Exercises: (Sections 2.2, 2.3)

1. Let $f\colon [0,1] \to [0,1]$ be the Cantor function, and define $g(x) := f(x) + x$.
   (a) Show that $g\colon [0,1] \to [0,2]$ is a bijection with continuous inverse.
   (b) If $C \subset [0,1]$ is the Cantor set, show that $m(g(C)) = 1$. [**Hint:** compute $m(g(C)^c)$.]
   (c) Show that there exists $A \subset g(C)$ such that $A \notin \mathcal{L}$ and $g^{-1}(A) \in \mathcal{L} \setminus \mathcal{B}_{\mathbb{R}}$.
   (d) Deduce that there exists a Lebesgue measurable function F and a continuous function G such that $F \circ G$ is not Lebesgue measurable.
2. Let $f \in L^+(X, \mathcal{M}, \mu)$ with $\int_X f\ d\mu < \infty$.
   (a) Show that $\{x \in X \colon f(x) = \infty\}$ is a μ-null set.
   (b) Show that $\{x \in X \colon f(x) > 0\}$ is σ-finite.
   (c) Show that for all $\epsilon > 0$, there exists $E \in \mathcal{M}$ with $\mu(E) < \infty$ and such that $\int_X f\ d\mu < \int_E f\ d\mu + \epsilon$.
   (d) Suppose $(f_n)_{n\in\mathbb{N}} \subset L^+(X, \mu)$ decreases to f and $\int_X f_1\ d\mu < \infty$. Show that
$$\lim_{n\to\infty} \int_X f_n\ d\mu = \int_X f\ d\mu.$$
3. For $f \in L^+(X, \mathcal{M}, \mu)$, define $\nu\colon \mathcal{M} \to [0,\infty]$ by $\nu(E) := \int_E f\ d\mu$. Show that ν is a measure satisfying
$$\int_X g\ d\nu = \int_X gf\ d\mu$$
for all $g \in L^+(X, \mathcal{M}, \mu)$.
4. Let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \in L^1(X,\mu)$ be sequences converging μ-almost everywhere to $f, g \in L^1(X,\mu)$, respectively. Suppose $|f_n| \leq g_n$ for each $n \in \mathbb{N}$ and $\int g_n\ d\mu \to \int g\ d\mu$. Show that
$$\lim_{n\to\infty} \int_X f_n\ d\mu = \int_X f\ d\mu.$$
5. Suppose $(f_n)_{n\in\mathbb{N}} \subset L^1(X,\mu)$ converges μ-almost everywhere to $f \in L^1(X,\mu)$. Show that
$$\lim_{n\to\infty} \int_X |f_n - f|\ d\mu = 0 \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \int_X |f_n|\ d\mu = \int_X |f|\ d\mu.$$

Solutions:

1. (a) If $x, y \in [0,1]$ satisfy $x < y$, then $f(x) \leq f(y)$ and hence
$$g(x) = f(x) + x \leq f(y) + x < f(y) + y = g(y).$$
Thus g is injective. Recall that f is continuous (since it is increasing and onto $[0,1]$), hence g is continuous as the sum of continuous functions. Since $g(0) = f(0) + 0 = 0$ and $g(1) = f(1) + 1 = 2$, the intermediate value theorem implies g is onto $[0,2]$. Thus g is a a bijection. Its inverse $g^{-1}\colon [0,2] \to [0,1]$ is increasing since g is increasing, which we showed above. Hence it is continuous since it is also onto $[0,1]$: any discontinuity would necessarily be a jump discontinuity, and hence contradict the surjectivity of g^{-1}. □

(b) Recall that $[0,1]\setminus C$ is a countable union of open intervals (namely $(\frac{1}{3},\frac{2}{3}),(\frac{1}{9},\frac{2}{9}),(\frac{7}{9},\frac{8}{9})$, etc.), and f is constant on each of these intervals. Denote the intervals by I_n, $n\in\mathbb{N}$, and let c_n be such that $f(x)=c_n$ for all $x\in I_n$. Then for $x\in I_n$ we have $g(x)=c_n+x$, and hence $g(I_n)=I_n+c_n$. Moreover, $\{I_n+c_n\colon n\in\mathbb{N}\}$ is a disjoint collection since g is a bijection. So using the translation invariance of the Lebesgue measure, we have

$$m(g([0,1]\setminus C))=\sum_{n=1}^\infty m(I_n+c_n)=\sum_{n=1}^\infty m(I_n)=m([0,1]\setminus C)=1,$$

where we have used $m(C)=0$. Since g is onto $[0,2]$, we also have

$$m(g([0,1]\setminus C))=m(g([0,1])\setminus g(C))=m([0,2]\setminus g(C))=2-m(g(C)).$$

