2.1 Measurable functions

Def Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable (or just measurable) if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$. □

Proposition 2.1 Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$, and $(Z, \mathcal{O})$ be measurable spaces and $f: X \to Y$, $g: Y \to Z$ functions.

1. If f is $(\mathcal{M}, \mathcal{N})$-measurable and g is $(\mathcal{N}, \mathcal{O})$-measurable, then $g \circ f: X \to Z$ is $(\mathcal{M}, \mathcal{O})$-measurable.
2. If $\mathcal{N}$ is generated by $\mathcal{E} \subset \mathcal{P}(Y)$, then $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$ measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Proof 1: This follows from $(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$ for all $E \subset Z$.

2 ($\Longrightarrow$): This is immediate from $\mathcal{E} \subset \mathcal{N}$.

($\Longleftarrow$): Since f^{-1} commutes with complements and unions, we see that $\{E \subset \mathcal{P}(Y) : f^{-1}(E) \in \mathcal{M}\}$ is a σ-algebra containing $\mathcal{E}$. Hence it contains $\mathcal{N}$ by Lemma 1.1. □

Corollary 2.2 If X and Y are topological spaces (e.g. metric spaces), then every continuous function $f: X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Proof For all U open, $f^{-1}(U)$ is open hence in $\mathcal{B}_X$. □

Of particular interest are functions valued in $\mathbb{C}$ or $\mathbb{R}$:

Def Let $(X, \mathcal{M})$ be a measurable space and let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. We say $f: X \to \mathbb{F}$ is $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{F}})$-measurable. In particular, we say $f: \mathbb{R} \to \mathbb{F}$ is Lebesgue measurable if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{F}})$-measurable, and say it is Borel measurable if it is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{F}})$-measurable. □

That is, we tacitly assume a target space of $\mathbb{C}$ or $\mathbb{R}$ has the Borel σ-algebra. However, we allow a domain of $\mathbb{R}$ to have either the Lebesgue or Borel σ-algebra.

⚠ For $f, g: \mathbb{R} \to \mathbb{R}$ Borel measurable, it follows from Proposition 2.1 that $g \circ f$ is Borel measurable. Also, if f is Lebesgue measurable and g is Borel measurable, then $g \circ f$ is Lebesgue measurable. However, if f, g are both Lebesgue measurable, then $g \circ f$ need not be Lebesgue measurable: there may exist $E \subset \mathbb{R}$ such that $g^{-1}(E) \in \mathcal{L} \setminus \mathcal{B}_{\mathbb{R}}$ and hence $f^{-1}(g^{-1}(E)) \notin \mathcal{L}$. This can happen even if f is continuous.

Proposition 2.3 For a measurable space $(X, \mathcal{M})$ and $f: X \to \mathbb{R}$, the following are equivalent:

1. f is $\mathcal{M}$-measurable
2. $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$
3. $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$
4. $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$
5. $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$

Proof Each type of ray generates $\mathcal{B}_{\mathbb{R}}$ by Proposition 1.2, so Proposition 2.1 finishes the proof. □

Let X be a set and let $(Y_i, \mathcal{N}_i)$, $i \in I$, be measurable spaces. Given functions $f_i : X \to Y_i$ for each $i \in I$, there is a unique smallest σ-algebra $\mathcal{M}$ on X such that each f_i is $(\mathcal{M}, \mathcal{N}_i)$-measurable:

$$\mathcal{M} := \mathcal{M}(\{f_i^{-1}(E) : i \in I, E \in \mathcal{N}_i\})$$

Def The σ-algebra defined above is called the σ-algebra generated by $\{f_i : i \in I\}$ □

Ex For $X = \prod_{i \in I} Y_i$ and $f_i = \pi_i$ the coordinate map, the σ-algebra generated by $\{\pi_i : i \in I\}$ is the product σ-algebra $\bigotimes_{i \in I} \mathcal{N}_i$. □

Proposition 2.4 Let $(X, \mathcal{M})$ and $(Y_i, \mathcal{N}_i)$, $i \in I$, be measurable spaces. Set $Y := \prod_{i \in I} Y_i$, $\mathcal{N} := \bigotimes_{i \in I} \mathcal{N}_i$, and $\pi_i : Y \to Y_i$ the coordinate maps. Then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $\pi_i \circ f$ is $(\mathcal{M}, \mathcal{N}_i)$-measurable for all $i \in I$.

Proof $(\Rightarrow)$: This follows from Proposition 2.1 since each π_i is $(\mathcal{N}, \mathcal{N}_i)$-measurable.
$(\Leftarrow)$: For $E \in \mathcal{N}_i$, $f^{-1}(\pi_i^{-1}(E)) = (\pi_i \circ f)^{-1}(E) \in \mathcal{M}$. Since such sets generate $\mathcal{N}$, the second part of Proposition 2.1 implies f is $(\mathcal{M}, \mathcal{N})$-measurable. □

Corollary 2.5 A function $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are $\mathcal{M}$-measurable.

Proof Since $\mathbb{C} \cong \mathbb{R}^2$ as a metric space, $\mathcal{B}_{\mathbb{C}} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ by Proposition 1.5. Under the identification $\mathbb{C} \cong \mathbb{R}^2$, $\mathrm{Re}(f) = \pi_1 \circ f$ and $\mathrm{Im}(f) = \pi_2 \circ f$. Hence this follows from the above proposition. □

Proposition 2.6 If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable, then so are $f+g$, $f \cdot g$.

Proof Define $F : X \to \mathbb{C} \times \mathbb{C}$, $\phi, \psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$

$$F(x) = (f(x), g(x)), \qquad \phi(z,w) = z+w, \qquad \psi(z,w) = zw.$$

Since $\mathcal{B}_{\mathbb{C} \times \mathbb{C}} = \mathcal{B}_{\mathbb{C}} \otimes \mathcal{B}_{\mathbb{C}}$ by Proposition 1.5, F is $(\mathcal{M}, \mathcal{B}_{\mathbb{C} \times \mathbb{C}})$-measurable by Proposition 2.4. We also know ϕ and ψ are continuous, hence $(\mathcal{B}_{\mathbb{C} \times \mathbb{C}}, \mathcal{B}_{\mathbb{C}})$-measurable by Corollary 2.2. Thus $f+g = \phi \circ F$ and $f \cdot g = \psi \circ F$ are $\mathcal{M}$-measurable by Proposition 2.1 □

We may occassionally need to consider functions valued in the extended real line $\overline{\mathbb{R}}$:

Def The Borel σ-algebra on $\overline{\mathbb{R}}$ is $\mathcal{B}_{\overline{\mathbb{R}}} := \{E \subset \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$. We say $f : \mathcal{M} \to \overline{\mathbb{R}}$ is $\mathcal{M}$-measurable (or just measurable) if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. □

Exercise
1. Show that $\mathcal{B}_{\overline{\mathbb{R}}}$ is the σ-algebra generated by $\{\arctan^{-1}(U \cap [-\frac{\pi}{2}, \frac{\pi}{2}]) : U \subset \mathbb{R} \text{ open}\}$ where we define $\arctan(\pm\infty) := \pm\frac{\pi}{2}$.
2. Show that $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the rays $\{(a, \infty] : a \in \mathbb{R}\}$ or $\{[-\infty, a) : a \in \mathbb{R}\}$.

The previous proposition is true for $f, g : \mathcal{M} \to \overline{\mathbb{R}}$, provided one redefines $(f+g)(x)$ whenever $f(x) = -g(x) = \pm\infty$ (see Exercise 4 on Homework 4)

Proposition 2.7 Let $f_n : X \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, be $\mathcal{M}$-measurable functions. Then the functions

$$g_1(x) := \sup_{n\in\mathbb{N}} f_n(x) \qquad g_2(x) := \inf_{n\in\mathbb{N}} f_n(x)$$

$$g_3(x) := \limsup_{n\to\infty} f_n(x) \qquad g_4(x) := \liminf_{n\to\infty} f_n(x)$$

are all $\mathcal{M}$-measurable. If $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$, then f is $\mathcal{M}$-measurable.

Proof Observe that

$$g_1^{-1}((a,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((a,\infty]) \quad \text{and} \quad g_2^{-1}([-\infty,a)) = \bigcup_{n=1}^{\infty} f_n^{-1}([-\infty,a))$$

and so g_1 and g_2 are $\mathcal{M}$-measurable by Proposition 2.1.2. This also implies $h_k(x) := \sup_{n\geq k} f_n(x)$ is measurable for each k, and $g_3(x) = \inf_k h_k(x)$ is measurable. Similarly for g_4. Finally, if $f(x)$ exists for each $x \in X$, then $f = g_3 = g_4$ is measurable. □

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By considering $f_1 = f$, $f_2 = g$, and $f_n \equiv \mp\infty$ $\forall n \geq 3$ we have:

Corollary 2.8 If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then so are $\max(f,g)$ and $\min(f,g)$.

Corollary 2.9 If $f_n : X \to \mathbb{C}$, $n \in \mathbb{N}$, are measurable and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$, then f is measurable.

Proof $\mathrm{Re}(f)(x) = \lim_{n\to\infty} \mathrm{Re}(f_n)(x)$ and $\mathrm{Im}(f)(x) = \lim_{n\to\infty} \mathrm{Im}(f_n)(x)$ are both measurable by Proposition 2.7, hence f is measurable by Corollary 2.5. □

Def For $f : X \to \overline{\mathbb{R}}$, its positive part is the function

$$f^+(x) := \max(f(x), 0).$$

The negative part of f is the function

$$f^-(x) := \max(-f(x), 0).$$

□

Observe that $f = f^+ - f^-$ and $f^+ \cdot f^- = 0$. Then f is measurable if and only if $f^\pm$ are measurable by Proposition 2.6 and Corollary 2.8.

Def For $f : X \to \mathbb{C}$, its polar decomposition is the product $f = (\mathrm{sgn}\, f)\,|f|$ where

$$\mathrm{sgn}\, z = \begin{cases} z/|z| & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

□

Observe that f is measurable if and only if $\mathrm{sgn}\, f$ and $|f|$ are measurable. Indeed, the "if" part follows from Proposition 2.6. For the "only if" part, note that $z \mapsto \mathrm{sgn}\, z$ and $z \mapsto |z|$ are Borel measurable: the latter by continuity, the former by

$$\mathrm{sgn}^{-1}(U) = \begin{cases} V & \text{if } 0 \notin U \\ V \cup \{0\} & \text{if } 0 \in U \end{cases}$$

for U open where $V = \{rz : z \in U, r > 0\}$. Hence $\mathrm{sgn}\, f$ and $|f|$ are measurable by Proposition 2.1.

Def For $E \subset X$, the indicator function on E is the function $\mathbb{1}_E : X \to \mathbb{C}$ defined by

$$\mathbb{1}_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

If $\mathcal{M}$ is a σ-algebra on X, then a simple function on X is a $\mathbb{C}$-linear combination of indicator functions $\mathbb{1}_E$ with $E \in \mathcal{M}$. □

Observe that $\mathbb{1}_E$ is $\mathcal{M}$-measurable iff $E \in \mathcal{M}$, since $\mathbb{1}_E^{-1}(U) \in \{\emptyset, E, E^c, X\}$ for all $U \subset \mathbb{C}$. Also note that f is simple iff it is measurable with $f(X) = \{z_1, \dots, z_n\} \subset \mathbb{C}$, in which case

$$f(z) = \sum_{j=1}^{n} z_j \mathbb{1}_{f^{-1}(\{z_j\})}.$$

Note that $z_1, \dots, z_n$ are distinct and hence $f^{-1}(\{z_1\}), \dots, f^{-1}(\{z_n\})$ are disjoint and their union is X. (We may have $z_j = 0$ for some $1 \le j \le n$, but we still include it the sum so that these preimages form a partition of X.)

Def The formula for the simple function f above is called its standard representation. □

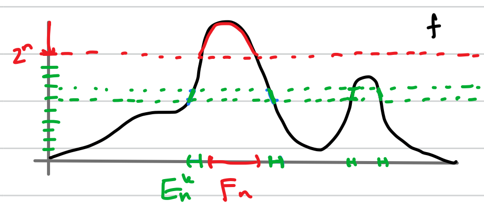
The collection $S(X)$ of simple functions on X is a unital $*$-algebra: $\mathbb{1} \in S(X)$ and $S(X)$ is closed under $\mathbb{C}$-linear combinations, products (since $\mathbb{1}_E \mathbb{1}_F = \mathbb{1}_{E \cap F}$), and complex conjugation. Moreover, it is dense in the collection of all measurable functions in a strong way:

Theorem 2.10 Let $(X, \mathcal{M})$ be a measurable space.

