Exercises: (Sections 2.3, 2.4)

1. Let $f \in L^1(\mathbb{R}, m)$. Show that $F\colon \mathbb{R} \to \mathbb{C}$ is continuous where $F(t) = \int_{(-\infty,t]} f\ dm$.
2. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and consider $h, H\colon [a,b] \to \mathbb{R}$ defined by

$$h(t) := \lim_{\delta\to 0} \inf_{|s-t|\leq\delta} f(s) \qquad H(t) := \lim_{\delta\to 0} \sup_{|s-t|\leq\delta} f(s).$$

   (a) Show that f is continuous at $t \in [a,b]$ if and only if $h(t) = H(t)$.
   (b) Show that $\int_{[a,b]} h\ dm$ and $\int_{[a,b]} H\ dm$ equal the lower and upper Darboux integrals of f, respectively.
   [**Hint:** show that $h = g$ and $H = G$ m-almost everywhere, where g and G are as in the proof of the Riemann–Lebesgue theorem.]
   (c) Deduce that f is Riemann integrable if and only if

$$m(\{t \in [a,b]\colon f \text{ is discontinuous at } t\}) = 0.$$

3. Let $\{q_n : n \in \mathbb{N}\} = \mathbb{Q}$ be an enumeration of the rationals, and for $x \in \mathbb{R}$ define

$$g(x) := \sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{x-q_n}} 1_{(q_n,q_n+1)}$$

   (a) Show that $g \in L^1(\mathbb{R}, m)$ and hence $g < \infty$ m-almost everywhere.
   (b) Show that g is discontinuous everywhere and unbounded on every open interval.
   (c) Show that the conclusions of (b) hold for any function equal to g m-almost everywhere.
   (d) Show that $g^2 < \infty$ m-almost everywhere, but g^2 is not integrable on any interval.
4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. For $f, g\colon X \to \mathbb{C}$ $\mathcal{M}$-measurable define

$$\rho(f,g) = \int_X \frac{|f-g|}{1+|f-g|}\ d\mu.$$

   (a) Show that ρ defines a metric on equivalence classes of $\mathbb{C}$-valued $\mathcal{M}$-measurable functions under the relation of μ-almost everywhere equality.
   (b) Show that $f_n \to f$ in measure if and only if $\rho(f_n, f) \to 0$.
5. (**Lusin's Theorem**) Let $f\colon [a,b] \to \mathbb{C}$ be Lebesgue measurable. Show that for all $\epsilon > 0$ there exists a compact set $K \subset [a,b]$ with $m(K) > (b-a) - \epsilon$ such that $f|_K$ is continuous.
   [**Hint:** use Egoroff's theorem and the L^1-density of continuous functions.]

Solutions:

1. It suffices to show F is separately left and right continuous. We will show left continuity, with the proof for right continuity being similar. Fix $t_0 \in \mathbb{R}$ and suppose $t_n \nearrow t_0$. Observe that

$$|F(t_0) - F(t_n)| = \left| \int_{(t_n,t_0]} f\ dm \right| \leq \int_{(t_n,t_0]} |f|\ dm = \int_{\mathbb{R}} 1_{(t_n,t_0]}|f|\ dm.$$

Now, the sequence $1_{(t_n,t_0]}|f|$ decreases to $1_{\{t_0\}}|f|$ and the first function has finite integral (since $f \in L^1(\mathbb{R}, m)$). Thus Exercise 2.(d) on Homework 5 implies

$$\lim_{n\to\infty} |F(t_0) - F(t_n)| = \lim_{n\to\infty} \int_{\mathbb{R}} 1_{(t_n,t_0]}|f|\ dm = \int_{\mathbb{R}} 1_{\{t_0\}}|f|\ dm = 0,$$

where in the last equality we have used that $1_{\{t_0\}}|f| = 0$ m-almost everywhere. □

2. (a) $(\Rightarrow)$: Given $\epsilon > 0$ let $\delta > 0$ be such that $|f(s) - f(t)| < \epsilon$ whenever $|s - t| < \delta$. In particular, for any $\delta' < \delta$ we have $f(t) - \epsilon < f(s) < f(t) + \epsilon$ when $|s - t| \le \delta'$. Thus

$$\inf_{|s-t|\le\delta'} f(s) \ge f(t) - \epsilon$$
$$\sup_{|s-t|\le\delta'} f(s) \le f(t) + \epsilon.$$

This implies $f(t) - \epsilon \le h(t) \le H(t) \le f(t) + \epsilon$ and so $|H(t) - h(t)| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we must have $h(t) = H(t)$.

