Exercises: (Sections 1.4-5)

1. Let $\mathcal{A}$ be an algebra on X, let μ_0 be a premeasure on $(X, \mathcal{A})$, and let μ^* be the outer measure defined by
$$\mu^*(E) := \inf\left\{ \sum_{n=1}^\infty \mu_0(A_n) \colon A_n \in \mathcal{A} \text{ for each } n \in \mathbb{N} \text{ and } E \subset \bigcup_{n=1}^\infty A_n \right\}.$$
We will call μ^* the outer measure **induced** by μ_0.
   (a) Let A_σ denote the collection of all countable unions of sets in $\mathcal{A}$. For $E \subset X$ and $\epsilon > 0$ show that there exists $A \in \mathcal{A}_\sigma$ satisfying $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.
   (b) Let $A_{\sigma\delta}$ denote the collection of all countable intersections of sets in $\mathcal{A}_\sigma$. For $E \subset X$ with $\mu^*(E) < \infty$, show that E is μ^*-measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ satisfying $E \subset B$ and $\mu^*(B \setminus E) = 0$.
   (c) Suppose μ_0 is σ-finite. Show that E is μ^*-measurable if and only if there exists $B \in A_{\sigma\delta}$ satisfying $E \subset B$ and $\mu^*(B \setminus E) = 0$.
2. Let $\mathcal{A}$ be an algebra on X, let μ_0 be a finite premeasure on $(X, \mathcal{A})$, and let μ^* be the outer measure induced by μ_0. The **inner measure** induced by μ_0 is the map defined by $\mu_*(E) := \mu_0(X) - \mu^*(E^c)$ for $E \subset X$. Show that $A \subset X$ is μ^*-measurable if and only if $\mu^*(A) = \mu_*(A)$.
[**Hint:** use Exercise 1.(b).]
3. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let μ^* be the outer measure induced by μ.
   (a) Show that the σ-algebra $\mathcal{M}^*$ of μ^*-measurable sets equals $\overline{\mathcal{M}}$, the completion of $\mathcal{M}$.
   [**Hint:** use Exercise 1.(c).]
   (b) Show that $\mu^*|_{\mathcal{M}^*} = \overline{\mu}$, the completion of μ.
4. Let $\mathcal{A}$ be the collection of finite disjoint unions of sets of the form $(a, b] \cap \mathbb{Q}$ with $-\infty \leq a < b \leq \infty$.
   (a) Show $\mathcal{A}$ is an algebra on $\mathbb{Q}$ by showing $\mathcal{E} := \{\emptyset\} \cup \{(a, b] \cap \mathbb{Q} \colon -\infty \leq a < b \leq b\}$ is an elementary family.
   (b) Show that $\mathcal{M}(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$.
   (c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for all nonempty $A \in \mathcal{A}$. Show that μ_0 is a premeasure.
   (d) Show that there exists more than one measure on $(\mathbb{Q}, \mathcal{P}(\mathbb{Q}))$ extending μ_0.
5. Let $F \colon \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous, and let μ_F be the associated Borel measure on $\mathbb{R}$. For $-\infty < a < b < \infty$, prove the following equalities:
$$\begin{aligned} \mu_F(\{a\}) &= F(a) - \lim_{x \nearrow a} F(x) \\ \mu_F([a, b]) &= F(b) - \lim_{x \nearrow a} F(x) \\ \mu_F((a, b)) &= \lim_{x \nearrow b} F(x) - F(a) \\ \mu_F([a, b)) &= \lim_{x \nearrow b} F(x) - \lim_{y \nearrow a} F(y). \end{aligned}$$

Solutions:

1. (a) By definition of μ^* there exists $\{A_n : n \in \mathbb{N}\} \subset \mathcal{A}$ which covers E and satisfies

$$\sum_{n=1}^{\infty} \mu_0(A_n) \leq \mu^*(E) + \epsilon.$$

Recall from Proposition 1.13 in lecture that $\mu^*|_{\mathcal{A}} = \mu_0$ and so $\mu_0(A_n) = \mu^*(A_n)$ for each $n \in \mathbb{N}$. Define $A := \bigcup A_n$, which contains E and lies in $\mathcal{A}_\sigma$. By countable subadditivity we have