1. If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ of simple functions satisfying $0 \le \phi_1 \le \phi_2 \le \cdots \le f$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.
2. If $f : X \to \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ of simple functions satisfying $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

Proof 1: For $n \in \mathbb{N}$ and $0 \le k \le 2^{2n} - 1$, let

$$E_n^k := f^{-1}((k2^{-n}, (k+1)2^{-n}]) \quad \text{and} \quad F_n := f^{-1}((2^n, \infty])$$



and define

$$\phi_n := \sum_{k=0}^{2^{2n}-1} k2^{-n} \mathbb{1}_{E_n^k} + 2^n \mathbb{1}_{F_n}$$

By construction, $0 \le f - \phi_n \le 2^{-n}$ on F_n^c. So $\phi_n \le f$, and $\phi_n \to f$ in the claimed ways. Since we partitioned $(0, 2^n]$ dyadically, we have $E_n^k = E_{n+1}^{2k} \cup E_{n+1}^{2k+1}$ and hence

$$k2^{-n} \mathbb{1}_{E_n^k} \le 2k2^{-(n+1)} \mathbb{1}_{E_{n+1}^{2k}} + (2k+1)2^{-(n+1)} \mathbb{1}_{E_{n+1}^{2k+1}}$$

It follows that $\phi_n \le \phi_{n+1}$.

2: Let $g = \mathrm{Re}(f)$, $h = \mathrm{Im}(h)$. Apply 1 to the positive and negative parts of g and h to obtain sequences $(\psi_n^{\pm})_{n \in \mathbb{N}}$, $(\omega_n^{\pm})_{n \in \mathbb{N}}$ of non-negative simple functions increasing to $g^{\pm}$ and $h^{\pm}$, respectively. Set $\phi_n := \psi_n^+ - \psi_n^- + i(\omega_n^+ - \omega_n^-)$. Then $\phi_n \to f$ in the claimed ways by the convergence of $\psi_n^\pm \to g^\pm$ and $\omega_n^\pm \to h^\pm$. Finally, note that

$$\mathrm{Re}(\phi_n)(x) = \begin{cases} \psi_n^+(x) & \text{if } g(x) \geq 0 \\ -\psi_n^-(x) & \text{if } g(x) < 0 \end{cases}$$

Since $g(x) \geq 0$ implies $0 = \psi_n^-(x) \leq g^-(x) = 0$, and similarly for $g(x) < 0$. Hence

$$|\mathrm{Re}(\phi_n)(x)| = \begin{cases} \psi_n^+(x) & \text{if } g(x) \geq 0 \\ \psi_n^-(x) & \text{if } g(x) < 0 \end{cases} \leq \begin{cases} g^+(x) & \text{if } g(x) \geq 0 \\ g^-(x) & \text{if } g < 0 \end{cases} = |g(x)|$$

Similarly, $|\mathrm{Im}(\phi_n)| \leq |h|$. It follows that $|\phi_n| \leq |f|$, and a similar argument shows that $|\phi_n| \leq |\phi_{n+1}|$. □

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The following proposition gives one advantage of working with complete measures:

Proposition 2.11 Let $(X, \mathcal{M}, \mu)$ be a measure space. If μ is complete, then

1. If f is $\mathcal{M}$-measurable and $f = g$ μ-almost everywhere, then g is $\mathcal{M}$-measurable.
2. If f_n is $\mathcal{M}$-measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-almost everywhere, then f is $\mathcal{M}$-measurable.

Conversely, 1 and 2 each imply μ is complete.

Proof First suppose μ is complete.

1: For any Borel set B, $g^{-1}(B) = f^{-1}(B) \cup N_1 \setminus N_2$ where $N_1, N_2 \subset \{x \in X : f(x) \neq g(x)\}$. The completeness of μ implies N_1, N_2 are measurable and hence $g^{-1}(B)$ is measurable.

2: Let $g = \limsup_{n\to\infty} \mathrm{Re}(f_n) + i \limsup_{n\to\infty} \mathrm{Im}(f_n)$, which is measurable by Corollary 2.5 and Proposition 2.7. Then $g = f$ μ-almost everywhere and hence f is measurable by 1.

Now, suppose 1 holds and let E be a subset of a μ-null set N. Then $1_E + 2 \cdot 1_{N \setminus E}$ agrees with the zero function on N^c; that is, μ-almost everywhere. Thus this simple function is $\mathcal{M}$-measurable. Since E is the preimage of $\{1\}$, we have $E \in \mathcal{M}$ and so μ is complete.

Finally if 2 holds, we consider the constant sequence $1_E + 2 \cdot 1_{N \setminus E}$ and argue as above. □

Proposition 2.12 Let $(X, \mathcal{M}, \mu)$ be a measure space with completion $(X, \overline{\mathcal{M}}, \overline{\mu})$. If f is $\overline{\mathcal{M}}$-measurable, then there exists a $\mathcal{M}$-measurable function g such that $f = g$ $\overline{\mu}$-almost everywhere.

Proof If $f = 1_E$ for $E \in \overline{\mathcal{M}}$, then $E = F \cup N$ where $F \in \mathcal{M}$ and N is a subset of a μ-null set. Thus $g = 1_F$ agrees with f except on $N \setminus F$, which is $\overline{\mu}$-null. So the theorem holds for simple functions by applying this argument to each term. Now let f be a general $\overline{\mathcal{M}}$-measurable function. Using Proposition 2.10 we can find a sequence $(\phi_n)_{n\in\mathbb{N}}$ of $\overline{\mathcal{M}}$-measurable simple functions that converges pointwise to f. For each $n \in \mathbb{N}$, let ψ_n be an $\mathcal{M}$-measurable simple function so that $\psi_n = \phi_n$ except on a set $E_n \in \overline{\mathcal{M}}$ with $\overline{\mu}(E_n) = 0$. Then $E_n \subset N_n \in \mathcal{M}$ with $\mu(N_n) = 0$, and thus $N := \bigcup_{n=1}^{\infty} N_n$ is μ-null. Set

$$g := \lim_{n\to\infty} \psi_n 1_{N^c}.$$

Then $\psi_n \to g$ pointwise and so g is $\mathcal{M}$-measurable by Corollary 2.9. Since $\psi_n 1_{N^c} = \phi_n 1_{N^c}$, we have $g = f$ except on N. □

2.2 Integration of nonnegative functions

In this section we fix a measure space $(X, \mathcal{M}, \mu)$ and denote

$$L^+(X,\mu) := \{ f : X \to [0,\infty] : f \text{ is } (\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})\text{-measurable}\} =: L^+$$

Def For a simple function $\phi \in L^+(X,\mu)$ with standard representation $\phi = \sum_{j=1}^n \alpha_j 1_{E_j}$, its integral with respect to μ is the quantity

$$\int_X \phi \, d\mu := \sum_{j=1}^n \alpha_j \, \mu(E_j)$$

(where by convention $0 \cdot \infty = 0$). $\square$

Note that $\phi \in L^+$ implies $\alpha_1, \ldots, \alpha_n \in [0,\infty)$. In particular, we may have $\int_X \phi \, d\mu = \infty$ if $\mu(E_j) = \infty$ and $\alpha_j > 0$ for some $1 \le j \le n$. For $A \in \mathcal{M}$, note that $\phi 1_A = \sum \alpha_j 1_{A \cap E_j}$ is simple. We write

$$\int_A \phi \, d\mu := \int_X \phi 1_A \, d\mu = \sum_{j=1}^n \alpha_j \, \mu(A \cap E_j)$$

We will also use the following variations on the notation:

$$\int_X \phi \, d\mu = \int \phi \, d\mu = \int_X \phi(x) \, d\mu(x).$$

Let $\phi \in L^+$ be a simple function with standard representation $\phi = \sum_{j=1}^n \alpha_j 1_{E_j}$. Then $\{E_1, \ldots, E_n\}$ is a partition of X. Suppose $\{F_1, \ldots, F_m\}$ is a partition of X refining $\{E_1, \ldots, E_n\}$ in the sense that for each $k = 1, \ldots, m$ one has $F_k \subset E_{j(k)}$ for exactly one $1 \le j(k) \le n$. Then $E_j = F_{k_1} \cup \cdots \cup F_{k_\ell}$ for each $j = 1, \ldots, n$ and so

$$\int \phi \, d\mu = \sum_{j=1}^n \alpha_j \, \mu(E_j) = \sum_{j=1}^n \alpha_j \left(\mu(F_{k_1}) + \cdots + \mu(F_{k_\ell}) \right) = \sum_{k=1}^m \alpha_{j(k)} \, \mu(F_k)$$

Proposition 2.13 Let $\phi, \psi \in L^+(X,\mu)$ be simple functions.

1. For $c \ge 0$
$$\int_X c\phi + \psi \, d\mu = c \int_X \phi \, d\mu + \int_X \psi \, d\mu.$$

2. If $\phi \le \psi$, then
$$\int_X \phi \, d\mu \le \int_X \psi \, d\mu.$$

3. The map
$$\mathcal{M} \ni A \mapsto \int_A \phi \, d\mu$$
is a measure.

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Proof 1: Let $\phi = \sum_{j=1}^n \alpha_j 1_{E_j}$ and $\psi = \sum_{k=1}^m \beta_k 1_{F_k}$ be the standard representations of ϕ and ψ. Then $\{E_1, \ldots, E_n\}$ and $\{F_1, \ldots, F_m\}$ both partition X. Observe that $\{E_j \cap F_k : 1 \le j \le n, 1 \le k \le m\}$ is a common refinement of these partitions, and so by the discussion preceding the proposition we have

$$c \int_X \phi \, d\mu + \int_X \psi \, d\mu = c \sum_{j=1}^n \sum_{k=1}^m \alpha_j \, \mu(E_j \cap F_k) + \sum_{j=1}^n \sum_{k=1}^m \beta_k \, \mu(E_j \cap F_k) = \sum_{j=1}^n \sum_{k=1}^m (c\alpha_j + \beta_k) \, \mu(E_j \cap F_k)$$

This equals $\int_X c\phi + \psi \, d\mu$ since

$$c\phi + \psi = \sum_{j=1}^n \sum_{k=1}^m (c\alpha_j + \beta_k) 1_{E_j \cap F_k}$$

and $\{E_j \cap F_k : 1 \le j \le n, 1 \le k \le m\}$ necessarily refines the partition coming from the standard representation for $c\phi + \psi$.

②: Let ϕ and ψ have standard representations as above. Then $\alpha_j \mu(E_j \cap F_k) \leq \beta_k \mu(E_j \cap F_k)$ for $1 \leq j \leq n$ and $1 \leq k \leq m$ since either $E_j \cap F_k \neq \emptyset$ implies $\alpha_j \leq \beta_k$ or $\mu(E_j \cap F_k) = 0$. So by the discussion preceding the proposition we have:

$$\int_X \phi \, d\mu = \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_j \mu(E_j \cap F_k) \leq \sum_{j=1}^{n} \sum_{k=1}^{m} \beta_k \mu(E_j \cap F_k) = \int_X \psi \, d\mu$$

③: Let ϕ have the above standard representation. If $A = \emptyset$, then $\mu(A \cap E_j) = 0$ for $1 \leq j \leq n$ so that $\int_A \phi \, d\mu = 0$. Let $\{A_k : k \in \mathbb{N}\} \subset \mathcal{M}$ be a disjoint collection, and denote $A := \bigcup A_k$. Then the countable additivity of μ gives:

$$\begin{aligned}\int_A \phi \, d\mu &= \sum_{j=1}^{n} \alpha_j \mu(A \cap E_j) \\ &= \sum_{j=1}^{n} \alpha_j \sum_{k=1}^{\infty} \mu(A_k \cap E_j) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{n} \alpha_j \mu(A_k \cap E_j) = \sum_{k=1}^{\infty} \int_{A_k} \phi \, d\mu.\end{aligned}$$

□

Observe that ② in the previous proposition implies $\int_X \psi \, d\mu = \sup\{\int_X \phi \, d\mu : 0 \leq \phi \leq \psi, \phi \text{ simple}\}$. We can then use this formula to extend the integral beyond simple functions:

Def For $f \in L^+(X, \mu)$, its integral with respect to μ is the quantity

$$\int_X f \, d\mu := \sup\left\{\int_X \phi \, d\mu : 0 \leq \phi \leq f, \phi \text{ simple}\right\}$$

