired extension. $\square$

III.10 Duals of Quotient Spaces.

Ex Let $\mathcal{H}$ be a Hilbert space and let $K \leq \mathcal{H}$. As a Banach space, $\mathcal{H}^* = \mathcal{H}$ by the Reisz representation theorem. In particular,

$$K^\perp = \{h \in \mathcal{H} : \langle k, h \rangle = 0 \ \forall k \in K\} = \{L \in \mathcal{H}^* : K \leq \ker L\}.$$ □

- Let X be a normed space and let $Y \leq X$. In light of the previous example we define
$$Y^\perp := \{f \in X^* : Y \leq \ker f\}.$$
By a theorem from Section III.5, we have
$$Y = \bigcap_{f \in Y^\perp} \ker f.$$

- Observe that $Y^\perp \leq X^*$, so we can form the quotient $X^*/Y^\perp$. For $f + Y^\perp, g + Y^\perp \in X^*/Y^\perp$ we have
$$f + Y^\perp = g + Y^\perp \iff f - g \in Y^\perp \iff f|_Y = g|_Y.$$
That is, each element $f + Y^\perp \in X^*/Y^\perp$ defines an element of Y^*, namely $f|_Y$. Moreover, by the Hahn-Banach theorem (or one of its corollaries), every element of Y^* arises as the restriction of some $f \in X^*$.

Thm For a normed space X and $Y \leq X$, the map
$$\rho : X^*/Y^\perp \to Y^*$$
$$f + Y^\perp \mapsto f|_Y$$
is an isometric isomorphism.

Proof The map is well-defined and bijective by the discussion preceding the theorem. It remains to show it is isometric. For $f \in X^*$ and $g \in Y^\perp$ we have $f|_Y = (f+g)|_Y$. So
$$\|f|_Y\| = \|(f+g)|_Y\| \leq \|f+g\|$$
Taking an infimum over $g \in Y^\perp$ yields $\|f|_Y\| \leq \|f + Y^\perp\|$. Conversely, by a Hahn-Banach corollary $\exists F \in X^*$ with $F|_Y = f|_Y$ and $\|F\| = \|f|_Y\|$. Therefore
$$\|f|_Y\| = \|F\| \geq \|F + Y^\perp\| = \|f + Y^\perp\|.$$ □

- We consider now the dual perspective of the above theorem.

Thm Let X be a normed space with $Y \leq X$, and let $Q : X \to X/Y$ be the quotient map. Then the map
$$\rho : (X/Y)^* \to Y^\perp$$
$$f \mapsto f \circ Q$$
is an isometric isomorphism.

Proof First note that for all $x \in X$
$$|f \circ Q(x)| = |f(x+Y)| \leq \|f\| \, \|x+Y\| \leq \|f\| \, \|x\|$$

So $f\circ Q \in X^*$ with $\|f\circ Q\| \le \|f\|$. Also, since $Y \subset \ker Q$ we have $f\circ Q \in Y^\perp$, and $f \mapsto f\circ Q$ is clearly linear. Let $(x_n+Y)_{n\in\mathbb{N}}$ be such that $\|x_n+Y\|=1$ and

$$\lim_{n\to\infty} |f(x_n+Y)| = \|f\|$$

Let $y_n \in Y$ be such that

$$\|x_n+y_n\| \le 1+\tfrac{1}{n}.$$

Then

$$\lim_{n\to\infty} \frac{|f\circ Q(x_n+y_n)|}{\|x_n+y_n\|} \ge \lim_{n\to\infty} \frac{|f(x_n+Y)|}{1+\frac{1}{n}} = \|f\|$$

which implies $\|f\circ Q\| = \|f\|$. So ρ is an isometry (and therefore injective). It remains to show it is surjective. Let $g \in Y^\perp \subseteq X^*$. Define $f: X/Y \to \mathbb{F}$ by

$$f(x+Y) := g(x)$$

which is well-defined since $g|_Y \equiv 0$. For $x\in X$, $y \in Y$ we have

$$|f(x+Y)| = |g(x)| = |g(x+y)| \le \|g\|\cdot\|x+y\|.$$

Taking an infimum over $y\in Y$ yields

$$|f(x+Y)| \le \|g\|\,\|x+Y\|$$

so that $f\in (X/Y)^*$ with $\|f\|\le\|g\|$. Clearly $\rho(f) = f\circ Q = g$. $\square$

III.11 Reflexive Spaces

- Recall that whenever X is a normed space, X^* is a Banach space. Consequently, we can consider its dual $(X^*)^* =: X^{**}$, the double dual of X.