Hence $m(g(C))=1$. ☐

(c) By Exercise 2.(b) on Homework 4, $m(g(C))>0$ implies there exists $A\subset g(C)$ which is not Lebesgue measurable. Then $B:=g^{-1}(A)\subset C$ is a subset of a null set and hence is Lebesgue measurable. However, if we had $B\in\mathcal{B}_{\mathbb{R}}$, then the continuity of g^{-1} would imply $A=g(B)=(g^{-1})^{-1}(B)$ is Borel measurable, a contradiction. ☐

(d) Let $F:=1_{g^{-1}(A)}$, which is Lebesgue measurable since $g^{-1}(A)\in\mathcal{L}$. Let $G:=g^{-1}$, which is continuous by part (a). Then $F\circ G$ is **not** Lebesgue measurable because

$$(F\circ G)^{-1}(\{1\})=G^{-1}(F^{-1}(\{1\}))=G^{-1}(g^{-1}(A))=g(g^{-1}(A))=A$$

is not Lebesgue measurable. ☐

2. (a) Denote $E=\{x\in X\colon f(x)=\infty\}$. Suppose, towards a contradiction, that $\mu(E)>0$ and denote $R:=\frac{1}{\mu(E)}\left(\int_X f\,d\mu+1\right)$. Then $\phi:=R1_E$ is a simple function satisfying $0\leq\phi\leq f$. Thus

$$\int_X f\,d\mu\geq\int_X\phi\,d\mu=R\mu(E)=\int_X f\,d\mu+1,$$

a contradiction. ☐

(b) Let $E_n=\{x\in X\colon f(x)\geq\frac{1}{n}\}$ so that

$$\bigcup_{n=1}^\infty E_n=\{x\in X\colon f(x)>0\}.$$

Then $\int_X f\,d\mu\geq\int_{E_n}f\,d\mu\geq\frac{1}{n}\mu(E_n)$ implies $\mu(E_n)<\infty$ for all $n\in\mathbb{N}$. ☐

(c) Let $\epsilon>0$ and let E_n be as in the previous part, and let $F=\{x\in X\colon f(x)>0\}$. Then $E_n\subset E_{n+1}$ implies the sequence $f_n:=1_{E_n}f$ increases to $1_Ff=f$ pointwise. So the monotone convergence theorem implies

$$\int_{E_n}f\,d\mu=\int_X 1_{E_n}f\,d\mu\to\int_X f\,d\mu.$$

Consequently there exists sufficiently large $n\in\mathbb{N}$ so that $\int_{E_n}f\,d\mu\geq\int_X f\,d\mu-\epsilon$. Then E_n is the desired set. ☐

(d) Since the sequence is decreasing, f_1 having finite integral implies each f_n has finite integral. Thus $E_n:=\{x\in X\colon f_n(x)=\infty\}$ is a μ-set for each $n\in\mathbb{N}$ by part (a). Now, define

$$g_n(x):=\begin{cases}f_1(x)-f_n(x) & \text{if } x\in E_n^c\\ 0 & \text{otherwise}\end{cases}.$$

Then $g_n\in L^+(X,\mu)$ by measurable by Exercise 4 on Homework 4 (note that f_1 is infinite whenever f_n is and neither ever equals $-\infty$). Now, $E:=\bigcup_n E_n$ is a μ-null set (it actually equals E_1) and for any $x\in E^c$ we have

$$g_n(x)=f_1(x)-f_n(x)\nearrow f_1(x)-f(x).$$

Thus the monotone convergence theorem implies

$$\lim_{n\to\infty} \int_{E^c} f_1 - f_n \; d\mu = \int_{E^c} f_1 - f(x) \; d\mu.$$