□

Recall that Theorem 2.10.① tells us any $f \in L^+$ is the limit of an increasing sequence of simple functions which indicates that it is reasonable to define $\int f \, d\mu$ using only simple functions dominated by f. Note that if $f, g \in L^+$ satisfy $f \leq g$ then $\{\phi \text{ simple} : 0 \leq \phi \leq f\} \subset \{\psi \text{ simple} : 0 \leq \psi \leq g\}$ and hence

$$\int_X f \, d\mu \leq \int_X g \, d\mu. \quad *$$

Also, for $c \in (0, \infty)$ we have $0 \leq \phi \leq f \iff 0 \leq c\phi \leq cf$ so that

$$\int_X cf \, d\mu = c \int_X f \, d\mu.$$

The above definition essentially defines $\int f \, d\mu$ as a limit (recall the Riemann integral is also technically a limit), and knowing when one can interchange two limits is a fundamental problem in analysis. For integrals, this means when do we have the equality

$$\lim_{n \to \infty} \int_X f_n \, d\mu \stackrel{?}{=} \int_X \lim_{n \to \infty} f_n \, d\mu$$

for a sequence of measurable functions. We will have three major theorems addressing this problem, starting with the following theorem:

Theorem 2.14 (The Monotone Convergence Theorem) If $(f_n)_{n \in \mathbb{N}} \subset L^+(X, \mu)$ satisfies $f_n \leq f_{n+1}$ for each $n \in \mathbb{N}$

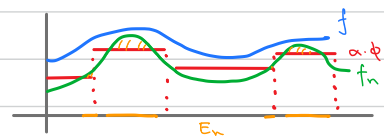
and $f := \lim_{n\to\infty} f_n \; (= \sup_n f_n)$, then

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Proof By $(*)$, $(\int_X f_n \, d\mu)_{n\in\mathbb{N}}$ is an increasing sequence that is dominated by $\int f \, d\mu$. Thus its limit exists and satisfies

$$\lim_{n\to\infty} \int_X f_n \, d\mu \le \int_X f \, d\mu.$$

Conversely, fix $0<\alpha<1$ and let $0 \le \phi \le f$ be simple. Then $\alpha\phi(x) < f(x)$ for all $x\in X$, and so if we let



$$E_n := \{ x\in X : \alpha\phi(x) \le f_n(x)\},$$

then $\bigcup_{n=1}^{\infty} E_n = X$. By definition of E_n and $(*)$

$$\int_X f_n \, d\mu \ge \int_{E_n} f_n \, d\mu \ge \int_{E_n} \alpha\phi \, d\mu = \alpha \int_{E_n} \phi \, d\mu.$$

Now, noting that $E_n \subset E_{n+1}$ for each $n\in\mathbb{N}$, we have $\int_{E_n} \phi \, d\mu \to \int_X \phi \, d\mu$ by continuity from below (Theorem 1.8 3). Hence

$$\lim_{n\to\infty} \int_X f_n \, d\mu \ge \alpha \int_X \phi \, d\mu.$$

Taking the supremum over all simple $0 \le \phi \le f$ and letting $\alpha \to 1$ yields the needed inequality. $\square$

The above theorem makes computing $\int_X f \, d\mu$ for $f\in L^+$ more tractable. Indeed, rather than computing the supremum of $\int_X \phi \, d\mu$ for all simple $0 \le \phi \le f$, one just needs to a find a sequence $(\phi_n)_{n\in\mathbb{N}}$ of simple functions satisfying $\phi_n \le \phi_{n+1}$ and $\phi_n \to f$, in which case

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X \phi_n \, d\mu.$$

Furthermore, Theorem 2.10 not only tells us such a sequence exists, but we explicitly constructed it in the proof.

Theorem 2.15 If $(f_n)_{n\in\mathbb{N}} \subset L^+(X,\mu)$ and $f := \sum f_n$, then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

In particular, $\int_X f_1 + \cdots + f_N \, d\mu = \int_X f_1 \, d\mu + \cdots + \int_X f_N \, d\mu$.

Proof We fill first use induction to show the integral is finitely additive. The base case is trivial, so assume

$$\int_X f_1 + \cdots + f_N \, d\mu = \int_X f_1 \, d\mu + \cdots + \int_X f_N \, d\mu.$$

Using Theorem 2.10 let $(\phi_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}} \subset L^+$ be increasing sequences of simple functions converging pointwise to $f_1 + \cdots + f_N$ and f_{N+1}, respectively. The monotone convergence theorem (Theorem 2.14) and Proposition 2.13.1 imply

$$\begin{aligned}\int_X f_1 + \cdots + f_N + f_{N+1} \, d\mu &= \lim_{n\to\infty} \int_X \phi_n + \psi_n \, d\mu \\ &= \lim_{n\to\infty} \int_X \phi_n \, d\mu + \int_X \psi_n \, d\mu \\ &= \int_X f_1 + \cdots + f_N \, d\mu + \int_X f_{N+1} \, d\mu = \int_X f_1 \, d\mu + \cdots + \int_X f_N \, d\mu + \int_X f_{N+1} \, d\mu\end{aligned}$$

M1: | Q1 | Q2 | Q3 | Total |
|---|---|---|---|
| 20.5 | 7.5 | 3.6 | 29 |
| 82% | 38% | 76% | 58% |

Hence the integral is finitely additive by induction. Now observe that the partial sums are increasing:

$$\sum_{n=1}^{N} f_n \leq \sum_{n=1}^{N+1} f_n$$

Thus using the monotone convergence theorem again we have

$$\int_X f\,d\mu = \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n\,d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n\,d\mu = \sum_{n=1}^{\infty} \int f_n\,d\mu. \quad \square$$

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Proposition 2.16 If $f \in L^+(X,\mu)$, then $\int_X f\,d\mu = 0$ if and only if $f = 0$ μ-a.e.

Proof If f is simple with standard form $f = \sum \alpha_j 1_{E_j}$, then $f = 0$ μ-a.e iff $\mu(E_j) = 0$ for any j with $\alpha_j > 0$, iff

$$\int_X f\,d\mu = \sum_{j=1}^{n} \alpha_j \mu(E_j) = 0.$$

For general f, suppose $f = 0$ μ-a.e. Then for simple $0 \leq \phi \leq f$, we have $\phi = 0$ μ-a.e. and so $\int \phi\,d\mu = 0$. Hence $\int f\,d\mu = \sup\{0\} = 0$. Conversely, suppose $\int f\,d\mu = 0$ and consider

$$E_n := \{x \in X : f(x) \geq \tfrac{1}{n}\}$$

Then $f \geq \frac{1}{n} 1_{E_n}$ implies

$$0 = \int_X f\,d\mu \geq \int_X \tfrac{1}{n} 1_{E_n}\,d\mu = \tfrac{1}{n}\mu(E_n).$$

Thus $\mu(E_n) = 0$, and therefore

$$\{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$$

is μ-null. That is, $f = 0$ μ-a.e. $\square$

Note that if $f \leq g$ with equality μ-a.e., then the previous theorem gives

$$\int_X f\,d\mu = \int_X f\,d\mu + \int_X g - f\,d\mu = \int_X g\,d\mu$$

Corollary 2.17 If $(f_n)_{n\in\mathbb{N}} \subset L^+(X,\mu)$ increases to some $f \in L^+(X,\mu)$ μ-a.e., then

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu.$$

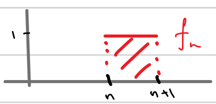
Proof Let $E \subset X$ be the set of x such that $f_n(x) \nearrow f(x)$. Then $\mu(E^c) = 0$, and thus so $f - f1_E = 0$ and $f_n - f_n 1_E = 0$, $n \in \mathbb{N}$, μ-a.e. Proposition 2.16 and the monotone convergence theorem (Theorem 2.14) imply:

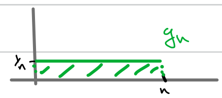
$$\int_X f\,d\mu = \int_X f 1_E\,d\mu = \lim_{n\to\infty} \int_X f_n 1_E\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu \quad \square$$

It is necessary that the sequences increase in the monotone convergence theorem and the previous corollary, as the following examples demonstrate.

$\boxed{\text{Ex}}$ On $(\mathbb{R}, \mathcal{L}, m)$, define for each $n \in \mathbb{N}$.

$$f_n = 1_{[n, n+1]} \qquad g_n = \frac{1}{n} 1_{[0,n]} \qquad h_n = n 1_{(0, \frac{1}{n}]}$$





Then each sequence tends to zero pointwise, but all have integrals equal to 1. We say "their mass escapes to infinity." However, note that one still has the inequality

$$\int_{\mathbb{R}} 0 \, dm \leq \lim_{n\to\infty} \int_X f_n \, dm$$

and similarly for g_n and h. □

The inequality in the above example holds more generally, and is the second of our major theorems on exchanging limits and integrals.

Theorem 2.18 (Fatou's Lemma) For any $(f_n)_{n\in\mathbb{N}} \subset L^+(X,\mu)$

$$\int_X (\liminf_{n\to\infty} f_n) \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Proof For each $N \in \mathbb{N}$,

$$\inf_{n\geq N} f_n \leq f_m \quad \forall m \geq N \implies \int_X \inf_{n\geq N} f_n \, d\mu \leq \int_X f_m \, d\mu \quad \forall m \geq N \implies \int_X \inf_{n\geq N} f_n \, d\mu \leq \inf_{n\geq N} \int_X f_n \, d\mu.$$

Now, $g_N := \inf_{n\geq N} f_n$ is an increasing sequence with $g_N \to \liminf f_n$. So the monotone convergence theorem implies:

$$\begin{aligned}\int_X (\liminf_{n\to\infty} f_n) \, d\mu &= \int_X \lim_{N\to\infty} g_N \, d\mu \\ &= \lim_{N\to\infty} \int_X g_N \, d\mu \\ &\leq \lim_{N\to\infty} \inf_{n\geq N} \int_X f \, d\mu = \liminf_{n\to\infty} \int_X f_n \, d\mu\end{aligned}$$

□

Proceeding as in the proof of Corollary 2.17 and using Fatou's lemma gives:

Corollary 2.19 If $(f_n)_{n\in\mathbb{N}} \in L^+(X,\mu)$ converges pointwise μ-a.e. to some $f \in L^+(X,\mu)$, then

$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$$

Proposition 2.20 If $f \in L^+(X,\mu)$ satisfies $\int_X f \, d\mu < \infty$, then $\{x \in X : f(x) = \infty\}$ is μ-null and $\{x : f(x) > 0\}$ is σ-finite with respect to μ.

Proof Homework 5. □

2.3 Integration of complex functions

We once again fix a measure space $(X, \mathcal{M}, \mu)$ throughout this section.

For a measurable function $f : X \to \mathbb{C}$, one can try to extend the integral from last section by considering

$$\int_X f\,d\mu := \int_X \mathrm{Re}(f)^+\,d\mu - \int_X \mathrm{Re}(f)^-\,d\mu + i\int_X \mathrm{Im}(f)^+\,d\mu - i\int_X \mathrm{Im}(f)^-\,d\mu$$

This will be well-defined provided each of the above integrals is finite. Since

$$|f| \le \mathrm{Re}(f)^+ + \mathrm{Re}(f)^- + \mathrm{Im}(f)^+ + \mathrm{Im}(f)^- = |\mathrm{Re}(f)| + |\mathrm{Im}(f)| \le 2|f|,$$

it follows that the above four integrals are all finite iff $\int_X |f|\,d\mu < \infty$.

Def We say an $\mathcal{M}$-measurable function $f : X \to \mathbb{C}$ is <u>integrable</u> with respect to μ if $\int_X |f|\,d\mu < \infty$, and we denote the collection of such functions by $L^1(X,\mu)$ (or $L^1(X,\mathcal{M},\mu)$ or $L^1(\mu)$). For $E \in \mathcal{M}$, we say f is <u>integrable on E</u> if $\int_E |f|\,d\mu < \infty$. $\square$

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Proposition 2.21 $L^1(X,\mu)$ is a $\mathbb{C}$-vector space and $L^1(X,\mu) \ni f \mapsto \int_X f\,d\mu$ is a linear functional. Similarly, the $\mathbb{R}$-valued functions form an $\mathbb{R}$-vector space.