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n) = \sum_{n=1}^{\infty} \mu_0(A_n) \leq \mu^*(E) + \epsilon.$$

$\square$

(b) $(\Rightarrow)$: Suppose E is μ^*-measurable. Then for any superset $F \supset E$ we have

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c) = \mu^*(E) + \mu^*(F \setminus E),$$

and so $\mu^*(F \setminus E) = \mu^*(F) - \mu^*(E)$. (Note that we have used $\mu^*(E) < \infty$ to make sense of the preceding expression.) Now, for each $n \in \mathbb{N}$ let $A_n \in \mathcal{A}_\sigma$ be such that $E \subset A_n$ and $\mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$, which exists by part (a). Then $B := \bigcap A_n$ contains in E and lies in $\mathcal{A}_{\sigma\delta}$. Moreover, monotonicity implies

$$\mu^*(E) \leq \mu^*(B) \leq \mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$$

for all $n \in \mathbb{N}$, and hence $\mu^*(B) = \mu^*(E)$. Therefore $\mu^*(B \setminus E) = \mu^*(B) - \mu^*(E) = 0$ by the above discussion.

$(\Leftarrow)$: Let $F \subset X$. We must show $\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$. Let $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$. Note that $\mathcal{A}_{\sigma\delta} \subset \mathcal{M}(\mathcal{A})$ and this latter set is contained in the collection of all μ^*-measurable sets. Thus B is μ^*-measurable and so

$$\mu^*(F) = \mu^*(F \cap B) + \mu^*(F \cap B^c).$$

Now $\mu^*(F \cap B) \geq \mu^*(F \cap E)$ by monotonicity. We also claim $\mu^*(F \cap B^c) = \mu^*(F \cap E^c)$, which will complete the proof. Indeed, observe that $E^c = B^c \cup (B \setminus E)$ and so by subadditivity and monotonicity we have

$$\mu^*(F \cap E^c) \leq \mu^*(F \cap B^c) + \mu^*(F \cap (B \setminus E)) \leq \mu^*(F \cap B^c) + 0 \leq \mu^*(F \cap E^c).$$

Thus $\mu^*(F \cap B^c) = \mu^*(F \cap E^c)$. $\square$

(c) Assume μ_0 is σ-finite: $X = \bigcup_{n=1}^{\infty} A_n$ with $A_n \in \mathcal{A}$ and $\mu_0(A_n) < \infty$. By replacing A_n with $A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$ we may assume the A_n's are disjoint.

Suppose E is μ^*-measurable. Since $\mathcal{A}$ is contained in the μ^*-measurable sets—a σ-algebra—it follows that $E \cap A_n$ is μ^*-measurable for each $n \in \mathbb{N}$. Additionally, $\mu^*(E \cap A_n) \leq \mu^*(A_n) = \mu_0(A_n) < \infty$ so by part (b) we can find $B_n \in \mathcal{A}_{\sigma\delta}$ containing $E \cap A_n$ with $\mu^*(B_n \setminus (E \cap A_n)) = 0$. Note that $B_n \cap A_n \in \mathcal{A}_{\sigma\delta}$ since this collection is closed under (countable) intersections and contains $\mathcal{A}$. Since this intersection also contains $E \cap A_n$ and satisfies $\mu^*((B_n \cap A_n) \setminus (E \cap A_n)) = 0$ by monotonicity, we may assume $B_n \subset A_n$ by replacing it with $B_n \cap A_n$ if necessary. Now, consider $B := \bigcup_n B_n$. This contains E and by countable subadditivity we have

$$\mu^*(B \setminus E) \leq \sum_{n=1}^{\infty} \mu^*(B_n \setminus E) = \sum_{n=1}^{\infty} \mu^*(B_n \setminus (E \cap A_n)) = 0.$$

Thus if $B \in \mathcal{A}_{\sigma\delta}$, then it is our desired set. Each $B_n = \bigcap_k C_k^{(n)}$ for $C_k^{(n)} \in \mathcal{A}_\sigma$. We claim

$$B = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} C_k^{(n)},$$

Indeed, a point x in the set in the right belongs to A_n for exactly one $n \in \mathbb{N}$, say n_0. So $x \in \bigcup_n C_k^{(n)}$ necessarily implies $x \in C_k^{(n_0)}$ for all $k \in \mathbb{N}$, hence $x \in B_k \subset B$. The reverse inclusion is immediate. Thus $B \in \mathcal{A}_{\sigma\delta}$ since $\bigcup_n C_k^{(n)} \in \mathcal{A}_\sigma$ for each $k \in \mathbb{N}$.
The other implication follows by the same proof as in part (b), since this never used the hypothesis $\mu^*(E) < \infty$. □

