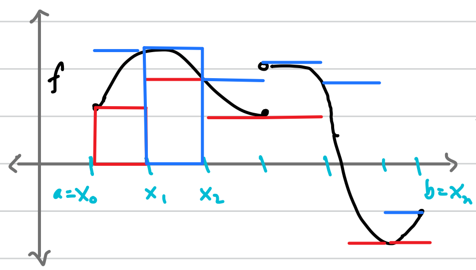
# 1.1 Motivation

Recall that the approach to defining / computing the Riemann integral was to partition the domain and then use max / min of the function to approximate the integral by rectangles:




$$\rightsquigarrow \sum_{j=1}^{n} (x_j - x_{j-1}) \cdot \min_{x_{j-1} \le t \le x_j} f(t) \le \int_a^b f(t)\,dt \le \sum_{j=1}^{n} (x_j - x_{j-1}) \max_{x_{j-1} \le t \le x_j} f(t)$$

This method typically relies on the continuity of $f$ to guarantee that the min's and max's are close, provided the domain is partitioned into narrow enough intervals. Actually, the Riemann integral can handle a surprising amount of discontinuity:

**Thm (Riemann-Lebesgue Theorem)** For $f:\mathbb{R}\to\mathbb{R}$ bounded, $f$ is Riemann integrable if and only if its set of discontinuities $D$ is a null set: for all $\varepsilon > 0$ there exists a countable collection $\{(a_i,b_i)\}_{i\in I}$ of open intervals satisfying

$$D \subset \bigcup_{i\in I} (a_i,b_i) \qquad \text{and} \qquad \sum_{i\in I} b_i - a_i \le \varepsilon.$$

In particular, this theorem implies a bounded function with even a countably infinite number of discontinuities is still Riemann integrable. For example, the "snowfall function"

$$f(t) = \begin{cases} \frac{1}{n} & \text{if } t = \frac{m}{n} \text{ with } m\in\mathbb{Z}, n\in\mathbb{N} \text{ having no common factors} \\ 0 & \text{if } t\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$$

has a discontinuity set of $\mathbb{Q}$, which is countable, and so is Riemann integrable with

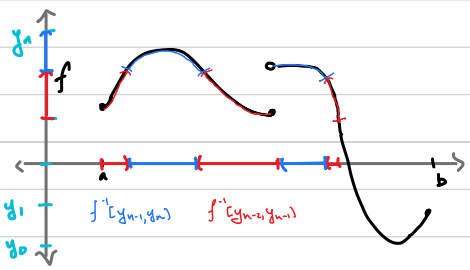
$$\int_a^b f(t)\,dt = 0$$

for all $a<b$. On the other hand, the much "simpler" function

$$1_{\mathbb{Q}}(t) = \begin{cases} 1 & \text{if } t\in\mathbb{Q} \\ 0 & \text{if } t\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$$

is discontinuous everywhere and so is not Riemann integrable.

In order to define an integral for functions of the above form, we need a method that does not rely on continuity. In light of how continuity is used to define the Riemann integral, the most direct way to get around it is to partition the range rather than the domain:




$$\rightsquigarrow \sum_{j=1}^{n} \text{"length}(f^{-1}[y_{j-1},y_j))\text{"} \cdot y_{j-1} \le \int_a^b f(t)\,dt \le \sum_{n\in\mathbb{Z}} \text{"length}(f^{-1}[y_{j-1},y_j))\text{"} \cdot y_j$$

Thus if we know how to assign a "length" to each $f^{-1}[y_{j-1},y_j]$, then we can approximate the integral by partioning the range into narrow intervals, and this does not rely on continuity. Which subsets $E\subset\mathbb{R}$ we can assign a length $\ell(E)$ to is then the only thing limiting which functions we can integrate. If we could assign a length to all subsets of $\mathbb{R}$, then we can integrate any bounded function.

Unfortunately, this is not possible if we want our length function to satisfy the following natural properties:

1. For a finite or infinite sequence of disjoint subsets $(E_n)_{n\in\mathbb{N}}$ of $\mathbb{R}$
$$\ell(E_1\cup E_2\cup\cdots) = \ell(E_1)+\ell(E_2)+\cdots .$$

2. For $E\subset\mathbb{R}$ and $t\in\mathbb{R}$, $\ell(E+t)=\ell(E)$.

3. $\ell([0,1))=1$.

Indeed, consider the equivalence relation on $\mathbb{R}$ defined by: $x\sim y$ iff $x-y\in\mathbb{Q}$. Using the axiom of choice let $N\subset[0,1)$ consist of one representative from each equivalence class. For $q\in\mathbb{Q}\cap[0,1)$ define
$$N_q := \{x+q : x\in N\cap[0,1-q)\}\cup\{x+q-1 : x\in N\cap[1-q,1)\}$$
That is, $N_q$ is $N+q$ but with the part to the right of 1 shifted back by 1. In particular, using 1 and 2 we have
$$\begin{aligned}\ell(N_q) &= \ell(\{x+q : x\in N\cap[0,1-q)\})+\ell(\{x+q-1 : x\in N\cap[1-q,1)\}) \quad * \\ &= \ell(\{x+q : x\in N\cap[0,1-q)\})+\ell(\{x+q : x\in N\cap[1-q,1)\}) = \ell(N+q)=\ell(N)\end{aligned}$$

Now, $N_q\subset[0,1)$ and
$$[0,1) = \bigsqcup_{q\in\mathbb{Q}\cap[0,1)} N_q$$
Indeed, for $x\in[0,1)$ let $y\in[x]\cap N$. Then $x\in N_q$ where
$$q=\begin{cases} x-y & \text{if } x\ge y\\ x-y+1 & \text{if } x<y\end{cases}$$
If also $x\in N_p$ for some $p\ne q$ then $x-q$ (or $x-q+1$) and $x-p$ (or $x-p+1$) are both in $N$ but are distinct representatives of $[x]$, contradicting the definition of $N$. Now, since $\mathbb{Q}\cap[0,1)$ is countable, we have
$$1 \overset{3}{=} \ell([0,1)) \overset{1}{=} \sum_{q\in\mathbb{Q}\cap[0,1)} \ell(N_q) \overset{*}{=} \sum_{q\in\mathbb{Q}\cap[0,1)} \ell(N)$$

If $\ell(N)=0$, then we obtain the contradiction $1=0$. If $\ell(N)\neq 0$, we obtain the contradiction $1=\infty$.

One might hope to remedy the situation by reducing 1 to only allow finite sums. But the Banach-Tarski paradox tells us there is still trouble even in this case. Moreover, allowing countable sums is where the analysis comes into the picture. Instead, one must restrict the domain of the length function $\ell$ to exclude pathological sets like $N$ above. As discussed above, this will reduce the class of functions we can integrate, but it will still vastly eclipse the class of Riemann integrable functions. Additionally, the theory can be developed in great generality with little additional effort, so we will consider arbitrary sets $X$ instead of just $\mathbb{R}$.

# 1.2 σ-Algebras

**Def** Let $X$ be a nonempty set. A <u>σ-algebra</u> of sets on $X$ is a nonempty collection $\mathcal{M}$ of subsets of $X$ such that:

① $E^c \in \mathcal{M}$ for all $E \in \mathcal{M}$.

② If $E_n \in \mathcal{M}$ for $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$. □

Observe that if $E_n \in \mathcal{M}$, $n \in \mathbb{N}$, then

$$\bigcap_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} E_n^c\right)^c \in \mathcal{M}.$$

Hence a σ-algebra is closed under countable intersections as well. Furthermore, for $E \in \mathcal{M}$ we have

$$X = E \cup E^c \in \mathcal{M} \quad \text{and} \quad \phi = E \cap E^c \in \mathcal{M}.$$

So a σ-algebra always contains $X$ and $\phi$.

**Ex** ① For any nonempty set $X$, the collections $\{\phi, X\}$ and $P(X)$ (the <u>power set</u> of $X$, i.e. the collection of <u>all</u> subsets of $X$) are σ-algebras.