Proof For $f, g \in L^1(X,\mu)$ and $\alpha \in \mathbb{C}$, $|\alpha f + g| \le |\alpha||f| + |g|$ implies $\alpha f + g \in L^1(X,\mu)$. Hence $L^1(X,\mu)$ is a $\mathbb{C}$-vector space. The equality

$$\alpha \int_X f\,d\mu = \int_X \alpha f\,d\mu$$

follows by matching real and imaginary parts (and their respective positive and negative parts). It remains to show $\int f + g\,d\mu = \int f\,d\mu + \int g\,d\mu$. Denote $h := f + g$. Then

$$\mathrm{Re}(h)^+ - \mathrm{Re}(h)^- = \mathrm{Re}(h) = \mathrm{Re}(f) + \mathrm{Re}(g) = \mathrm{Re}(f)^+ - \mathrm{Re}(f)^- + \mathrm{Re}(g)^+ - \mathrm{Re}(g)^-$$

so that $\mathrm{Re}(h)^+ + \mathrm{Re}(f)^- + \mathrm{Re}(g)^- = \mathrm{Re}(h)^- + \mathrm{Re}(f)^+ + \mathrm{Re}(g)^+$. Hence their integrals agree and using Theorem 2.15 we have

$$\int_X \mathrm{Re}(h)^+\,d\mu + \int_X \mathrm{Re}(f)^-\,d\mu + \int_X \mathrm{Re}(g)^-\,d\mu = \int_X \mathrm{Re}(h)^-\,d\mu + \int_X \mathrm{Re}(f)^+\,d\mu + \int_X \mathrm{Re}(g)^+\,d\mu.$$

Since all integrals are finite, subtracting terms gives

$$\int_X \mathrm{Re}(h)^+\,d\mu - \int_X \mathrm{Re}(h)^-\,d\mu = \int_X \mathrm{Re}(f)^+\,d\mu - \int_X \mathrm{Re}(f)^-\,d\mu + \int_X \mathrm{Re}(g)^+\,d\mu - \int_X \mathrm{Re}(g)^-\,d\mu.$$

Applying the same argument to the imaginary parts then yields $\int h\,d\mu = \int f\,d\mu + \int g\,d\mu$. $\square$

Proposition 2.22 For $f \in L^1(X,\mu)$, $\left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu$.

Proof First suppose f is $\mathbb{R}$-valued so that $f = f^+ - f^-$. Then

$$\left|\int_X f\,d\mu\right| = \left|\int_X f^+\,d\mu - \int_X f^-\,d\mu\right| \le \int_X f^+\,d\mu + \int_X f^-\,d\mu = \int_X f^+ + f^-\,d\mu = \int_X |f|\,d\mu.$$

For $\mathbb{C}$-valued f, let $\alpha := \operatorname{sgn}(\int f\,d\mu)$. Then $|\int f\,d\mu| = \bar{\alpha}\int f\,d\mu = \int \bar{\alpha} f\,d\mu$. Consequently, $\int \bar{\alpha} f\,d\mu \in \mathbb{R}$ so that

$$\begin{aligned}\left|\int_X f\,d\mu\right| &= \operatorname{Re}\int_X \bar{\alpha} f\,d\mu \\ &= \int_X \operatorname{Re}(\bar{\alpha}f)^+\,d\mu - \int_X \operatorname{Re}(\bar{\alpha}f)^-\,d\mu \\ &\le \int_X \operatorname{Re}(\bar{\alpha}f)^+\,d\mu + \int_X \operatorname{Re}(\bar{\alpha}f)^-\,d\mu \\ &= \int_X |\operatorname{Re}(\bar{\alpha}f)|\,d\mu \\ &\le \int_X |\bar{\alpha}f|\,d\mu = \int_X |f|\,d\mu.\end{aligned}$$

$\square$

Proposition 2.23

1. For $f \in L^1(X,\mu)$, $\{x \in X : f(x) \neq 0\}$ is σ-finite
2. For $f,g \in L^1(X,\mu)$, the following are equivalent:
   - i. $f = g$ μ-almost everywhere.
   - ii. $\int_X |f-g|\,d\mu = 0$
   - iii. For all $E \in \mathcal{M}$, $\int_E f\,d\mu = \int_E g\,d\mu$

Proof 1: Since $f(x) \neq 0$ iff $|f(x)| > 0$, this follows from Proposition 2.20.
2: If $f = g$ μ-a.e., then $|f-g| = 0$ μ-a.e. and so $\int |f-g|\,d\mu = 0$ by Proposition 2.16. If $\int |f-g|\,d\mu = 0$ then Proposition 2.22 implies for $E \in \mathcal{M}$ that

$$\left|\int_E f\,d\mu - \int_E g\,d\mu\right| = \left|\int_E f - g\,d\mu\right| \le \int_E |f-g|\,d\mu \le \int_X |f-g|\,d\mu = 0.$$

Thus $\int_E f\,d\mu = \int_E g\,d\mu$. Finally, suppose f does not equal g μ-a.e. Then

$$E := \{x \in X : h(x) > 0\}$$

has positive measure for one of $h \in \{\operatorname{Re}(f-g)^{\pm}, \operatorname{Im}(f-g)^{\pm}\}$. We will assume this holds $h = \operatorname{Re}(f-g)^+$, with the other cases being similar. Note that $\operatorname{Re}(f(x)-g(x))^+ > 0$ for $x \in E$ implies $\operatorname{Re}(f(x)-g(x))^- = 0$. Thus

$$\operatorname{Re}\left(\int_E f\,d\mu - \int_E g\,d\mu\right) = \int_E \operatorname{Re}(f-g)^+\,d\mu > 0$$

by Proposition 2.16. Hence $\int_E f\,d\mu \neq \int_E g\,d\mu$. $\square$

For $f,g \in L^1(X,\mu)$, define

$$\rho(f,g) := \int_X |f-g|\,d\mu$$

Then $\rho(f,g) = \rho(g,f)$, and $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$ for $h \in L^1(X,\mu)$ since
$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \quad \forall x \in X$.

So ρ is nearly a metric, but not quite: $\rho(f,g) = 0$ does not imply $f=g$, only that $f=g$ μ-a.e. by Proposition 2.23. The standard way to correct this is to define an equivalence relation

$$f \sim g \quad \text{iff} \quad f = g \ \mu\text{-a.e.}$$

and consider ρ as a metric on the quotient space $L^1(X,\mu)/\sim$. Since our primary interest is in the integrability and integrals of functions, rather than the functions themselves, we will actually redefine $L^1(X,\mu) := L^1(X,\mu)/\sim$. That is, elements of $L^1(X,\mu)$ are equivalence classes of functions rather than individual functions, though we will still write "$f \in L^1(X,\mu)$" when we really mean "$[f] \in L^1(X,\mu)$". This has the following advantages:

1. ρ as defined above is a metric on $L^1(X,\mu)$ and we say a sequence $(f_n)_{n\in\mathbb{N}} \subset L^1(X,\mu)$ converges in L^1 to $f \in L^1(X,\mu)$ if it converges with respect to this metric:
$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu \longrightarrow 0$$
2. If $f: X \to \overline{\mathbb{R}}$ is integrable, then it is finite μ-a.e. by Proposition 2.20 and so as an element of $L^1(X,\mu)$ we may treat it as finitely valued
3. If $\bar{\mu}$ is the completion of μ, then there is a one-to-one correspondence between $L^1(X,\bar{\mu})$ and $L^1(X,\mu)$ given by Proposition 2.12.

However, there is also a disadvantage: since $f \in L^1(X,\mu)$ is no longer a function, it does not make sense to talk about specific values of the function $f(x)$ for $x \in X$. Instead, we can only talk about the behavior of f on sets of positive measure. (This is reminiscent of the transition from discrete to continuous probability theory.)

We have now arrived at the third and final major Theorem on exchanging limits and integrals.

$\boxed{\text{Theorem 2.24}}$ (The Dominated Convergence Theorem) Let $(f_n)_{n\in\mathbb{N}} \subset L^1(X,\mu)$ be such that

1. f_n converges to some f μ-a.e.
2. there exists non-negative $g \in L^1(X,\mu)$ with $|f_n| \leq g$ for all $n \in \mathbb{N}$

Then $f \in L^1(X,\mu)$ with
$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu.$$

Proof Let $(\overline{\mathcal{M}}, \bar{\mu})$ be the completion of $(\mathcal{M},\mu)$. Then f is $\overline{\mathcal{M}}$-measurable by Proposition 2.11, and by Proposition 2.12 we can identify it with an $\mathcal{M}$-measurable function. (As a function this may require redefining f on a μ-null set, but its equivalence class already contains this new function.) So f is $\mathcal{M}$-measurable with $|f| \leq g$, thus $f \in L^1(X,\mu)$. Note that $|\mathrm{Re}(f_n)| \leq g$ implies $g \pm \mathrm{Re}(f_n) \geq 0$. Thus we can apply Fatou's Lemma:

$$\int_X g\,d\mu \pm \int_X \mathrm{Re}(f)\,d\mu = \int_X g \pm \mathrm{Re}(f)\,d\mu \le \liminf_{n\to\infty} \int_X g \pm \mathrm{Re}(f_n)\,d\mu = \int_X g\,d\mu \begin{cases} + \liminf\limits_{n\to\infty} \int_X \mathrm{Re}(f_n)\,d\mu \\ - \limsup\limits_{n\to\infty} \int_X \mathrm{Re}(f_n)\,d\mu \end{cases}$$

Since $\int_X g\,d\mu < \infty$, we can subtract it from the above and rearranging terms yields

$$\limsup_{n\to\infty} \int_X \mathrm{Re}(f_n)\,d\mu \le \int_X \mathrm{Re}(f) \le \liminf_{n\to\infty} \int_X \mathrm{Re}(f_n)\,d\mu$$

Thus $\int_X \mathrm{Re}(f_n)\,d\mu \to \int_X \mathrm{Re}(f)\,d\mu$. The same argument also shows that the imaginary parts converge. Hence $\int_X f_n\,d\mu \to \int_X f\,d\mu$. $\square$

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Remark Recall that our examples showing "increasing" was necessary in the monotone convergence theorem all had "mass escaping to infinity." The hypothesis $|f_n| \le g$ in the dominated convergence theorem prevents this escape of mass by trapping it under the graph of g. $\square$

Theorem 2.25 Suppose $(f_n)_{n\in\mathbb{N}} \subset L^1(X,\mu)$ satisfies $\sum_{n=1}^{\infty} \int_X |f_n|\,d\mu < \infty$. Then $\sum_{n=1}^{\infty} f_n$ converges μ-a.e. to a function in $L^1(X,\mu)$ and

$$\int_X \sum_{n=1}^{\infty} f_n\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu$$

Proof Let $g := \sum_{n=1}^{\infty} |f_n|$. Then by Theorem 2.15

$$\int_X g\,d\mu = \sum_{n=1}^{\infty} \int_X |f_n|\,d\mu < \infty.$$

Hence $g \in L^1(X,\mu)$. Proposition 2.20 implies $g(x) < \infty$ for μ-a.e. $x \in X$, and for each such x the series $\sum_{n=1}^{\infty} f_n(x)$ converges. The result then follows from the dominated convergence theorem applied to the partial sums: $\left|\sum_{n=1}^{N} f_n\right| \le g \quad \forall N \in \mathbb{N}$. $\square$

We next show that simple functions are dense in the L^1-metric. In the case of $X = \mathbb{R}$ and Lebesgue–Stieltjes measure, we show that compactly supported continuous functions are dense in the L^1-metric

Theorem 2.26 Let $f \in L^1(X,\mu)$. For all $\varepsilon > 0$, there exists a simple function $\phi = \sum_{j=1}^{n} \alpha_j 1_{E_j}$ such that $\int_X |\phi - f|\,d\mu < \varepsilon$. If $X = \mathbb{R}$ and μ is a Lebesgue-Stieltjes measure, then the sets E_j can be taken to be finite unions of bounded open intervals. Moreover, there is a continuous function g that vanishes outside a bounded interval such that $\int_X |g - f|\,d\mu < \varepsilon$.

Proof Let $(\phi_n)_{n\in\mathbb{N}}$ be simple functions converging pointwise to f, which are given by Theorem 2.10(2). Since $|\phi_n - f| \le 2|f|$, the dominated convergence theorem implies $\int |\phi_n - f|\,d\mu < \varepsilon$ for sufficiently large $n \in \mathbb{N}$.
Now let $X = \mathbb{R}$ and μ be a Lebesgue-Stieltjes measure. Let $\phi = \sum_{j=1}^{n} \alpha_j 1_{E_j}$ be such that $\int_X |\phi - f|\,d\mu < \frac{\varepsilon}{2}$. By the argument above, we may assume $|\phi| \le |f|$. Observe that for each non-zero α_j

$$\mu(E_j) = \frac{1}{|\alpha_j|} \int_{E_j} |\phi|\,d\mu \le \frac{1}{|\alpha_j|} \int_X |f|\,d\mu < \infty.$$

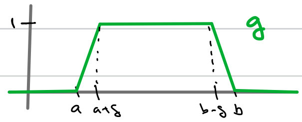
So we can use Proposition 1.20 to find a finite union of bounded open intervals A_j with $\mu(E_j \Delta A_j) \le \frac{\varepsilon}{2|\alpha_j| n}$. Noting that

$$\mu(E_j \Delta A_j) = \int_{\mathbb{R}} |1_{E_j} - 1_{A_j}|\,d\mu,$$

it follows that for $\psi := \sum_{j=1}^{n} \alpha_j 1_{A_j}$, we have

$$\int_X |\psi - f|\,d\mu \le \int_X |\psi - \phi|\,d\mu + \int_X |\phi - f|\,d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, given an open interval $(a,b) \subset \mathbb{R}$, observe that for $\delta > 0$ the continuous function



satisfies $\int_{\mathbb{R}} |g - 1_{(a,b)}|\,d\mu < \delta$. Since ψ is a finite sum of indicator functions of bounded open intervals, we can approximate each term by a continuous function in this way to obtain the desired continuous function approximating f in the L^1-metric. $\square$

Related to interchanging limits and integrals is differentiating integrals:

Theorem 2.27 Let $-\infty < a < b < \infty$ and suppose $f: X \times [a,b] \to \mathbb{C}$ is such that $f(\cdot, t) \in L^1(X,\mu)$ for each $a \le t \le b$. Denote $F(t) := \int_X f(x,t)\,d\mu(x)$ for $a \le t \le b$.

