1. Let μ be a **finitely additive** measure on a measurable space $(X, \mathcal{M})$. Show that the following statements are equivalent:

   (i) μ is a measure.

   (ii) For any sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$-measurable satisfying $0 \leq f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$, one has
   $$\int_X \sup_{n\in\mathbb{N}} f_n \, d\mu = \sup_{n\in\mathbb{N}} \int_X f_n \, d\mu.$$ [1]

   (iii) For any sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$-measurable functions satisfying $f_n \geq 0$ for all $n \in \mathbb{N}$, one has
   $$\int_X \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Proof. **(i)⇒(ii):** Note that $0 \leq f_n \leq f_{n+1}$ implies $\sup_n f_n = \lim_n f_n$ and that the integrals are likewise increasing. So by the monotone convergence theorem we have

$$\int_X \sup_{n\in\mathbb{N}} f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu = \sup_{n\in\mathbb{N}} \int_X f_n \, d\mu.$$

(ii)⇒(iii): Observe that $g_N := \sum_{n=1}^{N} f_n$ defines an increasing sequence of non-negative functions with $\sup_N g_N = \sum_{n=1}^{\infty} f_n$, and so by (ii) and the linearity of the integral we have

$$\int_X \sum_{n=1}^{\infty} f_n \, d\mu = \int_X \sup_{N\in\mathbb{N}} g_N \, d\mu = \sup_{N\in\mathbb{N}} \int_X g_N \, d\mu = \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} \int_X f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

(iii)⇒ (i): Let $E_n \in \mathcal{M}$, $n \in \mathbb{N}$ be a sequence of disjoint subsets. Noting that

$$\sum_{n=1}^{\infty} 1_{E_n} = 1_E,$$

where $E = \bigcup_{n=1}^{\infty} E_n$, (iii) then gives

$$\mu(E) = \int_X 1_E \, d\mu = \int_X \sum_{n=1}^{\infty} 1_{E_n} \, d\mu = \sum_{n=1}^{\infty} \int_X 1_{E_n} \, d\mu = \sum_{n=1}^{\infty} \mu(E_n).$$

Hence μ is countably additive and therefore a measure. □

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ are sequences of $\mathcal{M}$-measurable functions converging in measure to f and g, respectively. Suppose there exists $R > 0$ so that $|f_n(x)|, |g_n(x)| \leq R$ for μ-almost every $x \in X$. Show that $(f_n g_n)_{n\in\mathbb{N}}$ converges in measure to fg.
   [**Hint:** first show $|f(x)|, |g(x)| \leq R$ for μ-almost every $x \in X$.]

Proof. The convergence in measure of $(f_n)_{n\in\mathbb{N}}$ to f implies by a result from lecture that there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges to f μ-almost everywhere. Hence for μ-almost every $x \in X$ we have

$$|f(x)| = \lim_{k\to\infty} |f_{n_k}(x)| \leq R.$$

[1] For an $\mathcal{M}$-measurable function $f \colon X \to [0, +\infty]$, we define its integral with respect to μ in exactly the same way as with respect to a measure, and in particular you may assume integration is linear and monotone.

Similarly for g.

Now, fix $\epsilon > 0$ and observe that $|f_n(x)g_n(x) - f(x)g(x)| \geq \epsilon$ implies

$$\epsilon \leq |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)| \leq |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)|,$$

and for μ-almost every $x \in X$ we further have

$$\epsilon \leq |f_n(x) - f(x)|R + R|g_n(x) - g(x)|.$$

Also note that, in this case at least one of $|f_n(x) - f(x)|R \geq \frac{\epsilon}{2}$ or $R|g_n(x) - g(x)| \geq \frac{\epsilon}{2}$ must be true, since otherwise we obtain the following contradiction:

$$\epsilon \leq |f_n(x) - f(x)|R + R|g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

These observations imply that

$$\begin{aligned}
&\mu(\{x \in X \colon |f_n(x)g_n(x) - f(x)g(x)| \geq \epsilon\}) \\
&\qquad \leq \mu\left(\left\{x \in X \colon |f_n(x) - f(x)|R \geq \frac{\epsilon}{2}\right\} \cup \left\{x \in X \colon R|g_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}\right) \\
&\qquad \leq \mu\left(\left\{x \in X \colon |f_n(x) - f(x)| \geq \frac{\epsilon}{2R}\right\}\right) + \mu\left(\left\{x \in X \colon |g_n(x) - g(x)| \geq \frac{\epsilon}{2R}\right\}\right).
\end{aligned}$$

The last two terms tend to zero as $n \to \infty$ by the assumed convergence in measure, and consequently $(f_n g_n)_{n\in\mathbb{N}}$ converges to fg in measure. □

3. Let $\{\nu_n \colon n \in \mathbb{N}\}$ is a family of finite signed measures on $(X, \mathcal{M})$ satisfying

$$\sum_{n=1}^{\infty} |\nu_n|(X) < \infty.$$

Show that $\nu := \sum_{n=1}^{\infty} \nu_n$ is a finite signed measure on $(X, \mathcal{M})$.

Proof. First observe $\nu(\emptyset) = \sum_{n=1}^{\infty} \nu_n(\emptyset) = 0$. Next, suppose $E_k \in \mathcal{M}$, $k \in \mathbb{N}$, is a disjoint collection of subsets with union $E = \bigcup E_k$. Before showing ν is countable additive, we observe that Tonelli's theorem (applied to the counting measure) implies

$$\sum_{k=1}^{\infty} |\nu(E_k)| \leq \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\nu_n|(E_k) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\nu_n|(E_k) = \sum_{n=1}^{\infty} |\nu_n|(E) \leq \sum_{n=1}^{\infty} |\nu_n|(X) < \infty.$$

Consequently Fubini's theorem (applied to the counting measure) gives

$$\sum_{k=1}^{\infty} \nu(E_k) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \nu_n(E_k) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \nu_n(E_k) = \sum_{n=1}^{\infty} \nu_n(E) = \nu(E).$$

Additionally, the sum in the first expression is absolutely convergent by the above estimate. Hence ν is a signed measure. To see that ν is finite, we observe that

$$|\nu(E)| \leq \sum_{n=1}^{\infty} |\nu_n|(E) \leq \sum_{n=1}^{\infty} |\nu_n|(X) < \infty$$

for all $E \in \mathcal{M}$. □