Thm Let X be a normed space. For $x \in X$, define $\hat{x}: X^* \to \mathbb{F}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ with $\|\hat{x}\| = \|x\|$.

Proof $\hat{x}$ is clearly linear. Also

$$|\hat{x}(f)| = |f(x)| \leq \|f\| \cdot \|x\|$$

So $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leq \|x\|$. Recall that a Hahn-Banach corollary gives

$$\|x\| = \sup_{\|f\| \leq 1} |f(x)| = \sup_{\|f\| \leq 1} |\hat{x}(f)| = \|\hat{x}\|.$$ $\square$

Def The map $X \ni x \mapsto \hat{x} \in X^{**}$ is called the natural map of X into its double dual. We say X is reflexive if the natural map is surjective.

- Note that if X is reflexive then the natural map is a surjective isometry, hence an isometric isomorphism. However, X being isometrically isomorphic to X^{**} is not sufficient for X to be reflexive. We specifically need the natural map to implement the isomorphism.

Ex (1) For $1 < p < \infty$, $L^p(X, \Omega, \mu)$ is reflexive.

(2) $c_0(\mathbb{N})$ is not reflexive: $c_0(\mathbb{N})^* = \ell^1(\mathbb{N})$ and $\ell^1(\mathbb{N})^* = \ell^\infty(\mathbb{N})$. The natural map is then the inclusion $c_0(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})$, which is clearly not surjective. $\square$

We will revisit reflexivity in Chapter V, in the context of weak topologies

III. 12 The Open Mapping and Closed Graph Theorems

Def Let X be a topological space. A subset $S \subset X$ is called <u>nowhere dense</u> if $\overline{S}^\circ = \phi$.

- Exercise: Show S is nowhere dense iff $\forall U \subseteq X$ open $S \cap U$ is not dense in U.

- Observe that if S is nowhere dense, then $\forall U \subseteq X$ open $U \cap \overline{S}^c \neq \phi$ since otherwise $U \subseteq \overline{S}$, contradicting $\overline{S}^\circ = \phi$. Thus $\overline{S}^c$ is an open dense subset of X.

Thm (Baire Category Theorem)
Let X be a complete metric space.
1. If $\{U_n : n \in \mathbb{N}\}$ is a collection of open dense subsets of X, then $\bigcap_{n=1}^\infty U_n$ is dense in X.
2. X is not a countable union of nowhere dense sets.

Proof (1): Let $W \subseteq X$ be open and non-empty. We must show $W \cap \bigcap_{n=1}^\infty U_n \neq \phi$. By the density of U_1, $W \cap U_1 \neq \phi$. So let x_1 and $r_1 > 0$ be s.t. $\overline{B(x_1, r_1)} \subset W \cap U_1$. By the density of U_2, $B(x_1, r_1) \cap U_2 \neq \phi$. So let x_2 and $r_2 > 0$ be s.t. $\overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap U_2$. By shrinking r_2 if necessary, we may assume $r_2 < \frac{r_1}{2}$. We inductively find sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(r_n)_{n\in\mathbb{N}} \subset (0,\infty)$ s.t.

$$\overline{B(x_{n+1}, r_{n+1})} \subset B(x_n, r_n) \cap U_{n+1} \quad \text{and} \quad r_{n+1} < \frac{r_n}{2} < \cdots < \frac{r_1}{2^n}.$$

Note that for $N \in \mathbb{N}$ and $n, m \geq N$ we have $x_n, x_m \in B(x_N, r_N)$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence since $r_N \to 0$. X is complete, so let $x = \lim_{n\to\infty} x_n$. Since $x_n \in B(x_N, r_N)$ $\forall n \geq N$, it follows that

$$x \in \overline{B(x_N, r_N)} \subset B(x_1, r_1) \cap U_N \subset W \cap U_N.$$

Since this holds for all $N \in \mathbb{N}$, we have $x \in W \cap \bigcap_{n=1}^\infty U_n$.