1. Suppose there exists nonnegative $g \in L^1(X,\mu)$ such that $|f(x,t)| \le g(x)$ for μ-a.e. $x \in X$ and all $a \le t \le b$. If for μ-a.e. $x \in X$, $f(x,\cdot)$ is continuous at t_0 then F is continuous at t_0.
2. Suppose $\frac{\partial f}{\partial t}(x,\cdot)$ exists for μ-a.e. $x \in X$ and that there exists nonnegative $g \in L^1(X,\mu)$ such that $|\frac{\partial f}{\partial t}(x,t)| \le g(x)$ for μ-a.e. x and all $a \le t \le b$. Then F is differentiable with
$$F'(t) = \int_X \frac{\partial f}{\partial t}(x,t)\,d\mu(x).$$

Proof ①: Apply the dominated convergence theorem to $f_n := f(\cdot, t_n)$ where $(t_n)_{n\in\mathbb{N}} \subset [a,b]$ converges to t_0.

②: Let $(t_n)_{n\in\mathbb{N}} \subset [a,b]$ be a sequence converging to t_0. Then for μ-a.e. x
$$\frac{\partial f}{\partial t}(x,t_0) = \lim_{n\to\infty} \underbrace{\frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}}_{=:h_n(x)}$$

Then each h_n is measurable therefore so is $\frac{\partial f}{\partial t}(\cdot, t_0)$ (after potentially modifying it on a μ-null set). The mean value theorem implies for μ-a.e. x
$$|h_n(x)| \le \sup_{t\in[a,b]} \left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x)$$

Hence the dominated convergence theorem gives:
$$F'(t_0) = \lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t - t_0} = \lim_{n\to\infty} \int_X h_n\,d\mu = \int_X \frac{\partial f}{\partial t}(x,t_0)\,d\mu(x)$$
$\square$

Comparing the Lebesgue and Riemann integrals

A partition of $[a,b] \subset \mathbb{R}$ will mean a set $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$. Given a bounded function $f: [a,b] \to \mathbb{R}$, we denote:

$$L(f,P) := \sum_{j=1}^{n} (t_j - t_{j-1}) \cdot \inf_{t_{j-1} \le t \le t_j} f(t) \qquad \text{and} \qquad U(f,P) := \sum_{j=1}^{n} (t_j - t_{j-1}) \cdot \sup_{t_{j-1} \le t \le t_j} f(t)$$

$L(f,P)$ and $U(f,P)$ are called the lower and upper Darboux sums of f with respect to P, respectively. The lower and upper Darboux integrals of f are:

$$L(f) := \sup\{L(f,P) : P \subset [a,b] \text{ partition}\}$$
$$U(f) := \inf\{U(f,P) : P \subset [a,b] \text{ partition}\}$$

Recall that one always has $L(f) \leq U(f)$ with equality if and only if f is Riemann integrable, in which case their common value equals the Riemann integral of f: $\int_a^b f(t)\,dt$. When this occurs we say f is Riemann integrable.

$\boxed{\text{Theorem 2.28}}$ (Riemann–Lebesgue Theorem) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if

$$m(\{t \in [a,b] : f \text{ is discontinuous at } t\}) = 0.$$

In this case, f is Lebesgue measurable and integrable on $[a,b]$ with

$$\int_a^b f(t)\,dt = \int_{[a,b]} f\,dm$$

Proof The iff part is on Homework 6. Suppose f is Riemann integrable. For a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define

$$g(P) := \sum_{j=1}^n 1_{(t_{j-1}, t_j]} \cdot \inf_{t_{j-1} \leq t \leq t_j} f(t)$$
$$G(P) := \sum_{j=1}^n 1_{(t_{j-1}, t_j]} \cdot \sup_{t_{j-1} \leq t \leq t_j} f(t)$$

so that $g(P) \leq f \leq G(P)$ (except possibly at $t_0 = a$, but this is m-null) and

$$\int_{[a,b]} g(P)\,dm = L(f,P) \qquad \text{and} \qquad \int_{[a,b]} G(P)\,dm = U(f,P).$$

The Riemann integrability of f implies there is a sequence of partitions $(P_n)_{n\in\mathbb{N}}$ so that $L(f,P_n)$ increases to $\int_a^b f(t)\,dt$, and $U(f,P_n)$ decreases to it. Since $P \subset Q$ for partitions implies $L(f,P) \leq L(f,Q)$ and $U(f,P) \geq U(f,Q)$, we may assume $P_1 \subset P_2 \subset \cdots$. This also implies

$$g(P_1) \leq g(P_2) \leq \cdots \leq f \leq \cdots \leq G(P_2) \leq G(P_1),$$

and so we may consider the Lebesgue measurable functions $g := \sup_n g(P_n)$ and $G := \inf_n G(P_n)$, which satisfy $g \leq f \leq G$. The dominated convergence theorem implies

$$\int_{[a,b]} g\,dm = \lim_{n\to\infty} \int_{[a,b]} g(P_n)\,dm = \lim_{n\to\infty} L(f,P_n) = \int_a^b f(t)\,dt.$$

Similarly, $\int_{[a,b]} G\,dm = \int_a^b f(t)\,dt$. Thus $\int_{[a,b]} (G-g)\,dm = 0$, and since $G - g \geq 0$ Proposition 2.16 implies $G - g = 0$ m-a.e. Hence $g = f = G$ m-a.e. which implies f is Lebesgue measurable since m is a complete measure. This also yields the equality of the Lebesgue and Riemann integrals of f. $\square$

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Thus the Lebesgue integral subsumes all (proper) Riemann integrals. The Lebesgue integral is also defined for a wider class of functions: $\int_{\mathbb{R}} 1_{\mathbb{Q}}\,dm = m(\mathbb{Q}) = 0$ but $1_{\mathbb{Q}}$ is not Riemann integrable over any interval since it is discontinuous everywhere (by Exercise 1 on Homework 1). Moreover, our convergence theorems (monotone, Fatou, dominated) are much stronger than what one can obtain for the Riemann integral (where one typically needs uniform convergence of functions). However, improper Riemann integrals are <u>not</u> subsumed by the Lebesgue integral:

$\boxed{\text{Ex}}$ ① Consider the bounded function

$$f(t) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n} 1_{(n,n+1]}$$

Then

$$\int_0^\infty f(t)\,dt = \lim_{b\to\infty} \int_0^b f(t)\,dt = \lim_{b\to\infty} \sum_{n=1}^{\lfloor b\rfloor -1} \frac{(-1)^n}{n} + \frac{(-1)^{\lfloor b\rfloor}}{\lfloor b\rfloor}(b-\lfloor b\rfloor) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

However, f is <u>not</u> Lebesgue integrable on $[0,\infty)$ since

$$\int_{[0,\infty)} |f|\,dm = \int_{[0,\infty)} \sum_{n=1}^{\infty} \frac{1}{n} 1_{(n,n+1]}\,dm = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

② We claim $\frac{1}{\sqrt{t}} \in L^1([0,1],m)$ with

$$\int_{[0,1]} \frac{1}{\sqrt{t}}\,dm = \int_0^1 \frac{1}{\sqrt{t}}\,dt = 2$$

Indeed, for each $n \in \mathbb{N}$, $1_{[\frac{1}{n},1]}(t)\frac{1}{\sqrt{t}}$ is Riemann integrable on $[\frac{1}{n},1]$ and hence Lebesgue integrable. Moreover, this sequence is nonnegative and increases to $\frac{1}{\sqrt{t}}$. Thus the monotone convergence theorem and the Riemann-Lebesgue theorem imply

$$\int_{[0,1]} \frac{1}{\sqrt{t}}\,dt = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} \frac{1}{\sqrt{t}}\,dt = \lim_{n\to\infty} \int_{1/n}^1 \frac{1}{\sqrt{t}}\,dt = \int_0^1 \frac{1}{\sqrt{t}}\,dt = 2$$

□

In general, if a function f <u>is</u> Lebesgue integrable on an interval and the improper Riemann integral exists, then they necessarily agree by the dominated convergence theorem.

2.4 Modes of Convergence

So far we have discussed several ways that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions can converge to another function: pointwise, uniformly, almost everywhere, and in L^1. Let us consider another:

Def Let $(X, \mathcal{M}, \mu)$ be a measure space. We say a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathbb{C}$-valued $\mathcal{M}$-measurable functions is <u>Cauchy in measure</u> if for every $\varepsilon, \varepsilon' > 0$ there exists $N \in \mathbb{N}$ so that for all $m, n \geq N$

$$\mu(\{x \in X : |f_m(x) - f_n(x)| \geq \varepsilon\}) < \varepsilon'.$$

We say the sequence <u>converges in measure</u> to a function f if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$$

□

Ex We consider $(\mathbb{R}, \mathcal{L}, m)$ in each of the following examples:

① $f_n = 1_{[n,n+1]}$ converges to 0 pointwise, but not uniformly, or in L^1, or in measure: we have $[n, n+1] = \{x \in \mathbb{R} : |f_n(x) - 0| \geq \frac{1}{2}\}$ for all $n \in \mathbb{N}$ so that

$$\lim_{n\to\infty} m(\{x \in \mathbb{R} : |f_n(x) - 0| \geq \tfrac{1}{2}\}) = 1.$$

This sequence is also not Cauchy in measure since for $n \neq m$, $|f_n(x) - f_m(x)| = 1$ on $[n,n+1] \cup [m,m+1]$.

② $g_n = \frac{1}{n} 1_{[0,n]}$ converges to 0 pointwise, uniformly, and in measure: for $\varepsilon > 0$ if $n \geq \frac{1}{\varepsilon}$ then $\{x \in \mathbb{R} : |g_n(x) - 0| \geq \varepsilon\} = \emptyset$. However, g_n does not converge to zero in L^1.

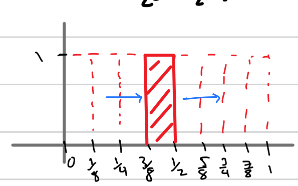
③ $h_n = n 1_{(0,\frac{1}{n}]}$ converges to 0 pointwise and in measure: for any $\varepsilon > 0$

$$m(\{x \in \mathbb{R} : |h_n(x) - 0| \geq \varepsilon\}) \leq \tfrac{1}{n} \longrightarrow 0.$$

However h_n does <u>not</u> converge to zero uniformly or in L^1

④ $k_1 = 1_{[0,1]}$, $k_2 = 1_{[0,\frac{1}{2}]}$, $k_3 = 1_{[\frac{1}{2},1]}$, $k_4 = 1_{[0,\frac{1}{4}]}$, $k_5 = 1_{[\frac{1}{4},\frac{1}{2}]}$, …, $k_n = 1_{[\frac{a}{2^d}, \frac{a+1}{2^d}]}$, … defines a sequence $(k_n)_{n\in\mathbb{N}}$ that converges to 0 in L^1

$$\int_{\mathbb{R}} |k_n - 0| \, dm = \frac{1}{2^d} \longrightarrow 0$$



and in measure

$$m(\{x \in \mathbb{R} : |k_n(x) - 0| \geq \varepsilon\}) \leq \frac{1}{2^d} \longrightarrow 0,$$

but it does not converge pointwise to 0 almost everywhere. Indeed, each $x \in [0,1]$ belongs to infinitely many dyadic intervals $[\frac{a}{2^d}, \frac{a+1}{2^d}]$ and so $k_n(x) = 1$ infinitely often. Likewise $k_n(x) = 0$ infinitely often.