$(\Leftarrow)$: Let $\epsilon > 0$. The equality $h(t) = H(t)$ implies there exists $\delta > 0$ so that

$$\sup_{|s-t|\le\delta} f(s) - \inf_{|s-t|\le\delta} f(s) < \epsilon.$$

The expression on the left dominates $|f(s) - f(s')|$ for any $s, s' \in [t - \delta, t + \delta]$. So, in particular, if $|s - t| < \delta$ then we have $|f(s) - f(t)| < \epsilon$. That is, f is continuous at t. □

(b) Let $(P_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[a, b]$ so that the lower and upper Darboux sums satisfied $L(f, P_n) \nearrow L(f)$ and $U(f, P_n) \searrow U(f)$. By taking the union of P_n with the "uniform partition"

$$\left\{a < a + \frac{(b-a)}{2^n} < a + \frac{2(b-a)}{2^n} < \cdots < b\right\}$$

we may assume the lengths of the subintervals determined by P_n tend to zero. For each $n \in \mathbb{N}$, if $P_n = \{a = t_0 < t_1 < \cdots < t_m = b\}$ define

$$g(P_n) = \sum_{j=1}^m 1_{(t_{j-1},t]} \inf_{t_{j-1}\le t\le t_j} f(t)$$
$$G(P_n) = \sum_{j=1}^m 1_{(t_{j-1},t]} \sup_{t_{j-1}\le t\le t_j} f(t)$$

so that $\int g(P_n)\ dm = L(f, P_n)$ and $\int G(P_n)\ dm = U(f, P_n)$. Note that $P_1 \subset P_2 \subset \cdots$ implies $g(P_n) \le g(P_{n+1})$ and $G(P_n) \ge G(P_{n+1})$ for each $n \in \mathbb{N}$, and so

$$g := \sup_{n\in\mathbb{N}} g(P_n) = \lim_{n\to\infty} g(P_n) \qquad \text{and} \qquad G := \inf_{n\in\mathbb{N}} G(P_n) = \lim_{n\to\infty} G(P_n).$$

The dominated convergence theorem (where our dominating function can always be taken to be $1_{[a,b]} \sup_t |f(t)|$) then implies

$$\int_{[a,b]} g\ dm = \lim_{n\to\infty} \int_{[a,b]} g(P_n)\ dm = \lim_{n\to\infty} L(f, P_n) = L(P),$$

and similarly $\int_{[a,b]} G\ dm = U(p)$. So it suffices to show $h = g$ and $H = G$ m-almost everywhere. We will show these functions agree outside of $P := \bigcup_n P_n$, which is countable and hence m-null. Fix $t \in [a, b] \setminus P$, let $\epsilon > 0$, and let $\delta > 0$ be such that

$$|h(t) - \inf_{|s-t|\le\delta} f(s)| < \epsilon.$$

Let $N \in \mathbb{N}$ be large enough so that P_n has subintervals of length at most $\frac{\delta}{2}$ for all $n \ge N$. Then for $n \ge N$, if t is in the subinterval (t_{j-1}, t_j) we have that $|s - t| \le \delta$ for all $s \in [t_{j-1}, t_j]$. Hence

$$g(P_n)(t) = \inf_{t_{j-1}\le t\le t_j} f(s) \ge \inf_{|s-t|\le\delta} f(s) > h(t) - \epsilon.$$

Since this holds for all $n \ge N$, we have $g(t) \ge h(t) - \epsilon$ and so $g(t) \ge h(t)$ since $\epsilon > 0$ was arbitrary. Conversely, given $\epsilon > 0$ let $n \in \mathbb{N}$ be such that $g(P_n)(t) \ge g(t) - \epsilon$. Since $t \notin P$, we can find $\delta > 0$ so that $[t - \delta, t + \delta]$ is entirely contained in some subinterval (t_{j-1}, t_j) of P_n. Then

$$\inf_{|s-t|\ge\delta} f(s) \ge \inf_{t_{j-1}\le t\le t_j} f(s) = g(P_n)(t) \ge g(t) - \epsilon.$$

Let $\delta \to 0$ and then $\epsilon \to 0$ yields $h(t) \geq g(t)$. Hence $h(t) = g(t)$ for all $t \in [a,b] \setminus P$. The proof for H and G is similar. □

(c) $(\Rightarrow)$: This implies the lower and upper Darboux integrals to agree, and hence $\int_{[a,b]} H - h\ dm = 0$ by part (b). Since $H - h \geq 0$, Proposition 2.16 implies $H - h = 0$ m-almost everywhere. Thus the set where H and h differ is m-null, but by part (a) this is precisely the set where f is discontinuous.