2. Recall from Proposition 1.13 in lecture that $\mu^*|_\mathcal{A} = \mu_0$, and hence $\mu_0(X) = \mu^*(X)$. Now, suppose A is μ^*-measurable. Then

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \cap A^c) = \mu^*(A) + \mu^*(A^c),$$

and hence $\mu_*(A) = \mu_0(X) - \mu^*(A^c) = \mu^*(X) - \mu^*(A^c) = \mu^*(A)$.
Conversely, suppose $\mu^*(A) = \mu_*(A)$. Let $B = \bigcap A_n \in \mathcal{A}_{\sigma\delta}$ where $A_n \in \mathcal{A}_\sigma$ contains A and satisfies $\mu^*(A_n) \leq \mu^*(A) + \frac{1}{k}$. It follows that $A \subset B$ and $\mu^*(B) = \mu^*(A)$. Now, since $\mathcal{A}_{\sigma\delta} \subset \mathcal{M}(\mathcal{A})$ we know B is μ^*-measurable by Proposition 1.13 and hence $\mu^*(B) + \mu^*(B^c) = \mu^*(X)$. On the other hand, $\mu^*(A) = \mu_*(A)$ implies

$$\mu^*(X) = \mu^*(A) + \mu^*(A^c) = \mu^*(B) + \mu^*(A^c).$$

Thus $\mu^*(B^c) = \mu^*(A^c)$ and using the μ^*-measurability of B yields

$$\mu^*(B^c) = \mu^*(A^c \cap B) + \mu^*(A^c \cap B^c) = \mu^*(B \setminus A) + \mu^*(B^c).$$

Consequently $\mu^*(B \setminus A) = 0$, and therefore A is μ^*-measurable by Exercise 1.(b). □

3. (a) Let $G \in \mathcal{M}$ with $\mu(G) = 0$. Then by Proposition 1.13, $G \in \mathcal{M}^*$ and $\mu^*(G) = 0$. Since $\mu^*|_{\mathcal{M}^*}$ is a complete measure by Carathéodory's Theorem, it follows that $\mathcal{P}(G) \subset \mathcal{M}^*$. Thus

$$\overline{\mathcal{M}} = \{E \cup F \colon E \in \mathcal{M},\ F \subset G \text{ where } \mu(G) = 0\} \subset \mathcal{M}^*.$$

Conversely, let $E \in \mathcal{M}^*$. By Exercise 1.(c) there exists $B \in \mathcal{M}_{\sigma\delta}$ containing E with $\mu^*(B \setminus E) = 0$. Denote $F := B \setminus E$ so that $E = B \setminus F$. To see that $E \in \overline{\mathcal{M}}$ it suffices to show $B, F \subset \overline{\mathcal{M}}$ since it is a σ-algebra. Since $\mathcal{M}$ is a σ-algebra, we have $B \subset \mathcal{M}_{\sigma\delta} \subset \mathcal{M} \subset \overline{\mathcal{M}}$. Next, applying Exercise 1.(c) to F we obtain a $B_0 \in \mathcal{M}_{\sigma\delta} \subset \mathcal{M}$ containing F so that $\mu^*(B_0 \setminus F) = 0$. But then

$$\mu(B_0) = \mu^*(B_0) \leq \mu^*(F) + \mu^*(B \setminus F) = 0.$$

That is, B_0 is μ-null set. Since $\overline{\mathcal{M}}$ contains all subsets of μ-null sets, it contains $F \subset B_0$. □