② Let $X$ be an uncountable set. Then

$$\mathcal{C} := \{E \subset X : E \text{ or } E^c \text{ is countable}\}$$

is a σ-algebra on $X$. <u>Exercise</u> check this.

③ For a set $X$, the collection

$$\mathcal{F} := \{E \subset X : E \text{ is finite}\} \cup \{X\}$$

is a σ-algebra on $X$ if and only if $X$ is finite. Indeed, if $X$ is finite then $\mathcal{F} = P(X)$. If $X$ is infinite, then for any $x_0 \in X$ we have $\{x\} \in \mathcal{F}$ but $\{x\}^c \notin \mathcal{F}$. Thus $\mathcal{F}$ fails to satisfy ① and is therefore <u>not</u> a σ-algebra.

④ Let $\{\mathcal{M}_i : i \in I\}$ be a nonempty family of σ-algebras on a common set $\mathcal{M}$. Then

$$\mathcal{M} := \bigcap_{i \in I} \mathcal{M}_i$$

is a σ-algebra. <u>Exercise</u> check this □

Using examples ① and ④, we can construct <u>many</u> new examples of σ-algebras:

**Def** Let $X$ be a set and let $\mathcal{E}$ be a collection of subsets of $X$. The

$\sigma$-algebra generated by $\mathcal{E}$, denoted $\mathcal{M}(\mathcal{E})$, is the intersection of all $\sigma$-algebras which contain $\mathcal{E}$. $\square$

Example ① tells us there is at least one $\sigma$-algebra containing $\mathcal{E}$, namely $\mathcal{P}(X)$, and then Example ④ implies $\mathcal{M}(\mathcal{E})$ is indeed a $\sigma$-algebra.

**Lemma 1.1** For $\mathcal{M}$ a $\sigma$-algebra on a set $X$ and $\mathcal{E} \subset \mathcal{P}(X)$, if $\mathcal{E} \subset \mathcal{M}$ then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}$.

*Proof* Since $\mathcal{M}$ is a $\sigma$-algebra containing $\mathcal{E}$, it also contains $\mathcal{M}(\mathcal{E})$ — the intersection of all such $\sigma$-algebras. $\square$

**Def** Let $X$ be a topological space (e.g. a metric space). The $\sigma$-algebra on $X$ generated by the collection of all open sets is called the Borel $\sigma$-algebra on $X$ and is denoted $\mathcal{B}_X$. We call $E \in \mathcal{B}_X$ a Borel set. $\square$

Thus all open sets are Borel sets. Since $\mathcal{B}_X$ is closed under complements, we see that all closed sets are Borel sets as well. (In fact, $\mathcal{B}_X$ is equal to the $\sigma$-algebra generated by all closed sets.) However, $\mathcal{B}_X$ is much larger than just open and closed sets: it contains all

- countable intersections of open sets, called $G_\delta$ sets
- countable unions of closed sets, called $F_\sigma$ sets
- countable unions of $G_\delta$ sets, called $G_{\delta\sigma}$ sets
- countable intersections of $F_\sigma$ sets, called $F_{\sigma\delta}$ sets

and so on.

**Rem** The symbols "$\delta$" and "$\sigma$" above stand for the German words Durchschitt (intersection) and Summe (union) $\square$

Of particular interest to us is the Borel $\sigma$-algebra on $\mathbb{R}$, which is a metric space with metric $d(x,y) = |x-y|$. It can be generated in a number of ways besides via open or closed sets:

**Proposition 1.2** $\mathcal{B}_{\mathbb{R}}$ is generated by each of the following collections:

① The open intervals $\mathcal{E}_1 = \{(a,b) : a,b \in \mathbb{R},\ a<b\}$,

② the closed intervals $\mathcal{E}_2 = \{[a,b] : a,b \in \mathbb{R},\ a<b\}$,

③ the half-open intervals $\mathcal{E}_3 = \{(a,b] : a,b \in \mathbb{R}, a<b\}$ or $\mathcal{E}_4 = \{[a,b) : a,b \in \mathbb{R}, a<b\}$,

④ the open rays $\mathcal{E}_5 = \{(a,\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty,a) : a \in \mathbb{R}\}$

⑤ the closed rays $\mathcal{E}_7 = \{[a,\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty,a] : a \in \mathbb{R}\}$

You will prove parts of this proposition on Homework 1, and the rest are left as exercises.

Let $\{X_i\}_{i\in I}$ be an indexed collection of non-empty sets. Denote their Cartesian product by
$$\prod_{i\in I} X_i = \{f : I \to \bigcup_{i\in I} X_i : f(i)\in X_i \text{ for all } i\in I\},$$
and for each $i\in I$, denote the ith coordinate map by
$$\begin{aligned}\pi_i : \prod_{j\in I} X_j &\to X_i\\ f &\mapsto f(i).\end{aligned}$$

**Def** For each $i\in I$, let $\mathcal{M}_i$ be a $\sigma$-algebra on $X_i$. The $\sigma$-algebra generated by
$$\{\pi_i^{-1}(E) : E\in\mathcal{M}_i,\ i\in I\}$$
is called a <u>product $\sigma$-algebra</u> on $\prod_{i\in I} X_i$, and is denoted $\bigotimes_{i\in I}\mathcal{M}_i$. If $I=\{1,\dots,n\}$, we also denote this $\sigma$-algebra by $\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n$. $\square$

The following propositions offer potentially more convenient characterizations of product $\sigma$-algebras.

**Proposition 1.3** If $I$ is countable, then $\bigotimes_{i\in I}\mathcal{M}_i$ is the $\sigma$-algebra generated by the collection
$$\{\prod_{i\in I} E_i : E_i\in\mathcal{M}_i,\ i\in I\}$$

**Proof** Denote the $\sigma$-algebra generated by this collection by $\mathcal{M}$. Observe that
$$\pi_i^{-1}(E) = \prod_{j\in I} E_j \in\mathcal{M}$$
where $E_j = E$ if $j=i$ and otherwise $E_j = X_j$. Thus $\bigotimes_{i\in I}\mathcal{M}_i\subset\mathcal{M}$ by Lemma 1.1. Conversely, if $E_i\in\mathcal{M}_i$ for each $i\in I$, then
$$\prod_{i\in I} E_i = \bigcap_{i\in I}\pi_i^{-1}(E_i).$$
This is a countable intersection (since $I$ is countable) and hence belongs to $\bigotimes_{i\in I}\mathcal{M}_i$. Therefore $\mathcal{M}\subset\bigotimes_{i\in I}\mathcal{M}_i$ by Lemma 1.1 again. $\square$

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**Proposition 1.4** For each $i\in I$, let $\mathcal{E}_i\subset\mathcal{P}(X_i)$. Then
$$\bigotimes_{i\in I}\mathcal{M}(\mathcal{E}_i) = \mathcal{M}(\{\pi_i^{-1}(E) : E\in\mathcal{E}_i,\ i\in I\})$$
If $I$ is countable and $X_i\in\mathcal{E}_i$ for all $i\in I$, then
$$\bigotimes_{i\in I}\mathcal{M}(\mathcal{E}_i) = \mathcal{M}(\{\prod_{i\in I} E_i : E_i\in\mathcal{E}_i,\ i\in I\})$$

**Proof** Denote
$$\mathcal{F} := \{\pi_i^{-1}(E) : E\in\mathcal{E}_i,\ i\in I\}.$$
Then $\mathcal{F}\subset\bigotimes_{i\in I}\mathcal{M}(\mathcal{E}_i)$ by definition of the product $\sigma$-algebra, and so $\mathcal{M}(\mathcal{F})\subset\bigotimes_{i\in I}\mathcal{M}(\mathcal{E}_i)$ by Lemma 1.1. Conversely, for each $i\in I$ note that
$$\mathcal{M}_i := \{E\subset X_i : \pi_i^{-1}(E)\in\mathcal{M}(\mathcal{F})\}$$

is a $\sigma$-algebra since $\pi_i^{-1}$ commutes with unions and complements and $\mathcal{M}(\mathcal{F})$ is closed under these operations. Also $\mathcal{E}_i \subset \mathcal{M}_i$ by definition of $\mathcal{F}$, so that $\mathcal{M}(\mathcal{E}_i) \subset \mathcal{M}_i$ by Lemma 1.1. This implies $\pi_i^{-1}(E) \in \mathcal{M}(\mathcal{F})$ for all $E \in \mathcal{M}(\mathcal{E}_i)$ and $i \in I$, but this collection generates $\bigotimes_{i\in I} \mathcal{M}(\mathcal{E}_i)$ and therefore

$$\bigotimes_{i\in I} \mathcal{M}(\mathcal{E}_i) \subset \mathcal{M}(\mathcal{F})$$

by Lemma 1.1.