$(\Leftarrow)$: This implies $H = h$ m-almost everywhere. Hence their integrals agree, and which by part (b) means the upper and lower Darboux integrals agree and f is therefore Riemann integrable. □

3. (a) Since the terms in the series defining g are all positive Lebesgue measurable functions, Theorem 2.15 implies

$$\int_{\mathbb{R}} g\ dm = \sum_{n=1}^{\infty} \int_{(q_n, q_n+1)} \frac{1}{2^n \sqrt{x - q_n}} 1_{(q_n, q_n+1)}\ dm.$$

Now, the integrand in each term is Riemann integrable and so by the Riemann–Lebesgue theorem we have

$$\int_{(q_n, q_n+1)} \frac{1}{2^n \sqrt{x - q_n}} 1_{(q_n, q_n+1)}\ dm = \int_{q_n}^{q_n+1} \frac{1}{2^n \sqrt{x - q_n}}\ dx = \left[\frac{\sqrt{x - q_n}}{2^{n-1}} \right]_{q_n}^{q_n+1} = \frac{1}{2^{n-1}}.$$

Hence

$$\int_{\mathbb{R}} g\ dm = \sum_{n=1}^{\infty} 2^{-(n-1)} = 2 < \infty.$$

So $g \in L^1(\mathbb{R}, m)$, and therefore $g < \infty$ m-almost everywhere by Proposition 2.20 (or Exercise 2.(a) on Homework 5). □

(b) For any q_n, and $0 < \epsilon < 1$

$$g(q_n + \epsilon) \geq \frac{1}{2^n \sqrt{\epsilon}},$$

which can be made arbitrarily large. Since $\mathbb{Q} \cap (a,b) \neq \emptyset$ for all open intervals, we see that g is unbounded. Moreover, this shows that for any $t \in \mathbb{R}$ where $g(t) < \infty$, we can first find a sequence of rationals $r_n \searrow t$ and then find $0 < \epsilon_n < 1$ so that $g(r_n + \epsilon_n) \geq n$. Thus

$$\lim_{n \to \infty} g(r_n + \epsilon_n) = \infty \neq g(t),$$

and so g is discontinuous at t. If $g(t) = \infty$, then it can only be continuous if g identically infinite on an interval around t, but such an interval would have positive measure and contradict part (a). □

(c) Suppose $h = g$ except on a subset $E \subset \mathbb{R}$ with $m(E) = 0$. Given a rational q_n and $R > 0$, there exists an open interval $(q_n, q_n + \epsilon)$ so that $g \geq R$ on this interval. Since this interval has positive measure, $h \geq R$ at some points on this interval. Then proceeding as in the previous part, we can show h is unbounded on any interval and discontinuous everywhere. □

(d) Whenever $g(x) < \infty$, we have $g(x)^2 < \infty$. Thus $g^2 < \infty$ m-almost everywhere by part (a). To see that g^2 is not integrable, note that if $n \in \mathbb{N}$ is such that $q_n = 0$, then

$$g^2 \geq \frac{1}{2^{2n} x} 1_{(0,1)}.$$

So it suffices to show $f(x) := \frac{1}{x} 1_{(0,1)}$ is not integrable. Consider the sequence of functions $f_n(x) := \frac{1}{x} 1_{(1/n,1)}$, which increase to f. The monotone convergence theorem and Riemann–Lebesgue theorem imply

$$\int_{\mathbb{R}} f\ dm = \lim_{n \to \infty} \int_{\mathbb{R}} f_n\ dm = \lim_{n \to \infty} \int_{1/n}^{1} \frac{1}{x}\ dx = \lim_{n \to \infty} [\ln(1) - \ln(1/n)] = \lim_{n \to \infty} \ln(n) = \infty.$$