Thus

$$\lim_{n\to\infty} \int_{E^c} f_n \; d\mu = \lim_{n\to\infty} \int_{E^c} f_1 \; d\mu - \int_{E^c} f_1 - f_n \; d\mu = \int_{E^c} f_1 \; d\mu - \int_{E^c} f_1 - f \; d\mu = \int_{E^c} f \; d\mu,$$

where we have used that $f_1 = (f_1 - f_n) + f_n$ and $f_1 = (f_1 - f) + f$ are sums are positive functions on E^c. Finally, since $\mu(E) = 0$, the above integrals of f_n and f over E^c equal the integrals over all of X. □

3. Since $1_\emptyset f = 0$, we have $\nu(\emptyset) = \int_\emptyset f \; d\mu = \int_X 1_\emptyset f \; d\mu = 0$. Now suppose $\{E_n : n \in \mathbb{N}\} \subset \mathcal{M}$ is a disjoint collection. Observe that $\sum 1_{E_n} = 1_{\bigcup E_n}$. Then by Theorem 2.15 from lecture we have

$$\sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} \int_X 1_{E_n} f \; d\mu = \int_X \sum_{n=1}^{\infty} 1_{E_n} f \; d\mu = \int 1_{\bigcup E_n} f \; d\mu = \nu\left(\bigcup_{n=1}^{\infty} E_n \right).$$

Hence ν is a measure.

Now, first suppose $g \in L^+(X, \mu)$ is simple with standard representation $g = \sum \alpha_j 1_{E_j}$. Then

$$\int_X g \; d\nu = \sum_{j=1}^{n} \alpha_j \nu(E_j) = \sum_{j=1}^{n} \alpha_j \int_X 1_{E_j} f \; d\mu = \int_X gf \; d\mu.$$

For general $g \in L^+(X, \mu)$, we use Theorem 2.10 to find a sequence of simple functions $(\phi_n)_{n\in\mathbb{N}} \subset L^+(X, \mu)$ which increase pointwise to g. Then above computation and the monotone convergence theorem imply

$$\int_X g \; d\nu = \lim_{n\to\infty} \int_X \phi_n \; d\nu = \lim_{n\to\infty} \int_X \phi_n f \; d\mu = \int_X gf \; d\mu,$$

where in the last equality we have used that $\phi_n f$ increases to gf. □

4. As in the proof of the dominated convergence theorem, by considering real and imaginary parts of these integrals it suffices to assume the f_n and f are real-valued. In this case, $|f_n| \le g_n$ implies $g_n \pm f_n \ge 0$, and similarly $g \pm f \ge 0$. Applying Fatou's Lemma, we have

$$\int_X g \; d\mu \pm \int_X f \; d\mu = \int_X g \pm f \; d\mu \le \liminf_{n\to\infty} \int_X g_n \pm f_n \; d\mu = \liminf_{n\to\infty} \int_X g_n \; d\mu \pm \int_X f_n \; d\mu.$$

Now, the convergence $\int g_n \; d\mu \to \int g \; d\mu$ implies the above is either $\int g \; d\mu + \liminf_n \int f_n \; d\mu$ or $\int g \; d\mu - \limsup_n \int_X f_n \; d\mu$. It follows that

$$\limsup_{n\to\infty} \int_X f_n \; d\mu \le \int_X f \; d\mu \le \liminf_{n\to\infty} \int_X f_n \; d\mu,$$

which implies the claimed convergence. □

5. $(\Longrightarrow)$: Note that $|f_n(x)| \le |f(x)| + |f_n(x) - f(x)|$ for all $x \in X$ and all $n \in \mathbb{N}$. Thus

$$\limsup_{n\to\infty} \int_X |f_n| \; d\mu \le \int_X |f| \; d\mu + \limsup_{n\to\infty} \int_X |f_n - f| \; d\mu = \int_X |f| \; d\mu.$$

Combining this with the inequality from Fatou's Lemma yields the desired convergence.

$(\Longleftarrow)$: Define $g_n := |f_n| + |f|$ and $g := 2|f|$. Then by assumption $\int g_n \to \int g$. Since $|f_n - f|$ converges to zero μ-almost everywhere and is dominated by g_n, Exercise 4 gives

$$\lim_{n\to\infty} \int_X |f_n - f| \; d\mu = \int_X 0 \; d\mu = 0.$$

□