(2): Let $\{S_n : n \in \mathbb{N}\}$ be a collection of nowhere dense subsets of X. Then $\{\overline{S_n}^c : n \in \mathbb{N}\}$ are open dense subsets. By (1) $\bigcap_{n=1}^\infty \overline{S_n}^c$ is dense, and in particular is nonempty. Thus

$$X \neq \left(\bigcap_{n=1}^\infty \overline{S_n}^c\right)^c = \bigcup_{n=1}^\infty \overline{S_n} \supset \bigcup_{n=1}^\infty S_n. \qquad \square$$

- Countable unions of nowhere dense sets are called <u>meagre</u>.

Thm (The Open Mapping Theorem)
If X, Y are Banach spaces and $A \in B(X,Y)$ is surjective, then $A(U)$ is open for any open $U \subseteq X$.

Proof It suffices to show $\forall x \in X$ and $\varepsilon > 0$, $\exists \delta > 0$ such that

$$B(Ax, \delta) \subset A(B(x, \varepsilon))$$

By linearity, it further suffices to show $B(0, \delta) \subset A(B(0, \varepsilon))$.

Fix $\varepsilon > 0$. First note that
$$X = \bigcup_{n=1}^{\infty} B(0, n\varepsilon)$$
Since A is surjective
$$Y = \bigcup_{n=1}^{\infty} A(B(0,n\varepsilon)) = \bigcup_{n=1}^{\infty} n\, A(B(0,\varepsilon)).$$
Since multiplication by n is a homeomorphism, $A(B(0,\varepsilon))$ is nowhere dense iff $n A(B(0,\varepsilon))$ is nowhere dense (Exercise: verify). Since Y is complete, the Baire category theorem implies $A(B(0,\varepsilon))$ is not nowhere dense. Thus $\exists\, W \subseteq Y$ open s.t. $W \subset \overline{A(B(0,\varepsilon))}$ and so $W \cap A(B(0,\varepsilon)) \neq \phi$.
Let $y_0 = Ax_0 \in W \cap A(B(0,\varepsilon))$ and let $r_0 > 0$ be s.t. $B(y_0, r_0) \subset W$. Observe that for $y \in B(0, r_0)$ we have
$$y = y + y_0 - Ax_0 \in B(y_0, r_0) + A(B(0,\varepsilon)) \subseteq \overline{A(B(0,2\varepsilon))}$$
Thus $B(0, \frac{r_0}{2}) \subset \overline{A(B(0,\varepsilon))}$. To complete the proof, we must remove the closure.
Set $\delta := \frac{r_0}{4}$, and suppose $y \in B(0,\delta)$. By the above $y \in \overline{A(B(0,\varepsilon/2))}$, so let $x_1 \in B(0, \varepsilon/2)$ be such that
$$\| y - Ax_1 \| < \frac{\delta}{2}$$
Since $B(0, \delta/2) \subset \overline{A(B(0,\varepsilon/4))}$, $\exists\, x_2 \in B(0, \varepsilon/4)$ such that
$$\| y - A(x_1 + x_2) \| = \| (y - Ax_1) - Ax_2 \| < \frac{\delta}{4}.$$
We inductively find $(x_n)_{n\in\mathbb{N}} \subseteq X$ satisfying $x_n \in B(0, \varepsilon/2^n)$ and
$$\| y - A(\sum_{i=1}^{n} x_i) \| < \delta/2^n.$$
It follows that $(\sum_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is a Cauchy sequence and if $x := \sum_{i=1}^{\infty} x_i$ then $x \in B(0,\varepsilon)$ and $Ax = y$. Thus $B(0,\delta) \subset A(B(0,\varepsilon))$. $\square$

- The following is an immediate corollary of the open mapping theorem.