Thus f is not integrable. □

4. (a) Symmetry follows from $|f-g| = |g-f|$, and the triangle inequality follows from $|f-g| \leq |f-h| + |h-g|$ and the observation that for all $s, t \geq 0$

$$\frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t}.$$

Finally, $\rho(f,g) = 0$ if and only if $\frac{|f-g|}{1+|f-g|} = 0$ μ-almost everywhere by Proposition 2.16. Since the dominator is bounded below by 1, this fraction is zero if and only if $|f(x) - g(x)| = 0$. Thus $\rho(f,g) = 0$ if and only if $f = g$ μ-almost everywhere. So ρ is a metric on the space of these equivalence classes. □

(b) $(\Rightarrow)$: Suppose $f_n \to f$ in measure. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $E_n := \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$ satisfies $\mu(E_n) < \epsilon$ for all $n \geq N$. Observe that for $x \in E_n^c$ we have

$$\frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} < \frac{\epsilon}{1 + |f_n(x) - f(x)|} \leq \epsilon,$$

and for $x \in E_n$ we can bound the above by 1. Thus

$$\begin{aligned}\rho(f_n, f) &= \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu + \int_{E_n^c} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \\ &\leq \int_{E_n} 1\, d\mu + \int_{E_n^c} \epsilon\, d\mu = \mu(E_n) + \epsilon\mu(E_n^c) < \epsilon(1 + \mu(X)).\end{aligned}$$

Thus $\rho(f_n, f) \to 0$ since $\mu(X) < \infty$.

$(\Leftarrow)$: Suppose $\rho(f_n, f) \to 0$. Let $\epsilon > 0$ and consider $E_n := \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$. Then since $t \mapsto \frac{t}{1+t}$ is increasing, we have

$$\frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \geq \frac{\epsilon}{1 + \epsilon}$$

for all $x \in E_n$. Thus

$$\rho(f_n, f) \geq \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \geq \int_{E_n} \frac{\epsilon}{1+\epsilon}\, d\mu = \frac{\epsilon}{1+\epsilon}\mu(E_n),$$

which implies $\mu(E_n) \leq \frac{1+\epsilon}{\epsilon}\rho(f_n, f) \to 0$. Thus $f_n \to f$ in measure. □

5. For each $n \in \mathbb{N}$, let $B_n = \{t \in [a,b] : |f(t)| \leq n\}$. Then $f_n := 1_{B_n} f$ converges everywhere to f since $[a,b] = \bigcup B_n$. So by Egoroff's theorem we can find $E_0 \subset [a,b]$ with $\mu(E_0) < \frac{\epsilon}{2}$ and such that $f_n \to f$ uniformly on $[a,b] \setminus E_0$.

Now, each f_n is bounded and therefore integrable on $[a,b]$. So using Theorem 2.26 we can find a sequence of continuous functions $(g_k^{(n)})_{k \in \mathbb{N}}$ so that

$$\int_{[a,b]} |f_n - g_k^{(n)}|\, dm \to 0.$$

By Corollary 2.32, there is a subsequence $(g_{k_\ell}^{(n)})_{\ell \in \mathbb{N}}$ that converges to f_n m-almost everywhere. Using Egoroff's theorem again we can find $E_n \subset [a,b]$ such that $\mu(E_n) < 2^{-(n+1)}\epsilon$ and the subsequence $(g_{k_\ell}^{(n)})_{\ell \in \mathbb{N}}$ converges to f_n uniformly on $[a,b] \setminus E_n$. Then $f_n|_{[a,b] \setminus E_n}$ is continuous as the uniform limit of continuous functions. Now

$$E := \bigcup_{n=0}^{\infty} E_n$$

has $m(E) < \epsilon$ by countable subadditivity, and using the regularity of the Lebesgue measure (Theorem 1.18) we can find an open set $U \supset E$ with $m(U) < \epsilon$. Then $K := [a,b] \setminus U$ is closed and bounded, hence compact, with $m(K) > m([a,b]) - m(U) = (b-a) - \epsilon$. Furthermore, for each $n \in \mathbb{N}$ we have $K = [a,b] \setminus U \subset [a,b] \setminus E_n$, and so $f_n|_K$ is continuous. Also $K \subset [a,b] \setminus E_0$, which means $f_n \to f$ uniformly on K, and therefore $f|_K$ is continuous as the uniform limit of continuous functions. □