(b) By part (a), a set in $\mathcal{M}^*$ can be written as $E \cup F$ where $E \in \mathcal{M}$ and F is a subset of μ-null set G. Since $\mu^* |_\mathcal{M} = \mu$ by Proposition 1.13, we know $\mu^*(G) = 0$ and so $\mu^*(F) = 0$ by monotonicity. Using this we see that

$$\mu^*(E \cup F) \leq \mu^*(E) + \mu^*(F) = \mu(E) + 0 \leq \mu^*(E \cup F).$$

Thus $\mu^*(E \cup F) = \mu(E) = \overline{\mu}(E \cup F)$. □

4. (a) We have $\emptyset \in \mathcal{E}$ by assumption. From

$$(a, b] \cap (c, d] = \begin{cases} (\max\{a, c\}, d] & \text{if } a < d \leq b \\ (c, \min\{b, d\}] & \text{if } a \leq c < b \\ \emptyset & \text{otherwise} \end{cases},$$

it follows that $\mathcal{E}$ is closed under (finite) intersections. From

$$(a, b]^c = \begin{cases} (-\infty, a] \cup (b, \infty) & \text{if } -\infty < a, b < \infty \\ (b, \infty) & \text{if } a = -\infty, b < \infty \\ (-\infty, a] & \text{if } -\infty < a, b = \infty \\ \emptyset & \text{otherwise} \end{cases}$$

it follows that the complement of a set in $\mathcal{E}$ is a finite union of sets in $\mathcal{E}$. Hence $\mathcal{E}$ is an elementary family and Proposition 1.12 from lecture implies $\mathcal{A}$ is an algebra on $\mathbb{Q}$. □

(b) For each $x \in \mathbb{Q}$,

$$\{x\} = \bigcap_{n=1}^{\infty} (x - \frac{1}{n}, x] \cap \mathbb{Q} \in \mathcal{M}(\mathcal{A}).$$

Thus $\mathcal{M}(\mathcal{A})$ contains all singleton subsets of $\mathbb{Q}$. This implies $\mathcal{M}(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$, since every subset of $\mathbb{Q}$ is the countable union of the singleton sets of its elements. □

(c) Let $E_1, E_2, \ldots \in \mathcal{A}$ be disjoint sets with $\bigcup_n E_n \in \mathcal{A}$. If $E_n = \emptyset$ for all $n \in \mathbb{N}$, we have $\mu_0(\bigcup_n E_n) = 0 = \sum_n \mu_0(E_n)$. Otherwise at least one E_n is nonempty, which in turn makes the union nonempty. Consequently $\mu_0(\bigcup E_n) = \infty = \sum_n \mu_0(E_n)$. Thus μ_0 is countably additive and therefore a premeasure. □

(d) One such measure is the induced outer measure $\mu^*|_{\mathcal{M}(\mathcal{A})} = \mu^*$. Observe that $\mu^*(E) = \infty$ unless $E = \emptyset$. Indeed, if E is non-empty, then any countable cover by elements from $\mathcal{A}$ will include a nonempty set, in which case the sum of the measures of the cover will be infinite. In particular, μ^* does not equal the counting measure on $\mathbb{Q}$, which we claim also extends μ_0. Indeed, the density of the rationals implies $(a, b] \cap \mathbb{Q}$ will always be infinite and so the counting measure gives infinite mass to any nonempty set in $\mathcal{A}$, which is precisely how μ_0 was defined. □

5. Note that

$$\{a\} = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a],$$

and so by continuity from above we have

$$\mu_F(\{a\}) = \lim_{n\to\infty} \mu_F((a - \frac{1}{n}, a]) = \lim_{n\to\infty} [F(a) - F(a - \frac{1}{n})] = F(a) - \lim_{x \nearrow a} F(x).$$

Using this we next have

$$\mu_F([a, b]) = \mu_F(\{a\}) + \mu_F((a, b]) = \left(F(a) - \lim_{x\nearrow a} F(x)\right) + F(b) - F(a) = F(b) - \lim_{x\nearrow a} F(x),$$

and

$$\mu_F((a, b)) = \mu_F((a, b]) - \mu_F(\{b\}) = F(b) - F(a) - \left(F(b) - \lim_{x\nearrow b} F(x)\right) = \lim_{x\nearrow b} F(x) - F(a).$$

Finally, combining the above equalities we have

$$\mu_F([a, b)) = \mu_F(\{a\}) + \mu_F((a, b)) = F(a) - \lim_{y\nearrow a} F(y) + \lim_{x\nearrow b} F(x) - F(a) = \lim_{x\nearrow b} F(x) - \lim_{y\nearrow a} F(y).$$

□