Midterm 2

Name: ____

NetID: ____

Instructions:

1. You will have 50 minutes to complete the exam.
2. The exam is a total of 3 questions, their respective points values are listed below.
3. Unless stated otherwise, you must justify your answers with proofs.
4. You may cite any results from lecture or the homework.
5. No books, notes, calculators, or electronic devices are permitted.
6. If you require additional space, please use the reverse side of the pages.
7. The exam has a total of 4 pages, please verify that your copy has all 4 pages.

Question	Score	Points
1.		15
2.		20
3.		15
Total		50

1. Let μ be a **finitely additive** measure on a measurable space $(X, \mathcal{M})$. Show that the following statements are equivalent:

   (i) μ is a measure.

   (ii) For any sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$-measurable satisfying $0 \leq f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$, one has

   $$\int_X \sup_{n\in\mathbb{N}} f_n \, d\mu = \sup_{n\in\mathbb{N}} \int_X f_n \, d\mu.[1]$$

   (iii) For any sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$-measurable functions satisfying $f_n \geq 0$ for all $n \in \mathbb{N}$, one has

   $$\int_X \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

[1] For an $\mathcal{M}$-measurable function $f\colon X \to [0, +\infty]$, we define its integral with respect to μ in exactly the same way as with respect to a measure, and in particular you may assume integration is linear and monotone.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ are sequences of $\mathcal{M}$-measurable functions converging in measure to f and g, respectively. Suppose there exists $R > 0$ so that $|f_n(x)|, |g_n(x)| \leq R$ for μ-almost every $x \in X$. Show that $(f_n g_n)_{n\in\mathbb{N}}$ converges in measure to fg.

   [**Hint:** first show $|f(x)|, |g(x)| \leq R$ for μ-almost every $x \in X$.]

3. Let $\{\nu_n : n \in \mathbb{N}\}$ is a family of finite signed measures on $(X, \mathcal{M})$ satisfying

$$\sum_{n=1}^{\infty} |\nu_n|(X) < \infty.$$

Show that $\nu := \sum_{n=1}^{\infty} \nu_n$ is a finite signed measure on $(X, \mathcal{M})$.