Now, suppose $I$ is countable and $X_i \in \mathcal{E}_i$ for all $i \in I$. Denote

$$\mathcal{G} := \left\{ \prod_{i\in I} E_i : E_i \in \mathcal{E}_i,\ i \in I \right\}$$

Then

$$\prod_{i\in I} E_i = \bigcap_{i\in I} \pi_i^{-1}(E_i)$$

implies $\mathcal{M}(\mathcal{G}) \subset \mathcal{M}(\mathcal{F})$, and

$$\pi_i^{-1}(E) = \prod_{j\in I} E_j \qquad \text{with} \quad E_j = \begin{cases} E & \text{if } j = i \\ X_j & \text{otherwise} \end{cases}$$

implies $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{G})$. Thus

$$\mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathcal{F}) = \bigotimes_{i\in I} \mathcal{M}(\mathcal{E}_i)$$

by the first part of the proposition. $\square$

Recall that a metric space is separable if it has a countable dense subset.

**Proposition 1.5** Let $(X_1, d_1), \dots, (X_n, d_n)$ be metric spaces and denote

$$X := X_1 \times \cdots \times X_n = \prod_{j=1}^{n} X_j,$$

which we equip with the product metric

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) := \max\{ d_1(x_1, y_1), \dots, d_n(x_n, y_n)\}.$$

Then

$$\bigotimes_{j=1}^{n} \mathcal{B}_{X_j} \subset \mathcal{B}_X.$$

If $X_1, \dots, X_n$ are all separable, then the above is an equality.

**Proof** Since

$$\mathcal{B}_{X_j} = \mathcal{M}(\{U \subset X_j : \text{open}\}),$$

Proposition 1.4 implies $\bigotimes_{j=1}^{n} \mathcal{B}_{X_j}$ is generated by $\{\pi_j^{-1}(U) : j = 1, \dots, n,\ U \subset X_j \text{ open}\}$. These sets are open in $X$ and hence

$$\bigotimes_{j=1}^{n} \mathcal{B}_{X_j} \subset \mathcal{B}_X$$

by Lemma 1.1.

Next, suppose each $X_j$ is separable with countable dense subset $C_j \subset X_j$. Then

$$C := \prod_{j=1}^{n} C_j$$

is a countable dense subset of $X$. Let $\mathcal{E}$ be the collection of open balls centered at some point in $C$ with rational radius. The density of $C$ implies every open set in $X$ is a union of open balls in $\mathcal{E}$, and in fact a countable union since $\mathcal{E}$ is a countable collection. Therefore $\mathcal{B}_X$ is generated by $\mathcal{E}$. Now, note that for $B((x_1,\dots,x_n), r) \in \mathcal{E}$ we have

$$B((x_1,\dots,x_n), r) = \prod_{j=1}^{n} B(x_j, r)$$

by definition of the product metric. So if we let $\mathcal{E}_j$ denote the collection of open balls in $X_j$ centered at some point in $C_j$ with rational radius, then

Proposition 1.4

$$\mathcal{B}_X = \mathcal{M}(\mathcal{E}) = \mathcal{M}(\{\prod_{j=1}^{n} E_j : E_j \in \mathcal{E}_j, j=1,\dots,n\}) = \bigotimes_{j=1}^{n} \mathcal{M}(\mathcal{E}_j)$$

Finally, the same argument as above with $X, \mathcal{E}$ replaced by $X_j, \mathcal{E}_j$ implies $\mathcal{M}(\mathcal{E}_j) = \mathcal{B}_{X_j}$ for each $j=1,\dots,n$. □

Since $\mathbb{R}$ is a seperable metric space ($\mathbb{Q} \subset \mathbb{R}$ is countable and dense) we obtain:

**Corollary 1.6** $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^{n} \mathcal{B}_{\mathbb{R}}$

This corollary will make it easier to generalize results from $\mathbb{R}$ to $\mathbb{R}^n$, $n \in \mathbb{N}$.

# 1.3 Measures

**Def** Let $X$ be a set equipped with a $\sigma$-algebra $\mathcal{M}$. A measure on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ satisfying

① $\mu(\phi) = 0$

② for a sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ of disjoint subsets (countable additivity)

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

We call $(X, \mathcal{M}, \mu)$ a measure space and the elements $E \in \mathcal{M}$ are called measurable sets. We say $\mu : \mathcal{M} \to [0,\infty]$ is a finitely additive measure if it satisfies ① and ②' for disjoint sets $E_1, \dots, E_n \in \mathcal{M}$ (finite additivity)

$$\mu(E_1 \cup \cdots \cup E_n) = \mu(E_1) + \cdots + \mu(E_n).$$

Note that ② $\Rightarrow$ ②' since one can take $E_j = \phi$ $\forall j \geq n$. □

**Ex** ① Let $X$ be an uncountable set which we equip with the $\sigma$-algebra

$$\mathcal{C} = \{E \subset X : E \text{ or } E^c \text{ is countable}\}$$

Then $\mu : \mathcal{C} \to \{0,1\}$ defined by

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{otherwise} \end{cases}$$

is a measure.

② Let $X$ be an infinite set equipped with $\sigma$-algebra $\mathcal{P}(X)$. Then

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

is a finitely additive measure but not a measure □

**Def** Let $(X, \mathcal{M}, \mu)$ be a measure space. We say $\mu$ is finite if $\mu(X) < \infty$, and otherwise say $\mu$ is infinite. We say $\mu$ is $\sigma$-finite if $X = \bigcup_{n=1}^{\infty} E_n$ with $E_n \in \mathcal{M}$ and $\mu(E_n) < \infty$. Finally, we say $\mu$ is semifinite if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ satisfying $F \subset E$ and $0 < \mu(F) < \infty$. □

Note that $\mu(X) < \infty$ implies $\mu(E) < \infty$ for all $E \in \mathcal{M}$ since $\mu(X) = \mu(E) + \mu(E^c)$. Also $\mu$ being $\sigma$-finite implies every $E \in \mathcal{M}$ is a countable union of finite measure sets. Using these observations, one can show

$$\mu \text{ finite} \Rightarrow \mu\ \sigma\text{-finite} \Rightarrow \mu \text{ semifinite},$$

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but the reverse converses are false.

**Ex** Let $X$ be a nonempty set equipped with the $\sigma$-algebra $\mathcal{P}(X)$. For any function $f : X \to [0,\infty]$ one can define a measure on $(X, \mathcal{P}(X))$ by

$$\mu(E) = \sum_{x\in E} f(x)$$

(Since $f$ is valued in $[0,\infty]$ we can make sense of this sum even for uncountable $E$, see section 0.5.) Then $\mu$ is semifinite if and only if $f(x)<\infty$ for all $x\in X$. Also $\mu$ is $\sigma$-finite if and only if it is semifinite and $\{x: f(x)>0\}$ is countable.

If $f(x)=1$ for all $x\in X$, then

$$\mu(E)=\begin{cases} |E| & \text{if } E \text{ is finite} \\ \infty & \text{otherwise} \end{cases}.$$

We call $\mu$ the counting measure, which we often denote by $\#$.