Thm (The Inverse Mapping Theorem)
If X and Y are Banach spaces and $A \in B(X,Y)$ is bijective, then $A^{-1} \in B(Y,X)$.

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Thm (The Closed Graph Theorem)
If X and Y are Banach spaces and $A: X \to Y$ is a linear transformation whose graph,
$$\text{graph}(A) := \{ (x, Ax) : x \in X \} \subseteq X \oplus_1 Y$$
is closed, then A is continuous.

Proof Since $X \oplus_1 Y$ is a Banach space, graph(A) is a Banach space. Define
$$P: \text{graph}(A) \to X \qquad\qquad Q: \text{graph}(A) \to Y$$
$$(x, Ax) \mapsto x \qquad\qquad (x, Ax) \mapsto Ax$$
These are both clearly bounded. Moreover, P is a bijection so $P^{-1}: X \to \text{graph}(A)$ is bounded by the Inverse Mapping Theorem. Noting that $A = Q \circ P^{-1}$, we see that A is bounded and hence continuous. $\square$

Prop Let X and Y be normed spaces and let $A: X \to Y$ be a linear transformation. Then $\text{graph}(A)$ is closed iff $x_n \to 0$ in X and $Ax_n \to y$ in Y implies $y = 0$.

Proof Homework 5. $\square$

- This proposition reveals to true utility of the closed graph theorem: suppose $A: X \to Y$ is a linear transformation between Banach spaces. We already know A is bounded iff it is continuous a zero; that is, $x_n \to 0$ in X implies $Ax_n \to 0$. But now the closed graph theorem and the above proposition imply we may assume Ax_n <u>converges to some</u> $y \in Y$, and then just need to show $y = 0$.

Ex Let (X, Ω, μ) be a σ-finite measure space. Fix $1 \leq p \leq \infty$, and suppose $\phi: X \to \mathbb{F}$ is a μ-measurable function such that $\forall f \in L^p(X, \mu)$, $\phi f \in L^p(X, \mu)$. We will show $\phi \in L^\infty(X, \mu)$. We first show $M_\phi : f \mapsto \phi f$ is bounded.

M_ϕ is clearly linear. Suppose $(f_n)_{n \in \mathbb{N}} \subseteq L^p(X, \mu)$ satisfies

$$f_n \to 0 \quad \text{and} \quad M_\phi f_n = \phi f_n \to g$$

in $L^p(X, \mu)$. We must show $g = 0$. Since $\|f_n\|_p \to 0$, there exists a subsequence f_{n_k} s.t. $f_{n_k}(x) \to 0$ for μ-a.e. $x \in X$ (for $p = \infty$ this holds for the sequence itself). Reducing to a further subsequence if necessary, we may assume $\phi(x) f_{n_k}(x) \to g(x)$ for μ-a.e. x. But this implies $g(x) = \phi(x) \cdot 0 = 0$ for μ-a.e. $x \in X$. Thus $g = 0$ and so M_ϕ is bounded.

Now, let $\varepsilon > 0$ and consider

$$E := \{ x \in X : |\phi(x)| \geq \|M_\phi\| + \varepsilon \}$$

For any $F \subseteq X$ with $\mu(F) < \infty$ we have

$$\mu(F \cap E) = \|1_{F \cap E}\|_p^p \geq \frac{1}{\|M_\phi\|^p} \|M_\phi 1_{F \cap E}\|_p^p \geq \frac{1}{\|M_\phi\|^p} (\|M_\phi\| + \varepsilon)^p \mu(F \cap E)$$

Thus we must have $\mu(F \cap E) = 0 \Rightarrow \mu(E) = 0$. Since $\varepsilon > 0$ was arbitrary, it follows that $\phi \in L^\infty(X, \mu)$ with $\|\phi\|_\infty \leq \|M_\phi\|$. $\square$

- We conclude this section by illustrating with a pair of examples that the completeness of X and Y in the closed graph theorem are both necessary.