□

Proposition 2.29 If $f_n \to f$ in L^1, then $f_n \to f$ in measure

Proof Fix $\varepsilon > 0$ and denote

$$E_n := \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}.$$

Then

$$\int_X |f_n - f| d\mu \geq \int_{E_n} |f_n - f| d\mu \geq \int_{E_n} \varepsilon \, d\mu = \varepsilon \mu(E_n)$$

Thus

$$\mu(E_n) \leq \frac{1}{\varepsilon} \int_X |f_n - f| d\mu \xrightarrow{n} 0,$$

and so $f_n \to f$ in measure. □

Note that the converse to this is false by Examples 2 and 3 above.
We next show convergence in measure induces a "complete topology."

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Theorem 2.30 Suppose $(f_n)_{n \in \mathbb{N}}$ is Cauchy in measure. Then there exists a measurable function f so that $f_n \to f$ in measure, and f is unique up to almost everywhere equivalence. Moreover, there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ that converges to f almost everywhere.

Proof Let $(f_{n_k})_{k \in \mathbb{N}}$ be a subsequence so that $E_k := \{x \in X : |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq 2^{-k}\}$ satisfies $\mu(E_k) < 2^{-k}$. Then $F_m := \bigcup_{k=m}^{\infty} E_k$ satisfies

$$\mu(F_m) \leq \sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1}$$

Also, for $x \notin F_m$ and $\ell \geq k \geq m$

$$|f_{n_k}(x) - f_{n_\ell}(x)| \leq \sum_{j=k}^{\ell-1} |f_{n_{j+1}}(x) - f_{n_j}(x)| < \sum_{j=k}^{\ell-1} 2^{-j} \leq \sum_{j=m}^{\infty} 2^{-j} = 2^{-m+1} \quad (*)$$

Thus $(f_{n_k}(x))_{k \in \mathbb{N}} \subset \mathbb{C}$ is a Cauchy sequence and therefore converges. Now, let

$$F := \bigcap_{m=1}^{\infty} F_m = \limsup_{k \to \infty} E_k$$

so that $\mu(F) = 0$. Also if $x \notin F$, then $x \notin F_m$ for some m and therefore $(f_{n_k}(x))_{k \in \mathbb{N}}$ converges by the above argument. Define $f(x) := \lim_{k \to \infty} f_{n_k}(x)$ for $x \notin F$, and $f(x) = 0$ otherwise. Then f is separately measurable on F^c (Corollary 2.7) and F, and hence is measurable on X (Exercise check this). Since $\mu(F) = 0$, $f_{n_k} \to f$ μ-almost everywhere. In fact for $x \in F_m^c$, $(*)$ implies

$$|f_{n_m}(x) - f(x)| = \lim_{\ell \to \infty} |f_{n_m}(x) - f_{n_\ell}(x)| < 2^{-m+1}$$

Hence, for $\varepsilon > 0$ if $2^{-m+1} < \varepsilon$ then

$$\mu(\{x \in X : |f_{n_m}(x) - f(x)| \geq \varepsilon\}) \leq \mu(F_m) \leq 2^{-m+1} \xrightarrow{m} 0.$$

So $f_{n_k} \to f$ in measure. But then $f_n \to f$ in measure:

$$\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} \subset \{x \in X : |f_n(x) - f_{n_k}(x)| \geq \tfrac{\varepsilon}{2}\} \cup \{x \in X : |f_{n_k}(x) - f(x)| \geq \tfrac{\varepsilon}{2}\}$$

and the measure of the sets on the right can be made arbitrarily small by Cauchy in measure and convergence in measure, respectively.

Now, suppose $f_n \to g$ in measure. Then

$$\{x \in X : |f(x) - g(x)| \geq \varepsilon\} \subset \{x \in X : |f(x) - f_n(x)| \geq \tfrac{\varepsilon}{2}\} \cup \{x \in X : |f_n(x) - g(x)| \geq \tfrac{\varepsilon}{2}\}$$

implies the set on the left is μ-null. Thus $\{x \in X : f(x) \neq g(x)\} = \bigcup_n \{x \in X : |f(x) - g(x)| \geq \frac{1}{n}\}$ is μ-null. $\square$

Ex Let k_n be as in Example 4 above, which we saw converged to 0 in measure but not almost everywhere. Theorem 2.30 says we should obtain almost everywhere convergence, but only after reducing to a subsequence. In this case the subsequence $(1_{[\frac{0}{2^d}, \frac{1}{2^d}]})_{d \in \mathbb{N}}$ converges to 0 almost everywhere, and is one of many such subsequences. $\square$

Combining Proposition 2.29 and Theorem 2.30 yields

Corollary 2.31 If $f_n \to f$ in L^1, then there exists a subsequence $f_{n_k} \to f$ almost everywhere.

The converse is false in general by Example 1 above. However it does hold on finite measure spaces, in which case one can prove something stronger:

Theorem 2.32 (Egoroff's Theorem) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. If $(f_n)_{n \in \mathbb{N}}, f$ are $\mathbb{C}$-valued $\mathcal{M}$-measurable functions such that $f_n \to f$ μ-a.e., then for all $\varepsilon > 0$ there exists $E \in \mathcal{M}$ with $\mu(E) < \varepsilon$ such that $f_n \to f$ uniformly on E^c.

Proof By adding the μ-null set $\{x \in X : f_n(x) \not\to f(x)\}$ to E, we may assume that $f_n \to f$ everywhere. For $k, n \in \mathbb{N}$ consider

$$E_n(k) := \bigcup_{j=n}^{\infty} \{x \in X : |f_j(x) - f(x)| \geq \tfrac{1}{k}\}.$$

Then $E_n(k) \supset E_{n+1}(k)$ and $\bigcap_{n=1}^{\infty} E_n(k) = \{x \in : |f_n(x) - f(x)| \geq \frac{1}{k} \ \forall n \in \mathbb{N}\} = \emptyset$. Since $\mu(X) < \infty$, we can use continuity from above to assert that $\mu(E_n(k)) \xrightarrow{n} 0$. Now, given $\varepsilon > 0$ and $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ be such that $\mu(E_{n_k}(k)) < \varepsilon 2^{-k}$. Define

$$E := \bigcup_{k=1}^{\infty} E_{n_k}(k)$$

so that $\mu(E) < \varepsilon$ by subadditivity. Also for $n > n_k$ and $x \notin E$, we have $x \notin E_{n_k}(k)$ and hence $|f_n(x) - f(x)| < \frac{1}{k}$. That is, for $n > n_k$

$$\sup_{x \in E^c} |f_n(x) - f(x)| \leq \tfrac{1}{k}$$

Thus $f_n \to f$ uniformly on E^c. $\square$

Def We say a sequence of functions $(f_n)_{n \in \mathbb{N}}$ converges <u>almost uniformly</u> to some f if for all $\varepsilon > 0$ there exists $E \subset X$ with $\mu(E) < \varepsilon$ such that $f_n \to f$ uniformly on E^c. $\square$

(Exercise show almost uniform convergence implies almost everywhere convergence and convergence in measure)

Remark The sets $E_n(k)$ in the previous proof are best understood through their complements:

$$E_n(k)^c = \{x \in X : |f_j(x) - f(x)| < \tfrac{1}{k} \ \forall j \geq n\}$$

That is, $E_n(k)^c$ is the set of points such that the sequence $(f_j(x))_{j \in \mathbb{N}}$ stays with $\frac{1}{k}$ of $f(x)$ from $j = n$ onward. Now, recall the definition of uniform convergence: $f_n \to f$ uniformly on a set A if $\forall k \in \mathbb{N}$, $\exists n_k \in \mathbb{N}$ such that $\forall n \geq n_k$

$$\sup_{x \in A} |f_n(x) - f(x)| < \tfrac{1}{k}.$$

This means $A \subset \bigcap_{k=1}^{\infty} E_{n_k}(k)^c$. In fact, for $\underline{\text{any}}$ increasing sequence $n_1 < n_2 < \cdots$, we will have $f_n \to f$ uniformly on

$$\bigcap_{k=1}^{\infty} E_{n_k}(k)^c = \left(\bigcup_{k=1}^{\infty} E_{n_k}(k) \right)^c.$$

What we accomplish in the proof of Egoroff's theorem is finding such a sequence $(n_k)_{k \in \mathbb{N}}$ so that

$$E := \bigcup_{k=1}^{\infty} E_{n_k}(k)$$

has small measure $\square$

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2.5 Product Measures

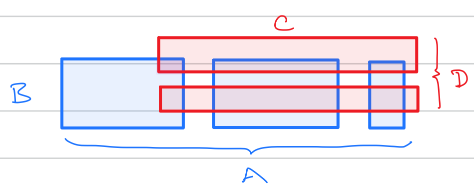
Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. In this section will build a measure on $\mathcal{M} \otimes \mathcal{N}$ from μ and ν, satisfying

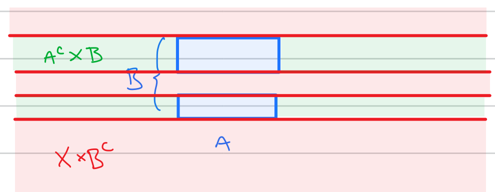
$$(\mu \times \nu)(E \times F) = \mu(E)\nu(F)$$

which is natural in the context of $X = Y = \mathbb{R}$ and E, F intervals. This will also lead us to an essential theorem about interchanging integrals.

For $A \in \mathcal{M}$, $B \in \mathcal{N}$, we will call the set $A \times B \subset X \times Y$ a (measurable) rectangle.
Observe that the collection of rectangles is an elementary family:

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \qquad (A \times B)^c = (X \times B^c) \cup (A^c \times B)$$





Thus if $\mathcal{A}$ is the collection of finite disjoint unions of rectangles, then it is an algebra by Proposition 1.12. Also note that

$$\mathcal{M}(\mathcal{A}) = \mathcal{M} \otimes \mathcal{N}$$

by Proposition 1.3. Thus we can build a measure on this σ-algebra using our techniques from Section 1.4. In fact, all that remains to do is define a premeasure on $\mathcal{A}$: for disjoint rectangles $A_1 \times B_1, \ldots, A_n \times B_n$ let

$$\pi\left(\bigcup_{j=1}^{n} A_j \times B_j\right) := \sum_{j=1}^{n} \mu(A_j)\nu(B_j).$$

To see that π is well-defined and a premeasure, we follow the same strategy we used for building Borel measures (see Proposition 1.15). It therefore suffices to show $\mu(A)\nu(B) = \sum \mu(A_i)\nu(B_i)$ when $A \times B$ is a countable disjoint union of rectangles $A_i \times B_i$, $i \in I$. For $x \in X$ and $y \in Y$, observe that

$$1_A(x) 1_B(y) = 1_{A \times B}(x,y) = \sum_{i \in I} 1_{A_i \times B_i}(x,y) = \sum_{i \in I} 1_{A_i}(x) 1_{B_i}(y)$$

Integrating with respect to x and using Theorem 2.15 gives:

$$\mu(A) 1_B(y) = \int_X 1_A(x) 1_B(y)\, d\mu(x) = \sum_{i \in I} \int_X 1_{A_i}(x) 1_{B_i}(y)\, d\mu(x) = \sum_{i \in I} \mu(A_i) 1_{B_i}(y).$$

Similarly, integrating with respect to y yields the necessary formula:

$$\mu(A)\nu(B) = \sum_{i \in I} \mu(A_i)\nu(B_i)$$

Thus Theorem 1.14 yields a measure on $\mathcal{M}(\mathcal{A}) = \mathcal{M} \otimes \mathcal{N}$ extending π:

Def The measure on $\mathcal{M} \otimes \mathcal{N}$ obtained from the above construction is called the product measure of μ and ν and is denoted $\mu\times\nu$. $\square$

If both spaces are σ-finite, say $X=\bigcup_{j=1}^{\infty}A_j$ with $\mu(A_n)<\infty$ and $Y=\bigcup_{k=1}^{\infty}B_k$ with $\nu(B_k)<\infty$, then

$$A\times B=\bigcup_{j,k=1}^{\infty}A_j\times B_k$$

and $\mu\times\nu(A_j\times B_k)=\mu(A_j)\nu(B_k)<\infty$. Thus $\mu\times\nu$ is σ-finite and is therefore the unique measure on $\mathcal{M}\otimes\mathcal{N}$ satisfying $\mu\times\nu(A\times B)=\mu(A)\nu(B)$ for all rectangles $A\times B$ (by Theorem 1.14).