If $f(x)=\delta_{x,x_0}$ for some $x_0\in X$, then

$$\mu(E)=\begin{cases} 1 & \text{if } x_0\in E \\ 0 & \text{otherwise.} \end{cases}$$

We call $\mu$ the point mass / Dirac measure at $x_0$, and denote it by $\delta_{x_0}$. □

$\boxed{\text{Theorem 1.8}}$ Let $(X,\mathcal{M},\mu)$ be a measure space. Then the measure $\mu$ satisfies the following properties

1. If $E,F\in\mathcal{M}$ with $E\subset F$, then $\mu(E)\le\mu(F)$. (monotonicity)
2. If $\{E_n : n\in\mathbb{N}\}\subset\mathcal{M}$, then
$$\mu\left(\bigcup_{n=1}^{\infty}E_n\right)\le\sum_{n=1}^{\infty}\mu(E_n)$$ (subadditivity)
3. If $\{E_n : n\in\mathbb{N}\}\subset\mathcal{M}$ with $E_1\subset E_2\subset\cdots$, then
$$\mu\left(\bigcup_{n=1}^{\infty}E_n\right)=\lim_{n\to\infty}\mu(E_n)$$ (continuity from below)
4. If $\{E_n : n\in\mathbb{N}\}\subset\mathcal{M}$ with $E_1\supset E_2\supset\cdots$ and $\mu(E_1)<\infty$, then
$$\mu\left(\bigcap_{n=1}^{\infty}E_n\right)=\lim_{n\to\infty}\mu(E_n)$$ (continuity from above)

Proof 1: Observe that $F\setminus E=F\cap E^c\in\mathcal{M}$ and $F$ is the disjoint union of $E$ and $F\setminus E$. Thus by countable additivity we have

$$\mu(F)=\mu(E)+\mu(F\setminus E)\ge\mu(E)$$

since $\mu(F\setminus E)\ge 0$.

2: Define $F_1:=E_1$ and

$$F_n:=E_n\setminus(E_1\cup\cdots\cup E_{n-1})$$

for $n\in\mathbb{N}$. Then $F_n\subset E_n$ and $\{F_n : n\in\mathbb{N}\}$ is a disjoint collection satisfying

$$\bigcup_{n=1}^{\infty}F_n=\bigcup_{n=1}^{\infty}E_n$$

(Note that $x$ in the union on the right belongs to $F_n$ where $n=\min\{k: x\in E_k\}$.) So by 1

$$\mu\left(\bigcup_{n=1}^{\infty}E_n\right)=\mu\left(\bigcup_{n=1}^{\infty}F_n\right)=\sum_{n=1}^{\infty}\mu(F_n)\le\sum_{n=1}^{\infty}\mu(E_n).$$

③: Let $E_0 := \emptyset$, then

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E_n \setminus E_{n-1})$$

and the latter is a disjoint union. Hence

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n \setminus E_{n-1}) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(E_n\setminus E_{n-1}) = \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} E_n\right) = \lim_{N\to\infty} \mu(E_N).$$

④: Define $F_n := E_1 \setminus E_n$ for each $n \in \mathbb{N}$. Then $F_1 \subset F_2 \subset \cdots$, $\mu(E_1) = \mu(E_n) + \mu(F_n)$, and

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_1 \setminus E_n = E_1 \setminus \bigcap_{n=1}^{\infty} E_n$$

So using ③ we have

$$\mu(E_1) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) + \mu\left(\bigcup_{n=1}^{\infty} F_n\right)$$

$$= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) + \lim_{n\to\infty} \mu(F_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) + \lim_{n\to\infty} [\mu(E_1) - \mu(E_n)]$$

Since $\mu(E_1) < \infty$, subtracting it from each side yields the desired equality. □

**Remark** The condition $\mu(E_1) < \infty$ in 4 in the previous theorem can be replaced by $\mu(E_{n_0}) < \infty$ for some $n_0 \in \mathbb{N}$, since $E_1 \supset E_2 \supset \cdots$ implies

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=n_0}^{\infty} E_n.$$

However, continuity from above fails without any finiteness assumption: Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \#)$ and the sets $E_n := \{k \in \mathbb{N} : k \geq n\}$. Then $\# E_n = \infty$ for all $n \in \mathbb{N}$, but

$$\#\left(\bigcap_{n=1}^{\infty} E_n\right) = \#(\emptyset) = 0.$$ □

**Def** Let $(X, \mathcal{M}, \mu)$ be a measure space. We call $E \in \mathcal{M}$ a <u>$\mu$-null set</u> (or just a <u>null set</u>) if $\mu(E) = 0$. If some statement about points $x \in X$ holds true except for $x$ in some $\mu$-null set, we say the statement is true <u>$\mu$-almost everywhere</u> (or just <u>almost everywhere</u>) which we abbreviate "$\mu$-a.e." (or a.e.) □

By subadditivity, a countable union of null sets is a null set. Also, by monotonicity any subset of a null set is a null set — so long as it is measurable. It turns out one can always force such subsets to be measurable by adding them to the σ-algebra:

**Theorem 1.9** Let $(X, \mathcal{M}, \mu)$ be a measure space and denote $\mathcal{N} := \{E \in \mathcal{M} : \mu(E) = 0\}$. Then

$$\overline{\mathcal{M}} := \{E \cup F : E \in \mathcal{M} \text{ and } F \subset G \text{ for some } G \in \mathcal{N}\}$$

is a σ-algebra and

$$\overline{\mu}(E \cup F) := \mu(E) \qquad E \in \mathcal{M},\ F \subset G \in \mathcal{N}$$

defines the unique extension of $\mu$ to $\overline{\mathcal{M}}$. Moreover, if $\overline{\mu}(E) = 0$ for $E \in \overline{\mathcal{M}}$ then $\mathcal{P}(E) \subset \overline{\mathcal{M}}$.

**Proof** Note that $\mathcal{N}$ is closed under countable unions by subadditivity. Thus for $E_n \cup F_n \in \overline{\mathcal{M}}$, $n \in \mathbb{N}$, with $E_n \in \mathcal{M}$ and $F_n \subset G_n \in \mathcal{N}$

$$\bigcup_{n\in\mathbb{N}} E_n \cup F_n = \underbrace{\Big(\bigcup_{n\in\mathbb{N}} E_n\Big)}_{\in\,\mathcal{M}} \cup \underbrace{\Big(\bigcup_{n\in\mathbb{N}} F_n\Big)}_{\subset\, \bigcup G_n \in \mathcal{N}} \in \overline{\mathcal{M}}.$$

For $E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}$ and $F \subset G \in \mathcal{N}$, note that

$$E \cup F = E \cup (F \setminus E)$$

and $F \setminus E \subset G \setminus E \in \mathcal{N}$. So we may assume $E \cap G = \emptyset$. Hence $E \subset G^c$ and $F \subset G$ imply

$$E \cup F = (E \cup F) \cap (G^c \cup F) = (E \cup G) \cap (G^c \cup F).$$

Thus

$$(E \cup F)^c = (E \cup G)^c \cup (G^c \cup F)^c = \underbrace{(E \cup G)^c}_{\in\,\mathcal{M}} \cup \underbrace{(G \cap F^c)}_{\subset\, G \in \mathcal{N}} \in \overline{\mathcal{M}},$$

and so $\overline{\mathcal{M}}$ is a $\sigma$-algebra.

Now, for $E \cup F \in \overline{\mathcal{M}}$ define $\overline{\mu}(E \cup F) := \mu(E)$. To see that this is well-defined, suppose

$$E_1 \cup F_1 = E_2 \cup F_2$$

for $E_1, E_2 \in \mathcal{M}$ and $F_1 \subset G_1$, $F_2 \subset G_2 \in \mathcal{N}$. Then $E_1 \subset E_2 \cup F_2 \subset E_2 \cup G_2$, and so

$$\mu(E_1) \leq \mu(E_2 \cup G_2) \leq \mu(E_2) + \mu(G_2) = \mu(E_2).$$

and similarly $\mu(E_2) \leq \mu(E_1)$. Hence $\mu(E_1) = \mu(E_2)$, and $\overline{\mu}$ is well-defined. Note this is an extension of $\mu$ since $E = E \cup \emptyset \in \overline{\mathcal{M}}$ for all $E \in \mathcal{M}$. That $\overline{\mu}$ is a measure then follows immediately from $\mu$ being a measure.