Ex ① Let $X = C^1[0,1]$ and $Y = C[0,1]$. However, consider both X and Y with the norm

$$\|f\| = \sup_{0 \leq t \leq 1} |f(t)|$$

Then Y is complete while X is not. Define $A: X \to Y$ by $Af = f'$. Observe that for all $n \in \mathbb{N}$, $\|A(t^n)\| / \|t^n\| = n$, so A is <u>not</u> bounded. However, its graph is closed.

Indeed, suppose for $(f_n)_{n\in\mathbb{N}} \subset X$ that $(f_n, f_n') \longrightarrow (f,g)$. Then $f_n \to f$ and $f_n' \to g$ uniformly on $[0,1]$ so that

$$f(t)-f(0) = \lim_{n\to\infty} f_n(t) - f_n(0) = \lim_{n\to\infty} \int_0^t f_n'(t)\,dt = \int_0^t g(t)\,dt$$

Consequently $f'=g$ and $(f,g) \in \text{graph}(A)$.

② Let X be a separable infinite dimensional Banach space. Let $\{e_i : i\in I\}$ be a unit Hamel basis. We claim I is uncountable. Suppose, towards a contradiction, that it is countable, and re-index $\{e_n : n\in\mathbb{N}\}$. Define

$$X_n := \text{span}\{e_1, \dots, e_n\}.$$

Note that X_n is closed since $\dim(X_n) < \infty$. Also $\forall x \in X_n$, $\varepsilon \cdot e_{n+1} + x \notin X_n$ so $X_n^\circ = \emptyset$. Thus X_n is nowhere dense. But

$$X = \bigcup_{n=1}^{\infty} X_n$$

which contradicts the Baire category theorem. So I is uncountable.

For $x = \alpha_1 e_{i_1} + \dots + \alpha_d e_{i_d}$, define

$$\|x\|_1 := \sum_{j=1}^{d} |\alpha_j|$$

Then $\|\cdot\|_1$ is a norm on X (Exercise: verify). Let $Y = X$ equipped with the $\|\cdot\|_1$-norm. Define $T: Y \to X$ by $T(x) = x$. Since $\|e_i\| = 1$ $\forall i\in I$, we have for $x = \sum_{j=1}^{d} \alpha_j e_{i_j}$

$$\|T(x)\| = \|x\| \le \sum_{j=1}^{d} |\alpha_j| = \|x\|_1$$

So T is bounded and consequently $\text{graph}(T)$ is closed. It follows that $T^{-1}: X \to Y$ also has a closed graph since $(x,y) \in \text{graph}(T^{-1})$ iff $(y,x) \in \text{graph}(T)$. However, T^{-1} is not continuous. If it were, then Y would be separable by virtue of X being separable. This is a contradiction because

$$\|e_i - e_j\|_1 = 2 \quad \text{for } i \ne j$$

and so $\{B(e_i, 1) : i \in I\}$ is an uncountable collection of disjoint open sets. $\square$

III.13 Complemented Subspaces of a Banach Space

Def Let X be a normed space. We say $Y \leq X$ is *algebraically complemented* if $\exists Z \leq X$ such that $Y \cap Z = \{0\}$ and $Y + Z = X$.

Ex $X = \mathbb{R}^2$, $Y = \operatorname{span}\begin{pmatrix}1\\0\end{pmatrix}$ then Y is algebraically complemented by $Z = \operatorname{span}\begin{pmatrix}t\\1\end{pmatrix}$ $\forall t \in \mathbb{R}$.
so the complement is not unique

- With $X, Y,$ and Z as above we can define

$$A : Y \oplus^1 Z \to X$$
$$(y, z) \mapsto y + z$$

Then A is a bijection and

$$\|A(y,z)\| = \|y+z\| \leq \|y\| + \|z\| = \|(y,z)\|_1$$

so that A is bounded.