The above construction works for any finite number of factors: if $(X_i,\mathcal{M}_i,\mu_i)$, $i=1,\dots,n$ are measure spaces then there exists a measure $\mu_1\times\cdots\times\mu_n$ on $\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n$ satisfying

$$\mu_1\times\cdots\times\mu_n(A_1\times\cdots\times A_n)=\mu_1(A_1)\cdots\mu_n(A_n) \qquad *$$

for all $A_i\in\mathcal{M}_i$, $i=1,\dots,n$. If each μ_i is σ-finite then so is $\mu_1\times\cdots\times\mu_n$, and moreover it is the unique measure on $\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n$ satisfying ($*$). The construction is also associative (e.g. $\mu_1\times(\mu_2\times\mu_3)=(\mu_1\times\mu_2)\times\mu_3$). However, as in Folland we will restrict ourselves in the following to the case of $n=2$ for simplicity

$\boxed{\text{Def}}$ Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces. For $E\subset X\times Y$, $x\in X$, and $y\in Y$ the x-section E_x and y-section E^y of E are the sets:

$$E_x:=\{y\in Y:(x,y)\in E\}=\pi_2(\pi_1^{-1}(\{x\})\cap E)$$
$$E^y:=\{x\in X:(x,y)\in E\}=\pi_1(\pi_2^{-1}(\{y\})\cap E)$$

For a function f on $X\times Y$, the x-section f_x and y-section f^y of f are the functions:

$$f_x(y)=f(x,y)=f^y(x)$$

$\square$

$\boxed{\text{Ex}}$ ① For $E=A\times B\subset X\times Y$, $x\in X$, and $y\in Y$

$$(A\times B)_x=\begin{cases}B & \text{if } x\in A\\ \emptyset & \text{otherwise}\end{cases} \qquad (A\times B)^y=\begin{cases}A & \text{if } y\in B\\ \emptyset & \text{otherwise.}\end{cases}$$

② Let $E\subset X\times Y$. Then

$$(1_E)_x(y)=1_E(x,y)=1_{(E_x)}(y)$$

and similarly $(1_E)^y=1_{(E^y)}$. $\square$

$\boxed{\text{Proposition 2.33}}$ Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces.

① If $E\in\mathcal{M}\otimes\mathcal{N}$, then $E_x\in\mathcal{N}$ for all $x\in X$ and $E^y\in\mathcal{M}$ for all $y\in Y$.

② If f is $(\mathcal{M}\otimes\mathcal{N})$-measurable, then f_x is $\mathcal{N}$-measurable for all $x\in X$ and f^y is $\mathcal{M}$-measurable for all $y\in Y$.

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Proof ①: Let $\mathcal{R}=\{E\subset X\times Y: E_x\in\mathcal{N}\ \forall x\in X \text{ and } E^y\in\mathcal{M}\ \forall y\in Y\}$. Then $\mathcal{R}$ contains all measurable rectangles by Example ① above. Thus if we show $\mathcal{R}$ is a σ-algebra, then by Lemma 1.1 it will contain the σ-algebra generated by all rectangles, which is $\mathcal{M}\otimes\mathcal{N}$ by Proposition 1.3. For $E\in\mathcal{R}$ and $x\in X$ we have

$$(E^c)_x = \{y\in Y : (x,y)\in E^c\} = \{y\in Y : (x,y)\in E\}^c = (E_x)^c \in \mathcal{N}$$

Similarly $(E^c)^y = (E^y)^c\in\mathcal{N}$ for all $y\in Y$. Hence $E^c\in\mathcal{R}$. For $E_n\in\mathcal{R}$, $n\in\mathbb{N}$, and $x\in X$ we have

$$\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \{y\in Y : (x,y)\in \bigcup_{n=1}^{\infty} E_n\} = \bigcup_{n=1}^{\infty}\{y\in Y : (x,y)\in E_n\} = \bigcup_{n=1}^{\infty}(E_n)_x\in\mathcal{N}.$$

Similarly for the y-sections, and so $\bigcup E_n\in\mathcal{R}$. Hence $\mathcal{R}$ is a σ-algebra.

②: For a Borel set B, we have:

$$f_x^{-1}(B) = \{y\in Y : f(x,y)\in B\} = \{y\in Y : (x,y)\in f^{-1}(B)\} = (f^{-1}(B))_x\in\mathcal{N}$$

by ①. Hence f_x is $\mathcal{N}$-measurable. Similarly, $f_y^{-1}(B) = (f^{-1}(B))^y\in\mathcal{M}$, and so f^y is $\mathcal{M}$-measurable. □

Before proceeding to the main results of this section, we need some new terminology and and additional characterization of σ-algebras.

Def Let X be a set. A monotone class on X is a collection of subsets $\mathcal{C}\subset\mathcal{P}(X)$ satisfying:

① For $E_1\subset E_2\subset\cdots$ with $E_n\in\mathcal{C}$ for all $n\in\mathbb{N}$, $\bigcup_{n=1}^{\infty}E_n\in\mathcal{C}$.

② For $E_1\supset E_2\supset\cdots$ with $E_n\in\mathcal{C}$ for all $n\in\mathbb{N}$, $\bigcap_{n=1}^{\infty}E_n\in\mathcal{C}$. □

$\mathcal{P}(X)$ is trivially always a monotone class. Also if $\{\mathcal{C}_i : i\in I\}$ is a family of monotone classes on X then $\bigcap_{i\in I}\mathcal{C}_i$ is a monotone class on X. Consequently for any $\mathcal{E}\subset\mathcal{P}(X)$ there is a smallest monotone class containing $\mathcal{E}$ formed by taking the intersection of all monotone classes containing $\mathcal{E}$. We call this the monotone class generated by $\mathcal{E}$.

Any σ-algebra $\mathcal{M}$ is a monotone class since it is closed under countable unions and intersections. The converse is not true in general, but it does hold when the monotone class is generated by an algebra:

Lemma 2.34 (The Monotone Class Lemma) For an algebra $\mathcal{A}$ of subsets of X, the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ equals the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$.

Proof Since $\mathcal{M}$ is a monotone class containing $\mathcal{A}$, we have $\mathcal{C}\subset\mathcal{M}$. It then suffices to show $\mathcal{C}$ is a σ-algebra. For $E\in\mathcal{C}$, define

$$\mathcal{C}(E) := \{F\in\mathcal{C} : E\setminus F,\ F\setminus E,\ E\cap F\in\mathcal{C}\}.$$

For $E\in\mathcal{A}$, we have $\mathcal{A}\subset\mathcal{C}(E)$ because $\mathcal{A}\subset\mathcal{C}$ is an algebra. Since $\mathcal{C}(E)$ is a monotone class (Exercise check this), it follows that $\mathcal{C}\subset\mathcal{C}(E)$. Thus for all $F\in\mathcal{C}$ we have $F\in\mathcal{C}(E)$, but this implies $E\in\mathcal{C}(F)$ by the symmetry of the definition. Since this held for any $E\in\mathcal{A}$, we see that $\mathcal{A}\subset\mathcal{C}(F)$ for all $F\in\mathcal{C}$, and hence $\mathcal{C}\subset\mathcal{C}(F)$. So for any $E,F\in\mathcal{C}$ we have $E\in\mathcal{C}(F)$ and hence $E\setminus F, F\setminus E, E\cap F\in\mathcal{C}$. In particular, $\mathcal{C}$ is closed under finite intersections, and since $X\in\mathcal{A}\subset\mathcal{C}$ it is also closed under complements.

Hence $\mathcal{C}$ is an algebra. But this coupled with being closed under increasing unions means $\mathcal{C}$ is a σ-algebra: for $E_n \in \mathcal{C}$, $n \in \mathbb{N}$,
$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{N=1}^{\infty} (E_1 \cup \cdots \cup E_N) \in \mathcal{C}. \qquad \square$$

Theorem 2.35 Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. For $E \in \mathcal{M} \otimes \mathcal{N}$, the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are $\mathcal{M}$ and $\mathcal{N}$-measurable, respectively, and
$$\mu \times \nu(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y).$$

Proof First suppose μ and ν are finite. If $E = A \times B$ is a measurable rectangle, then by Example 1 above, we have $\nu(E_x) = 1_A(x)\nu(B)$ and $\mu(E^y) = 1_B(y)\mu(A)$. Thus
$$\int_X \nu(E_x)\, d\mu(x) = \mu(A)\nu(B) = (\mu \times \nu)(E)$$
and similarly for $\mu(E^y)$. Let $\mathcal{C}$ be the collection of all $E \in \mathcal{M} \otimes \mathcal{N}$ for which the conclusion of the theorem holds. We just showed $\mathcal{C}$ contains all rectangles, and by finite additivity it contains the algebra $\mathcal{A}$ of all finite disjoint unions of rectangles.
If we show $\mathcal{C}$ is a monotone class, then it contains the monotone class generated by $\mathcal{A}$, which is $\mathcal{M}(\mathcal{A}) = \mathcal{M} \otimes \mathcal{N}$ by the monotone class lemma (Lemma 2.34).
Let $E_1 \supset E_2 \supset \cdots$ with $E_n \in \mathcal{C}$ for all $n \in \mathbb{N}$, and denote $E := \bigcap_{n=1}^{\infty} E_n$. Then the functions $f_n(x) := \nu((E_n)_x)$ are measurable and decrease to $f(x) := \nu(E_x)$. Thus f is measurable by Proposition 2.7. Since $\nu((E_n)_x) \leq \nu(Y) \equiv g$, the dominated convergence theorem and continuity from above imply
$$\int_X \nu(E_x)\, d\mu(x) = \lim_{n\to\infty} \int_X \nu((E_n)_x)\, d\mu(x) = \lim_{n\to\infty} \mu \times \nu(E_n) = \mu \times \nu(E).$$

Similarly for $\mu(E^y)$. Hence $E \in \mathcal{C}$, and so $\mathcal{C}$ is closed under countable decreasing intersections. Showing that $\mathcal{C}$ is closed under countable increasing unions is similar. Thus the theorem holds for all $E \in \mathcal{M} \otimes \mathcal{N}$ when μ and ν are finite.
Now suppose μ and ν are σ-finite. Then $X = \bigcup_{n=1}^{\infty} X_n$ and $Y = \bigcup_{n=1}^{\infty} Y_n$ with $X_1 \subset X_2 \subset \cdots$, $Y_1 \subset Y_2 \subset \cdots$, and $\mu(X_n), \nu(Y_n) < \infty$ for all $n \in \mathbb{N}$. Consequently
$$X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n)$$
and $\mu \times \nu(X_n \times Y_n) = \mu(X_n)\nu(Y_n) < \infty$. For $E \in \mathcal{M} \otimes \mathcal{N}$, the first part of the proof implies the theorem holds for $E \cap (X_n \times Y_n)$. So by the monotone convergence theorem and continuity from below, we have
$$\int_X \nu(E_x)\, d\mu(x) = \lim_{n\to\infty} \int_X \nu([E \cap (X_n \times Y_n)]_x)\, d\mu(x) = \lim_{n\to\infty} \mu \times \nu(E \cap (X_n \times Y_n)) = \mu \times \nu(E).$$

Similarly for $\mu(E^y)$. $\square$

Theorem 2.36 (The Fubini–Tonelli Theorem) Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

1. (Tonelli) For $f \in L^+(X\times Y, \mu\times\nu)$, we have $\int f_x \, d\nu \in L^+(X,\mu)$, $\int f^y \, d\mu \in L^+(Y,\nu)$, and

$$\int_{X\times Y} f \, d(\mu\times\nu) = \int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x) = \int_Y \int_X f(x,y) \, d\mu(x) \, d\nu(y) \quad {**}$$

2. (Fubini) If $f \in L^1(X\times Y, \mu\times\nu)$, then $f_x \in L^1(Y,\nu)$ for μ-a.e. $x \in Y$, and $f^y \in L^1(X,\mu)$ for ν-a.e. $y \in Y$. Moreover, $\int f_x \, d\nu \in L^1(X,\mu)$, $\int f^y \, d\mu \in L^1(Y,\nu)$, and (**) holds.

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Proof 1: If $f = 1_E$ for $E \in \mathcal{M}\otimes\mathcal{N}$, then Tonelli's theorem follows from Theorem 2.35. By linearity it also holds for simple functions. For general $f \in L^+(X\times Y, \mu\times\nu)$, we use Theorem 2.10 to obtain a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions that increases pointwise to f. Then $\int (f_n)_x \, d\nu \nearrow \int f_x \, d\nu$ and $\int (f_n)^y \, d\mu \nearrow \int f^y \, d\mu$, and so by the monotone convergence theorem

$$\int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x) = \int_X \int_Y f_x \, d\nu \, d\mu$$
$$= \lim_{n\to\infty} \int_X \int_Y (f_n)_x \, d\nu \, d\mu$$
$$= \lim_{n\to\infty} \int_{X\times Y} f_n \, d(\mu\times\nu) = \int_{X\times Y} f \, d(\mu\times\nu).$$

Similarly for integration in the other order.