Next, suppose $E \cup F \in \overline{\mathcal{M}}$ satisfies $\overline{\mu}(E \cup F) = 0$ for $E \in \mathcal{M}$, $F \subset G \in \mathcal{N}$. Then evidently $\mu(E) = 0$ and so $E \in \mathcal{N}$. Thus for any $H \in \mathcal{P}(E \cup F)$ we have

$$H \subset E \cup F \subset E \cup G \in \mathcal{N}$$

and therefore $H = \emptyset \cup H \in \overline{\mathcal{M}}$.

Finally, suppose $\nu$ is another measure on $(X, \overline{\mathcal{M}})$ extending $\mu$. Then for any $E \in \mathcal{M}$ and $F \subset G \in \mathcal{N}$ we have

$$\overline{\mu}(E \cup F) = \mu(E) = \nu(E) \leq \nu(E \cup F) \leq \nu(E \cup G) = \mu(E \cup G) = \mu(E) = \overline{\mu}(E \cup F).$$

Hence $\overline{\mu} = \nu$ is unique. $\square$

**Def** Let $(X, \mathcal{M}, \mu)$ be a measure space. We say $\mu$ is complete if for all $E \in \mathcal{M}$ with $\mu(E) = 0$ one has $\mathcal{P}(E) \subset \mathcal{M}$. The measure $\overline{\mu}$ (resp the $\sigma$-algebra $\overline{\mathcal{M}}$) in the previous theorem is called the completion of $\mu$ (resp. $\mathcal{M}$). $\square$

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# 1.4 Outer Measures

So far we have only seen relatively simple examples of measures. Outer measures will give us a power way to generate interesting examples, including the famous Lebesgue measure on $\mathbb{R}$. The idea is to loosen the definition of a measure, making it easier to find examples. To produce honest measures, one then restricts an outer measure to a smaller $\sigma$-algebra.

**Def** An outer measure on a nonempty set $X$ is a function $\mu^*: \mathcal{P}(X) \to [0,\infty]$ satisfying:

1. $\mu^*(\emptyset) = 0$
2. $\mu^*(A) \le \mu^*(B)$ for $A \subseteq B$ (monotonicity)
3. $\mu^*(\bigcup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty \mu^*(A_n)$ for $\{A_n : n \in \mathbb{N}\} \subset \mathcal{P}(X)$. (countable subadditivity) □

Unlike measures, outer measures are extremely easy to construct:

**Proposition 1.10** Let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of subsets with $\emptyset, X \in \mathcal{E}$, and let $\rho: \mathcal{E} \to [0,\infty]$ be a function satisfying $\rho(\emptyset) = 0$. Then

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^\infty \rho(E_n) : E_n \in \mathcal{E} \text{ for all } n \in \mathbb{N} \text{ and } A \subset \bigcup_{n=1}^\infty E_n \right\}$$

for $A \in \mathcal{P}(X)$ defines an outer measure on $X$.

**Proof** Since $\emptyset, X \in \mathcal{E}$, one can cover any $A \in \mathcal{P}(X)$ by $E_1 = X$, $E_2 = E_3 = \cdots = \emptyset$. Thus the definition of $\mu^*(A)$ makes sense, and since $\rho$ is valued in $[0,\infty]$ so is $\mu^*$.

For the emptyset, $E_1 = E_2 = \cdots = \emptyset$ gives a countable cover by $\mathcal{E}$ and so

$$0 \le \mu^*(\emptyset) \le \sum \rho(\emptyset) = 0.$$

Thus $\mu^*(\emptyset) = 0$.

If $A \subset B$, then any countable cover of $B$ by $\mathcal{E}$ is also a countable cover of $A$. It follows that $\mu^*(A) \le \mu^*(B)$.

Finally, let $\{A_n : n \in \mathbb{N}\} \subset \mathcal{P}(X)$ and $\varepsilon > 0$. Since $\mu^*$ is defined via an infimum, we can find $\{E_k^{(n)} : k \in \mathbb{N}\} \subset \mathcal{E}$ such that $A_n \subset \bigcup_k E_k^{(n)}$ and

$$\sum_{k=1}^\infty \rho(E_k^{(n)}) \le \mu^*(A_n) + \frac{\varepsilon}{2^n}$$

for each $n \in \mathbb{N}$. Then $\bigcup_n \{E_k^{(n)} : k \in \mathbb{N}\} \subset \mathcal{E}$ is a countable cover of $\bigcup_n A_n$ and hence

$$\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \le \sum_{n,k=1}^\infty \rho(E_k^{(n)}) \le \sum_{n=1}^\infty \mu^*(A_n) + \frac{\varepsilon}{2^n} = \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields countable subadditivity. □

**Ex** For $X = \mathbb{R}$, let $\mathcal{E} = \{(a,b) : a,b \in \overline{\mathbb{R}},\ a \le b\}$ and define

$$\rho((a,b)) = \begin{cases} 0 & \text{if } a = b \\ b-a & \text{if } -\infty < a,b < \infty \\ \infty & \text{otherwise} \end{cases}$$

Then in particular, $\rho(\emptyset) = \rho((a,a]) = 0$ and so the previous proposition gives us an outer measure $\mu^*$ on $\mathbb{R}$. Observe that $\mu^*(A) = 0$ if and only if $A \subset \mathbb{R}$ is a null set (see the Riemann–Lebesgue theorem in Section 1.1). We will later see that this $\mu^*$ is an extension of the Lebesgue measure, and in particular for any Borel subset $A \in \mathcal{B}_{\mathbb{R}}$ its Lebesgue measure will equal $\mu^*(A)$. □

Suppose $(X, \mathcal{M}, \mu)$ is a measure space. For $A \in \mathcal{M}$, we have

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c) \qquad \forall E \in \mathcal{M}.$$

This seemingly innocuous condition (sometimes called the Carathéodory condition) turns out to be the key to restricting outer measures to measures, as we will now show.

**Def** Let $\mu^*$ be an outer measure on $X$. We say $A \subset X$ is <u>$\mu^*$-measurable</u> if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \forall E \subset X.$$

□

We always have $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ by subadditivity, and equality holds trivially if $\mu^*(E) = \infty$. Thus $A$ being $\mu^*$-measurable is actually equivalent to:

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \forall E \subset X \text{ with } \mu^*(E) < \infty.$$

**Theorem 1.11** (Carathéodory's Theorem) If $\mu^*$ is an outer measure on $X$, then the collection $\mathcal{M}$ of all $\mu^*$-measurable sets is a $\sigma$-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

**Proof** Since the definition of being $\mu^*$-measurable is symmetric with respect to $A$ and $A^c$, it follows that $\mathcal{M}$ is closed under taking complements. Towards showing $\mathcal{M}$ is closed under countable unions, we will show

1. $\mathcal{M}$ is closed under finite unions
2. $\mathcal{M}$ is closed under countable disjoint unions.

This suffices because if $\{E_n : n \in \mathbb{N}\} \subset \mathcal{M}$, then defining

$$F_n := E_n \setminus (E_1 \cup \cdots \cup E_{n-1}) \in \mathcal{M},$$

we have $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n \in \mathcal{M}$ since the latter is disjoint.

1: By an easy induction argument, it suffices to show $A \cup B \in \mathcal{M}$ for $A, B \in \mathcal{M}$. Let $E \subset X$, then

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c)\\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c)\end{aligned}$$

Noting that $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, the first three terms above are bounded below by $\mu^*(E \cap (A \cup B))$ by subadditivity. Also $(A \cup B)^c = A^c \cap B^c$.

Thus we have
$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c),$$
and so $A \cup B$ is $\mu^*$-measurable by the discussion preceding the theorem.

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2: Let $\{A_n : n \in \mathbb{N}\} \subset \mathcal{M}$ be a disjoint collection, and define
$$B := \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad B_n := A_1 \cup \cdots \cup A_n.$$

Then for any $E \subset X$ and $n \in \mathbb{N}$
$$\begin{aligned}
\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\
&= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\
&\;\vdots \\
&= \mu^*(E \cap A_n) + \mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2}) \\
&\;\vdots \quad \text{induction} \\
&= \mu^*(E \cap A_n) + \cdots + \mu^*(E \cap A_1)
\end{aligned}$$
Since $B_n \in \mathcal{M}$ by 1 we then have
$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\
&= \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap B_n^c) \geq \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu(E \cap B^c)
\end{aligned}$$

Letting $n \to \infty$ yields
$$\begin{aligned}
\mu^*(E) &\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu(E \cap B^c) \\
&\geq \mu^*\Big(\bigcup_{j=1}^{\infty} E \cap A_j\Big) + \mu(E \cap B^c) \\
&= \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E)
\end{aligned}$$
Hence $B$ is $\mu^*$-measurable.