Def Let X be a normed space and $Y, Z \leq X$ algebraically complementary subspaces. We say Y and Z are *topologically complemented* if $A : Y \oplus^1 Y \to X$ defined as above is a homeomorphism.

- The following is an immediate corollary of the Inverse Mapping Theorem:

Thm If X is a Banach space, all algebraically complemented subspaces are topologically complemented.

- Thus in a Banach space we will henceforth just say "complemented" or "complementary".

Thm Let X be a Banach space

1. If Y, Z are complementary subspaces and $E : X \to X$ is defined by $E(y+z) = y$ for all $y \in Y$, $z \in Y$, then $E \in B(X)$ with $E^2 = E$, $\operatorname{ran} E = Y$, and $\ker E = Z$.
2. If $E \in B(X)$ satisfies $E^2 = E$, then $Y := \operatorname{ran} E$ and $Z := \ker E$ are complementary.

Facts
- $c_0(\mathbb{N})$ is not complemented in $\ell^\infty(\mathbb{N})$.
- If $Y \leq \ell^\infty(\mathbb{N})$ is infinite dimensional and complemented then Y is isomorphic to $\ell^\infty(\mathbb{N})$
- For X a Banach space, X is isomorphic to a Hilbert space iff every $Y \leq X$ is complemented.

III.14 The Principle of Uniform Boundedness

- The titular result says that any set of bounded operators on a Banach space that is pointwise bounded is uniformly bounded.

Thm (The Principle of Uniform Boundedness)
Let X be a Banach space and Y a normed space. For $\mathcal{C} \subseteq B(X,Y)$ if
$$\sup_{A\in\mathcal{C}} \|Ax\| < \infty \qquad \forall x \in X$$
then
$$\sup_{A\in\mathcal{C}} \|A\| < \infty.$$

Proof For $x \in X$, define
$$M(x) := \sup_{A\in\mathcal{C}} \|Ax\|.$$
Suppose, towards a contradiction, that $\sup_{A\in\mathcal{C}} \|A\| = \infty$. Then $\exists (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ such that $\|A_n\| > 4^n$, $\forall n \in \mathbb{N}$. Let $x_n \in X$ be s.t. $\|x_n\| = 1$ and $\|A_n x_n\| \geq 4^n$, $\forall n \in \mathbb{N}$. Define $y_n := 2^{-n} x_n$ so that $\|y_n\| = 2^{-n} \to 0$ and $\|A_n y_n\| \geq 2^n \to \infty$. We can then inductively find a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that the
$$\|y_{n_{k+1}}\| < \frac{1}{2^{k+1} \max_{1\leq j\leq k} \|A_{n_j}\|}$$
and
$$\|A_{n_{k+1}} y_{n_{k+1}}\| > 1 + k + \sum_{j=1}^{k} \sup_{A\in\mathcal{C}} \|A y_{n_j}\|$$

Since $\sum_{k=1}^{\infty} \|y_{n_k}\| < \infty$, we can consider $y := \sum_{k=1}^{\infty} y_{n_k} \in X$ since X is complete.
For any $k \in \mathbb{N}$ we have
$$\|A_{n_{k+1}} y\| = \| A_{n_{k+1}} \Big(\sum_{j=1}^{k} y_{n_j} + y_{n_{k+1}} + \sum_{j=k+2}^{\infty} y_{n_j} \Big) \|$$
$$\geq \|A_{n_{k+1}} y_{n_{k+1}}\| - \sum_{j=1}^{k} \|A_{n_{k+1}} y_j\| - \sum_{j=k+2}^{\infty} \|A_{n_{k+1}}\| \cdot \|y_{n_j}\|$$
$$\geq \Big(1 + k + \sum_{j=1}^{k} \sup_{A\in\mathcal{C}} \|A y_{n_j}\|\Big) - \sum_{j=1}^{k} \|A_{n_{k+1}} y_j\| - \sum_{j=k+2}^{\infty} \frac{1}{2^j} \geq 1 + k - \frac{1}{2^{k+1}} \geq k.$$
But this contradicts $\sup_{A\in\mathcal{C}} \|Ay\| < \infty$. □