2: If $g \in L^+(X\times Y, \mu\times\nu)$ with $\int g \, d(\mu\times\nu) < \infty$, then 1 implies $\int g_x \, d\nu \in L^1(X,\mu)$ and and $\int g^y \, d\mu \in L^1(Y,\nu)$. So by Proposition 2.20 we have $\int g_x \, d\nu < \infty$ for μ-a.e. $x \in X$, or in other words $g_x \in L^1(Y,\nu)$ for μ-a.e. $x \in X$. Similarly $g^y \in L^1(X,\mu)$ for ν-a.e. $y \in Y$. Applying this to $g \in \{\mathrm{Re}(f)^{\pm}, \mathrm{Im}(f)^{\pm}\}$ yields Fubini's theorem. □

Remark Typically one uses both parts together: one shows $f \in L^1(X\times Y, \mu\times\nu)$ by using Tonelli's theorem to compute $\int |f| \, d(\mu\times\nu)$ as an iterated integral, and then one uses Fubini's theorem to compute $\int f \, d(\mu\times\nu)$ as an iterated integral. □

Ex We show that $f \in L^1(X\times Y, \mu\times\nu)$ is necessary in Fubini's theorem. Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, and $\mu = \nu = \#$ the counting measure. Consider

$$f(m,n) = \begin{cases} 1 & \text{if } m=n \\ -1 & \text{if } m=n+1 \\ 0 & \text{otherwise} \end{cases}$$

Then $f \notin L^1(X\times Y, \mu\times\nu)$ since $|f| = 1$ on the infinite measure set $\{(n,n) : n \in \mathbb{N}\}$. However

$$\int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n) \, d\mu(m) \, d\nu(n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} f(n,n) + f(n+1,n) = 0$$

while

$$\int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n) \, d\nu(n) \, d\mu(m) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = f(1,1) + \sum_{m=2}^{\infty} f(m,m) + f(m,m-1) = 1$$

□

Note that a product measure $\mu\times\nu$ is usually not complete, even if μ and ν are complete. Indeed, suppose there exists $A\in\mathcal{M}\setminus\{\emptyset\}$ with $\mu(A)=0$ and $E\in\mathcal{P}(Y)\setminus\mathcal{N}$. (This occurs for $\mu=\nu=m$ the Lebesgue measure.) Then Proposition 2.37, $A\times E\notin\mathcal{M}\otimes\mathcal{N}$ since $(A\times E)_x=E\notin\mathcal{N}$ for any $x\in A$. However, $A\times E\subset A\times Y$ and $\mu\times\nu(A\times Y)=0$, so $\mu\times\nu$ is not complete. However, one can always consider the completion $\overline{\mu\times\nu}$ of $\mu\times\nu$, and it turns out the Fubini-Tonelli theorem extends (though Proposition 2.37 does not):

$\boxed{\text{Theorem 2.37}}$ (The Fubini-Tonelli Theorem for Completed Measures) Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be complete, σ-finite measure spaces, and let $(X\times Y,\mathcal{L},\lambda)$ be the completion of $(X\times Y,\mathcal{M}\otimes\mathcal{N},\mu\times\nu)$.

1. If $f\in L^+(X\times Y,\lambda)$, then $f_x\in L^+(Y,\nu)$ for μ-a.e. $x\in X$ and $f^y\in L^+(X,\mu)$ for ν-a.e. $y\in Y$. Moreover $\int f_x\,d\nu\in L^+(X,\mu)$ and $\int f^y\,d\mu\in L^+(Y,\nu)$ and
$$\int_{X\times Y} f\,d\lambda=\int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x)=\int_Y\int_X f(x,y)\,d\mu(x)\,d\nu(y)\qquad (***)$$
2. If $f\in L^1(X\times Y,\lambda)$, then $f_x\in L^1(Y,\nu)$ for μ-a.e. $x\in X$ and $f^y\in L^1(X,\mu)$ for ν-a.e. $y\in Y$. Moreover $\int f_x\,d\nu\in L^1(X,\mu)$ and $\int f^y\,d\mu\in L^1(Y,\nu)$ and $(***)$ holds.

The proof is outlined in Folland Exercise 2.49

The n-dimensional Lebesgue measure (video recording)

$\boxed{\text{Def}}$ For $n\in\mathbb{N}$, one defines $(\mathbb{R}^n,\mathcal{L}^n,m^n)$ as the completion of $(\mathbb{R}^n,\underbrace{\mathcal{L}\otimes\cdots\otimes\mathcal{L}}_{n\text{-times}},\underbrace{m\times\cdots\times m}_{n\text{ times}})$, and calls m^n the Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{L}^n$ the the Lebesgue measurable sets in $\mathbb{R}^n$. □

Observe that $\mathcal{B}_{\mathbb{R}^n}=\mathcal{B}_{\mathbb{R}}\otimes\cdots\otimes\mathcal{B}_{\mathbb{R}}\subset\mathcal{L}\otimes\cdots\otimes\mathcal{L}\subset\mathcal{L}^n$, and in fact one can show $(\mathcal{L}^n,m^n)$ is also the completion of $(\mathcal{B}_{\mathbb{R}^n},m\times\cdots\times m)$. We also note that it is common to write m for m^n when n is clear from context.

For $E=E_1\times\cdots\times E_n\subset\mathbb{R}^n$, we will call E a <u>rectangle</u> and E_j a <u>side</u> of E for $j=1,\dots,n$.

$\boxed{\text{Theorem 2.38}}$ Let $E\in\mathcal{L}^n$.

1. $m(E)=\inf\{m(U): U\supset E \text{ open}\}=\sup\{m(K): K\subset E \text{ compact}\}$
2. $E=V\setminus N_1=H\cup N_2$ for V a G_δ set, H an F_σ set, and $m(N_1)=m(N_2)=0$.
3. If $m(E)<\infty$, then for any $\varepsilon>0$ there is a finite collection $\{R_1,\dots,R_N\}$ of disjoint rectangles whose sides are intervals such that $m(E\,\Delta\,(R_1\cup\cdots\cup R_N))<\varepsilon$.

<u>Proof</u> ①: By definition of the product measure (and its completion), for $\varepsilon>0$ there exists a countable family $(T_n)_{n\in\mathbb{N}}$ of rectangles such that $E\subset\bigcup_{n=1}^\infty T_n$ and $\sum_{n=1}^\infty m(T_n)\le m(E)+\varepsilon$.

For each $n \in \mathbb{N}$, we apply Theorem 1.18 to the sides of T_n to find $U_n \supset T_n$ whose sides are open (and so U_n is open) and $m(U_n) < m(T_n) + \frac{\varepsilon}{2^n}$. Then

$$U := \bigcup_{n=1}^{\infty} U_n$$

is an open set containing E with $m(U) < m(E) + 2\varepsilon$. This implies the first equality. The second equality follows by the same argument as in the proof of Theorem 1.18.

2: Using 1, this follows by the same argument as in the proof of Theorem 1.19.

3: Let U_n, $n \in \mathbb{N}$ be as in part 1. Then $m(E) < \infty$ implies $m(U_n) < \infty$ for all $n \in \mathbb{N}$. Since the sides of U_n are open, they are a countable union of open intervals. By considering a sufficiently large finite union for each side, we can find $V_n \subset U_n$ whose sides are finite unions of intervals and $m(V_n) \geq m(U_n) - \frac{\varepsilon}{2^n}$. Then for sufficiently large $N \in \mathbb{N}$ we have

$$m\left(E \setminus \bigcup_{n=1}^{N} V_n\right) \leq m\left(\bigcup_{n=1}^{N} (U_n \setminus V_n)\right) + m\left(\bigcup_{n=N+1}^{\infty} U_n\right) < 2\varepsilon,$$

and

$$m\left(\bigcup_{n=1}^{N} V_n \setminus E\right) \leq m\left(\bigcup_{n=1}^{\infty} U_n \setminus E\right) < 2\varepsilon,$$

so that $m(E \,\Delta\, \bigcup_{n=1}^{N} V_n) < 4\varepsilon$. We then let $\bigcup_{n=1}^{N} V_n = R_1 \cup \cdots \cup R_M$ be a decomposition into disjoint rectangles whose sides are intervals. □

Theorem 2.39 Let $f \in L^1(\mathbb{R}^n, m)$. For all $\varepsilon > 0$ there exists a simple function ϕ with standard representation $\phi = \sum_{j=1}^{l} \alpha_j 1_{R_j}$, where each R_j is a product of intervals, such that

$$\int_{\mathbb{R}^n} |f - \phi| \, dm < \varepsilon.$$

Furthermore, there exists a continuous function $g : \mathbb{R}^n \to \mathbb{C}$ that vanishes outside a bounded set such that

$$\int_{\mathbb{R}^n} |f - g| \, dm < \varepsilon.$$

Proof First approximate f by simple functions using Theorem 2.10. Then use Theorem 2.38.3 to approximate the indicator function 1_E by sums of indicator functions of products of intervals. Proceeding as in the proof of Theorem 2.26, one approximates 1_{R_j} by a continuous function of the form $g(x_1, \ldots, x_n) = g_1(x_1) \cdots g_n(x_n)$. □

In the following theorem for $E \subset \mathbb{R}^n$ and $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ we denote

$$E + t = \{(x_1 + t_1, \ldots, x_n + t_n) : (x_1, \ldots, x_n) \in E\} \quad \text{and} \quad Et = \{(x_1 t_1, \ldots, x_n t_n) : (x_1, \ldots, x_n) \in E\}$$

and call the sets translations and dilations, respectively.

Theorem 2.40 If $E \in \mathcal{L}^n$, then $E + t, Et \in \mathcal{L}^n$ for all $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ with

$$m(E + t) = m(E) \quad \text{and} \quad m(Et) = |t_1 \cdots t_n| \, m(E).$$

Proof Using Exercise 1 on Homework 2, one argues exactly as in the proof of Theorem 1.21 (using Theorem 2.38.1 in place of Theorem 1.18). □

Ex ① For $x \in \mathbb{R}^n$ and $r > 0$, recall $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. There exists $0 < c < \infty$ (independent of x and r, but not n) so that

$$m(B(x,r)) = cr^n$$

Indeed, let $c := m(B((0,\ldots,0),1))$. Then

$$(-\tfrac{1}{\sqrt{n}}, \tfrac{1}{\sqrt{n}})^n \subset B((0,\ldots,0),1) \subset (-1,1)^n$$

so that $(\frac{2}{\sqrt{n}})^n \leq c \leq 2^n$. Since $B(x,r) = [B((0,\ldots,0),1)\cdot(r,\ldots,r)] + x$, Theorem 2.40 yields $m(B(x,r)) = cr^n$.

② For $x \in \mathbb{R}^n$ and $r > 0$, denote $S(x,r) := \{y \in \mathbb{R}^n : |x-y| = r\}$. Then

$$m(S(x,r)) = 0$$

Indeed, note that

$$\bigcup_{k=2}^{\infty} S(x, \tfrac{r}{k^{1/n}}) \subset B(x,r)$$

and the union is disjoint. So by Theorem 2.40

$$\infty > m(B(x,r)) \geq \sum_{k=2}^{\infty} m(S(x,\tfrac{r}{k^{1/n}})) = \sum_{k=2}^{\infty} \frac{1}{k} m(S(x,r)).$$

Since $\sum_{k=2}^{\infty} \frac{1}{k} = \infty$, we must have $m(S(x,r)) = 0$.
Note that

$$\overline{B(x,r)} = B(x,r) \cup S(x,r),$$

and so $m(\overline{B(x,r)}) = m(B(x,r)) = cr^n$ for c as in ① $\square$

The behavior of m under dilations is a special case of the following theorem, which we will not prove (but one can find a proof in Section 2.6 of Folland). Recall that

$$GL(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A \text{ is invertible}\},$$

which we view as linear transformations $A : \mathbb{R}^n \to \mathbb{R}^n$.

Theorem 2.41 If $E \in \mathcal{L}^n$, then $A(E) \in \mathcal{L}^n$ for all $A \in GL(n,\mathbb{R})$ with

$$m(A(E)) = |\det(A)|\, m(E).$$

If f is $\mathcal{L}^n$-measurable, then so is $f \circ A$, and if $f \geq 0$ or $f \in L^1(\mathbb{R}^n, m)$ then

$$\int_{\mathbb{R}^n} f\, dm = |\det(A)| \int_{\mathbb{R}^n} f \circ A\, dm$$

Note that for $t_1, \ldots, t_n \in \mathbb{R}\setminus\{0\}$,

$$A = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \in GL(n,\mathbb{R})$$

corresponds to dilation by $T = (t_1, \ldots, t_n)$, and $|\det(A)| = |t_1 \cdots t_n|$.