Thus $\mathcal{M}$ is a $\sigma$-algebra. Also observe that letting $E = B = \bigcup_{j=1}^{\infty} A_j$ above yields
$$\mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu^*(A_j).$$
Hence $\mu^*$ is countably additive on $\mathcal{M}$ and therefore a measure. Finally, to see that $\mu^*|_{\mathcal{M}}$ is complete, suppose $A \in \mathcal{M}$ satisfies $\mu^*(A) = 0$ and let $B \subset A$. For any $E \subset X$ one has by monotonicity
$$\mu^*(E) \leq \mu^*(E \cap B) + \mu^*(E \cap B^c) \leq 0 + \mu^*(E \cap B^c) \leq \mu^*(E).$$
Thus $B$ is $\mu^*$-measurable and therefore $B \in \mathcal{M}$. $\square$

## Premeasures

Using Proposition 1.10 given, $\mathcal{E} \subset \mathcal{P}(X)$ containing $\phi, X$ and a function $\rho: \mathcal{E} \to [0,\infty]$ satisfying $\rho(\phi) = 0$ we can construct an outer measure $\mu^*$ such that $\mu^*(E) \le \rho(E)$ for all $E \in \mathcal{E}$ (Exercise check this). Then using Theorem 1.11 we can obtain a complete measure $\mu$ from $\mu^*$. However, it may be that $\mathcal{E}$ is not contained in the domain of $\mu$ (i.e. $E \in \mathcal{E}$ may not be $\mu^*$-measurable). In order to avoid this, one must consider $\mathcal{E}$ and $\rho$ with a bit more structure: $\mathcal{E}$ must be an "algebra" and $\rho$ must be a "premeasure".

**Def** Let $X$ be a nonempty set. An algebra of sets on $X$ is a nonempty collection $\mathcal{A}$ of subsets of $X$ such that

1. $E^c \in \mathcal{A}$ for all $E \in \mathcal{A}$.
2. If $E_1, \dots, E_n \in \mathcal{A}$ then $E_1 \cup \cdots \cup E_n \in \mathcal{A}$. $\square$

Arguing the same way as for $\sigma$-algebras, one sees that an algebra is also closed under finite intersections and always contains $\phi$ and $X$. Moreover, an algebra $\mathcal{A}$ is a $\sigma$-algebra if and only if it is closed under countable disjoint unions (Exercise).

**Ex** Let $X$ be a nonempty set with $X = E_1 \sqcup E_2 \sqcup E_3$. Then

$$\mathcal{A} := \{\phi, E_1, E_2, E_3, E_1 \cup E_2, E_1 \cup E_3, E_2 \cup E_3, X\}$$

is an algebra. $\square$

Note that $E_1, E_2, E_3$ are the "building blocks" of $\mathcal{A}$, and they satisfy $X \setminus E_i = E_j \cup E_k$ distinct $i,j,k$.

**Proposition 1.12** Let $X$ be a nonempty set and $\mathcal{E} \subset \mathcal{P}(X)$ such that

1. $\phi \in \mathcal{E}$
2. for $F, E \in \mathcal{E}$, $E \cap F \in \mathcal{E}$
3. for $E \in \mathcal{E}$, $E^c = E_1 \sqcup \cdots \sqcup E_d$ for some $E_1, \dots, E_d \in \mathcal{E}$.

(We call $\mathcal{E}$ an elementary family.) Then the collection $\mathcal{A}$ of all finite disjoint unions of sets in $\mathcal{E}$ is an algebra.

**Proof** We first show $\mathcal{A}$ is closed under finite unions. Let $A, B \in \mathcal{E}$ with $B^c = E_1 \sqcup \cdots \sqcup E_d$ and $E_1, \dots, E_d \in \mathcal{E}$. Then

$$A \cup B = (A \cap B) \sqcup (A \cap B^c) = (A \cap B) \sqcup (A \cap E_1) \sqcup \cdots \sqcup (A \cap E_d) \in \mathcal{A}$$

Induction implies $A_1 \cup \cdots \cup A_n \in \mathcal{A}$ for $A_1, \dots, A_n \in \mathcal{E}$. Thus $\mathcal{A}$ contains all finite unions of sets in $\mathcal{E}$, and therefore is closed under finite unions.

Next, let $A = E_1 \sqcup \cdots \sqcup E_d \in \mathcal{A}$ with $E_1, \dots, E_d \in \mathcal{E}$. For each $1 \le j \le d$, $E_j^c = \bigsqcup_{k=1}^{D_j} F_k^{(j)}$ with $F_k^{(j)} \in \mathcal{E}$. Set $D := \max\{D_1, \dots, D_d\}$ and $F_k^{(j)} = \phi$ for $D_j < k \le D$. Then

$$A^c = \bigcap_{j=1}^{d} E_j^c = \bigcap_{j=1}^{d} \bigsqcup_{k=1}^{D_j} F_k^{(j)} = \bigcap_{j=1}^{d} \bigsqcup_{k=1}^{D} F_k^{(j)} = \bigsqcup_{k=1}^{D} \underbrace{\bigcap_{j=1}^{d} F_k^{(j)}}_{\in \mathcal{E}} \in \mathcal{A}$$

$\square$

Given an algebra $\mathcal{A}$ on a set $X$, the notion of a finitely additive measure on $(X,\mathcal{A})$ is quite natural. But in our pursuit of measures we require something a bit stronger:

$\boxed{\text{Def}}$ Let $\mathcal{A}$ be an algebra on $X$. A premeasure on $(X,\mathcal{A})$ is a function $\mu_0 : \mathcal{A} \to [0,\infty]$ satisfying

1. $\mu_0(\emptyset) = 0$
2. for a sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ of disjoint sets with $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$,
$$\mu_0\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu_0(E_n)$$

$\square$

Premeasures are finitely additive since $\mathcal{A} \ni E_1 \cup \cdots \cup E_n = E_1 \cup \cdots \cup E_n \cup \emptyset \cup \emptyset \cup \cdots$, and consequently satisfy monotonicity: for $A, B \in \mathcal{A}$ with $A \subset B$, $\mu_0(B) = \mu_0(A) + \mu_0(B \setminus A) \geq \mu_0(A)$. Since $\emptyset, X \in \mathcal{A}$ and $\mu_0(\emptyset) = 0$, we obtain an outer measure via Proposition 1.10:
$$\mu^*(E) := \inf\Big\{ \sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A} \text{ for all } j \in \mathbb{N} \text{ and } E \subset \bigcup_{j=1}^{\infty} A_j \Big\} \quad (*)$$

Restricting to $\mu^*$-measurable sets then yields a measure by Carathéodory's Theorem (Theorem 1.11). Our goal is to show that this measure extends $\mu_0$.

$\boxed{\text{Proposition 1.13}}$ Let $\mu_0$ be a premeasure on $(X,\mathcal{A})$. For the outer measure $\mu^*$ defined by $(*)$, one has:

1. $\mu^*|_{\mathcal{A}} = \mu_0$
2. every $A \in \mathcal{A}$ is $\mu^*$-measurable.

Proof. Let $A \in \mathcal{A}$. Then $E_1 := A$, $E_2 = E_3 = \cdots = \emptyset$ gives a countable cover of $A$ by $\mathcal{A}$, hence $\mu^*(A) \leq \mu_0(A)$. Conversely, if $\{E_n : n \in \mathbb{N}\} \subset \mathcal{A}$ is a countable cover of $A$, define $F_n := A \cap (E_n \setminus E_1 \cdots E_{n-1}) \in \mathcal{A}$. Then the $F_n$ are disjoint with $\bigcup F_n = A \in \mathcal{A}$. Thus 2 and monotonicity give
$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(F_n) \leq \sum_{n=1}^{\infty} \mu_0(E_n)$$
Thus $\mu_0(A) \leq \mu^*(A)$, which establishes 1.