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Cor For X a normed space and $A \subseteq X$, A is bounded if and only if
$$\sup_{a\in A} |f(a)| < \infty \qquad \forall f \in X^*$$

Proof Identify $\hat{X} \subseteq X^{**} = B(X^*, \mathbb{F})$ via the natural map. Then since X^* is a Banach space and
$$\sup_{a\in A} |f(a)| = \sup_{\hat{a}\in\hat{A}} |\hat{a}(f)|,$$
the Principle of Uniform Boundedness implies the above being bounded for all $f \in X^*$ iff $\hat{A}$ is bounded. But $\|\hat{a}\| = \|a\|$, so the result follows. □

Cor For X a Banach space and $B \subseteq X^*$, B is bounded if and only if

$$\sup_{f \in B} |f(x)| < \infty \qquad \forall x \in X.$$

Proof Since $X^* = B(X, \mathbb{F})$, this is a special case of the Principle of Unif. Bddness. $\square$

Cor Let X be a Banach space and Y a normed space. For $\mathcal{C} \subseteq B(X,Y)$ if

$$\sup_{A \in \mathcal{C}} |g(Ax)| < \infty \qquad \forall x \in X, g \in Y^*$$

then

$$\sup_{A \in \mathcal{C}} \|A\| < \infty$$

Proof By the corollary before last, $\{Ax : A \in \mathcal{C}\}$ is a bounded set in Y for all $x \in X$. But then the Principle of Uniform Boundedness implies $\mathcal{C}$ is bounded. $\square$

Thm (Banach - Steinhaus Theorem)
Let X and Y be Banach spaces. Suppose $(A_n)_{n \in \mathbb{N}} \subseteq B(X,Y)$ is such that for all $x \in X$ $A_n x$ converges to some $y \in Y$. Then $(A_n)_{n \in \mathbb{N}}$ is uniformly bounded and converges in the strong operator topology to some $A \in B(X,Y)$ satisfying $\|A\| \leq \limsup_{n \to \infty} \|A_n\|$.

Proof Since $(A_n x)_{n \in \mathbb{N}} \subseteq Y$ is a convergent sequence it is bounded. Hence

$$\sup_{n \in \mathbb{N}} \|A_n x\| < \infty \qquad \forall x \in X.$$

Therefore $(A_n)_{n \in \mathbb{N}}$ is uniformly bounded. Define $A : X \to Y$ by

$$Ax := \lim_{n \to \infty} A_n x.$$

Then A is clearly linear, and moreover

$$\|Ax\| = \lim_{n \to \infty} \|A_n x\| \leq \limsup_{n \to \infty} \|A_n\| \, \|x\| \leq \left(\sup_{n \in \mathbb{N}} \|A_n\|\right) \cdot \|x\|$$

Thus A is bounded and the above shows $\|A\| \leq \limsup_{n \to \infty} \|A_n\|$. $\square$

⚠ The Banach Steinhaus theorem does not apply to nets. This is because a convergent net need not be bounded. Indeed, $\left(\frac{1}{t}\right)_{t \in (0,\infty)}$ converges as a net to zero but clearly is not bounded.

Prop Let X be a locally compact Hausdorff space. For $(f_n)_{n \in \mathbb{N}} \subseteq C_0(X)$ and $f \in C_0(X)$, $\int f_n \, d\mu \to \int f \, d\mu$ $\forall \mu \in M(X)$ if and only if $\sup_n \|f_n\| < \infty$ and $f_n \to f$ pointwise on X.

Proof ($\Rightarrow$) Since $M(X) = C_0(X)^*$, we have $\sup_n \|f_n\| < \infty$ by one of the corollaries to the Principle of Uniform Boundedness. Fix $x \in X$ and let $\mu = \delta_x$. Then $f_n(x) = \int f_n \, d\mu \longrightarrow \int f \, d\mu = f(x)$. So $f_n \to f$ pointwise.
($\Leftarrow$) This is the dominated convergence theorem. $\square$