To see 2, let $A \in \mathcal{A}$, $E \subset X$, and $\varepsilon > 0$. Let $\{E_n : n \in \mathbb{N}\} \subset \mathcal{A}$ be a cover of $E$ such that
$$\sum_{n=1}^{\infty} \mu_0(E_n) \leq \mu^*(E) + \varepsilon$$
Since $\mu_0$ is additive on $\mathcal{A}$, we obtain
$$\mu^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu_0(E_n) = \sum_{n=1}^{\infty} \mu_0(E_n \cap A) + \mu_0(E_n \cap A^c)$$
Now $\{E_n \cap A : n \in \mathbb{N}\}$, $\{E_n \cap A^c : n \in \mathbb{N}\} \subset \mathcal{A}$ are covers of $E \cap A$ and $E \cap A^c$, respectively, thus the above is bounded below by $\mu^*(E \cap A) + \mu^*(E \cap A^c)$. Letting $\varepsilon \to 0$ yields that $A$ is $\mu^*$-measurable.

$\square$

$\boxed{\text{Theorem 1.14}}$ Let $\mathcal{A}$ be an algebra on $X$ and $\mu_0$ a premeasure on $(X,\mathcal{A})$. Then there exists a measure $\mu$ on $(X, \mathcal{M}(\mathcal{A}))$ extending $\mu_0$: $\mu|_{\mathcal{A}} = \mu_0$. If $\nu$ is another measure on $(X, \mathcal{M}(\mathcal{A}))$ extending $\mu_0$, then

$$\nu(E) \leq \mu(E) \qquad \forall E \in \mathcal{M}(\mathcal{A})$$

with equality when $\mu(E) < \infty$. If $\mu_0$ is $\sigma$-finite, then $\mu$ is the unique extension of $\mu_0$ to $\mathcal{M}(\mathcal{A})$.

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Proof Let $\mu^*$ be the outer measure defined by (*) above and let $\mathcal{M}$ be the collection of $\mu^*$-measurable sets. We have $\mathcal{A} \subset \mathcal{M}$ by Proposition 1.13, and hence $\mathcal{M}(\mathcal{A}) \subset \mathcal{M}$ by Lemma 1.1. By Carathéodory's theorem (Theorem 1.11), $\mu^*|_{\mathcal{M}}$ is a measure and hence so is $\mu := \mu^*|_{\mathcal{M}(\mathcal{A})}$. We have $\mu|_{\mathcal{A}} = \mu^*|_{\mathcal{A}} = \mu_0$ by Proposition 1.13.

Now, suppose $\nu$ also extends $\mu_0$. Let $E \in \mathcal{M}(\mathcal{A})$ and let $\{A_n : n \in \mathbb{N}\} \subset \mathcal{A}$ be a cover for $E$. Then

$$\nu(E) \leq \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu_0(A_n),$$

and hence $\nu(E) \leq \mu^*(E) = \mu(E)$. Now suppose $\mu(E) < \infty$ and let $\varepsilon > 0$. We can choose $\{A_n : n \in \mathbb{N}\} \subset \mathcal{A}$ so that $\sum \mu_0(A_n) \leq \mu^*(E) + \varepsilon$. Denote $A := \bigcup A_n$, then

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu_0(A_n) \leq \mu^*(E) + \varepsilon = \mu(E) + \varepsilon$$

Since $\mu(A \setminus E) = \mu(A) - \mu(E)$, we have $\mu(A \setminus E) < \varepsilon$. Using and continuity from below we have

$$\mu(E) \leq \mu(A) = \lim_{N \to \infty} \mu\left(\bigcup_{n=1}^{N} A_n\right) = \lim_{N \to \infty} \nu\left(\bigcup_{n=1}^{N} A_n\right) = \nu(A)$$

$$= \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E) < \nu(E) + \varepsilon$$

Letting $\varepsilon \to 0$ yields $\mu(E) = \nu(E)$.

Finally, suppose $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu_0(A_n) < \infty$. By the usual trick, we may assume the $A_n$'s are disjoint. Then for any $E \in \mathcal{M}(\mathcal{A})$, we have $\mu(E \cap A_n) \leq \mu(A_n) = \mu_0(A_n) < \infty$ for all $n \in \mathbb{N}$. Thus

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E \cap A_n) = \sum_{n=1}^{\infty} \nu(E \cap A_n) = \nu(E).$$

Hence $\mu$ is the unique extension of $\mu_0$ to $\mathcal{M}(\mathcal{A})$. □

Remark In the above theorem and the discussion preceding it, we only considered extending $\mu_0$ to $\mathcal{M}(\mathcal{A})$. But Carathéodory's theorem actually gives an extension of $\mu_0$ to all $\mu^*$-measurable sets, which may be strictly larger than $\mathcal{M}(\mathcal{A})$. □

# 1.5 Borel Measures on the real line

**Def** For a metric space $X$, a measure on $(X, \mathcal{B}_X)$ is called a Borel measure. For $X = \mathbb{R}$, if $\mu$ is a Borel measure then $F(x) := \mu((-\infty, x])$ is called the cumulative distribution function of $\mu$. $\square$

Note that a cumulative distribution function $F$ is increasing by monotonicity: for $x \le y$

$$F(x) = \mu((-\infty, x]) \le \mu((-\infty, y]) = F(y).$$

Also

$$F(y) - F(x) = \mu((-\infty, y] \setminus (-\infty, x]) = \mu((x, y])$$

If $\mu$ is finite, then it is also right-continuous by continuity from above: for $x_n \searrow x$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \mu((-\infty, x_n]) = \mu\left(\bigcap_{n=1}^{\infty} (-\infty, x_n]\right) = \mu((-\infty, x]) = F(x).$$

Our first goal in this section is to construct a Borel measure on $\mathbb{R}$ from an arbitrary increasing and right-continuous function. We will use the premeasure technique developed in the previous section. Consider the collection of left-open, right-closed intervals (and $\emptyset$):

$$\mathcal{E} := \{\emptyset\} \cup \{(a,b] : -\infty \le a < b < \infty\} \cup \{(a, \infty) : -\infty \le a < \infty\} \subset \mathcal{P}(\mathbb{R})$$

We will refer to sets in $\mathcal{E}$ as h-intervals in this section. We will also abuse notation and write $(a, \infty]$ for $(a, \infty)$ so that $\mathcal{E} = \{\emptyset\} \cup \{(a,b] : -\infty \le a < b \le \infty\}$. This will also reduce uninteresting complexity caused by having to consider two "types" of sets in $\mathcal{E}$ in our proofs below. Observe that the intersection of two h-intervals is again an h-interval, and the complement is the disjoint union of two or fewer h-intervals. Thus $\mathcal{E}$ satisfies the hypotheses of Proposition 1.12 (i.e. it is an elementary family) and therefore the collection $\mathcal{A}$ of finite disjoint unions of h-intervals is an algebra on $\mathbb{R}$. Moreover, $\mathcal{M}(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$ by Proposition 1.2.3. So if we can define a premeasure $\mu_0$ on $\mathcal{A}$, then Theorem 1.14 will give us a Borel measure.

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**Proposition 1.15** Let $F: \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous. Extend $F$ to $\overline{\mathbb{R}}$ (and possibly valued in $\overline{\mathbb{R}}$) by

$$F(\infty) := \sup_{x\in\mathbb{R}} F(x) \qquad \text{and} \qquad F(-\infty) := \inf_{x\in\mathbb{R}} F(x)$$

For disjoint h-intervals $(a_j, b_j]$, $j = 1, \ldots, n$ define

$$\mu_0\left(\bigcup_{j=1}^{n} (a_j, b_j]\right) = \sum_{j=1}^{n} [F(b_j) - F(a_j)]$$

and $\mu_0(\emptyset) = 0$. Then $\mu_0$ is a premeasure on $(\mathbb{R}, \mathcal{A})$.

**Proof** We first need to check that $\mu_0$ is well-defined since a subset of $\mathcal{A}$ can potentially be written as the union of two distinct collections of disjoint h-intervals. First note that if $(a_1, b_1], \ldots, (a_n, b_n]$ are disjoint with union equal $(a, b]$, then (after relabeling indices) we must have

$$a = a_1 < b_1 = a_2 < b_2 = \cdots < b_n = b$$

and therefore

$$\sum_{j=1}^{n} [F(b_j) - F(a_j)] = F(b) - F(a).$$

So if $\{I_i\}_{i=1}^n$, $\{J_j\}_{j=1}^m$ are disjoint collections of h-intervals with the same union, then

$$\sum_{i=1}^{n} \mu_0(I_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mu_0(I_i \cap J_j) = \sum_{j=1}^{m} \mu_0(J_j)$$

Thus $\mu_0$ is well-defined. It is also finitely additive by definition.
Now let $\{I_n : n\in\mathbb{N}\} \subset \mathcal{A}$ be a disjoint collection with $\bigcup_n I_n \in \mathcal{A}$. By definition of $\mathcal{A}$, $\bigcup_n I_n$ is a finite disjoint union of h-intervals and so we can partition $\{I_n : n\in\mathbb{N}\}$ into finitely many subcollections each of whose unions is a single h-interval. By considering these subcollections separately and using the finite additivity of $\mu_0$, we may assume

$$\bigcup_{n=1}^{\infty} I_n = (a,b] =: I$$

Now,

$$\mu_0(I) = \mu_0(I_1) + \cdots + \mu_0(I_n) + \mu_0(I \setminus \bigcup_{j=1}^{n} I_n) \geq \mu_0(I_1) + \cdots + \mu_0(I_n),$$

and letting $n\to\infty$ gives

$$\mu_0(I) \geq \sum_{n=1}^{\infty} \mu_0(I_n).$$

To prove the reverse inequality we consider some cases separately:

$\boxed{1}$ Assume $a,b$ finite. Let $\varepsilon>0$, then the right continuity of $F$ implies there exists $\delta>0$ so that $F(a+\delta) - F(a) < \varepsilon$. If $I_n = (a_n, b_n]$ for each $n\in\mathbb{N}$, then there also exists $\delta_n>0$ so that $F(b_n+\delta_n) - F(b_n) < \frac{\varepsilon}{2^n}$. Now, $\{(a_n, b_n+\delta_n) : n\in\mathbb{N}\}$ is an open cover for the compact set $[a+\delta, b]$, so there exists a finite subcover. After discarding any $(a_n, b_n+\delta_n)$ that is contained in a larger interval in the subcover and relabeling the indices $n$, we may assume our subcover is $(a_1, b_1+\delta_1), \ldots, (a_N, b_N+\delta_N)$ with

$$b_n + \delta_n \in (a_{n+1}, b_{n+1}+\delta_{n+1}) \qquad n=1,\ldots,N-1$$

We then have

$$\begin{aligned}
\mu_0(I) &= F(b) - F(a) \\
&< F(b) - F(a+\delta) + \varepsilon \\
&\leq F(b_N+\delta_N) - F(a_1) + \varepsilon \\
&= F(b_N+\delta_N) - F(a_N) + \sum_{n=1}^{N-1} [F(a_{n+1}) - F(a_n)] + \varepsilon \\
&\leq \sum_{n=1}^{N} [F(b_n+\delta_n) - F(a_n)] + \varepsilon \\
&\leq \sum_{n=1}^{\infty} [F(b_n) - F(a_n) + \tfrac{\varepsilon}{2^n}] + \varepsilon = \sum_{n=1}^{\infty} \mu_0(I_n) + 2\varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$ completes this case.

$\boxed{2}$ Assume either $a = -\infty$ or $b = \infty$. Let $a \le M < N \le b$ be finite. Then the intervals $(a_n, b_n + \delta_n)$ from case $\boxed{1}$ cover $[M, N]$, and so the same proof shows
$$F(N) - F(M) \le \sum_{n=1}^{\infty} \mu_0(I_n) + 2\varepsilon.$$
Letting $N \to b$, $M \to a$, and $\varepsilon \to 0$ yields the desired inequality. $\square$

$\boxed{\text{Theorem 1.16}}$ If $F: \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous, then there exists a unique $\sigma$-finite Borel measure $\mu_F$ on $\mathbb{R}$ satisfying
$$\mu_F((a,b]) = F(b) - F(a) \qquad \forall a, b \in \mathbb{R} \text{ with } a < b.$$
If $G$ is another such function, then $\mu_F = \mu_G$ if and only if $F - G$ is constant.
Conversely, if $\mu$ is a Borel measure on $\mathbb{R}$ that is finite on bounded Borel sets, then
$$F_\mu(x) := \begin{cases} \mu((0,x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x,0]) & \text{if } x < 0 \end{cases}$$
is increasing and right-continuous with $\mu_{F_\mu} = \mu$.

Proof. Each $F$ gives a Borel measure $\mu_F$ by Proposition 1.15 and the discussion at the beginning of the section. Since $\mu_F$ extends the premeasure from Proposition 1.15, we have $\mu_F((a,b]) = F(b) - F(a)$. Moreover, $\mu_F$ is $\sigma$-finite since $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+1]$ and so $\mu_F$ is unique by Theorem 1.14.
If $\mu_F = \mu_G$, then let $C := F(0) - G(0)$. Then
$$F(x) - G(x) = F(x) - F(0) - (G(x) - G(0)) + C = \left\{ \begin{matrix} \mu_F((0,x]) - \mu_G((0,x]) + C & \text{if } x > 0 \\ C & \text{if } x = 0 \\ -\mu_F((x,0]) - \mu_G((x,0]) + C & \text{if } x < 0 \end{matrix} \right\} = C$$
Thus $F - G \equiv C$. Conversely, if $F - G$ is constant, then
$$F(b) - F(a) = G(b) - G(a)$$
for all $a < b$. This means $F$ and $G$ define the same premeasure and hence $\mu_F = \mu_G$.
Finally, let $\mu$ be a Borel measure with $\mu(E) < \infty$ for all bounded $E \subset \mathcal{B}_\mathbb{R}$. Then $F_\mu(x)$ is finite for all $x$. We also have for $0 < a < b$
$$F(b) - F(a) = \mu((0,b]) - \mu((0,a]) = \mu((0,b] \setminus (0,a]) = \mu((a,b]) \ge 0.$$

Similarly for $a < b \le 0$ and $a \le 0 < b$. So $F_\mu$ is increasing. This also shows $\mu$ and $\mu_{F_\mu}$ agree on h-intervals and so are both extensions of the same premeasure. The uniqueness in Theorem 1.14 therefore implies $\mu = \mu_{F_\mu}$. Lastly, the right-continuity for $F_\mu$ follows from continuity from above for $\mu$. $\square$

$\boxed{\text{Remark}}$ ① $F_\mu$ is not the cumulative distribution function of $\mu$ (which may be equal to $\infty$ for all $x \in \mathbb{R}$). But if $\mu$ is finite then $F_\mu$ differs from this function by $\mu((-\infty, 0])$.

2. One could just as easily used intervals of the form $[a,b)$ to construct these Borel measures. However, open or closed intervals would yield a lot of technical issues since they do not form an elementary family.

3. If $\mu_F^*$ is the outer measure induced by the premeasure $F$ defines, then $\mu_F$ is extended by the complete measure given by restricting $\mu_F^*$ to the $\sigma$-algebra of $\mu_F^*$-measurable sets. Since $\mu_F$ is $\sigma$-finite, Exercise 3 on Homework 3 implies this extension of $\mu_F$ is its completion. $\square$

## Lebesgue - Stieltjes Measures

**Def** Let $F: \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous. The completion of the measure $\mu_F$ from Theorem 1.16 is called the Lebesgue - Stieltjes measure associated to $F$, and we typically abuse notation to denote this by $\mu_F$. $\square$

Fix a Lebesgue-Stieltjes measure $\mu$ associated to some $F: \mathbb{R} \to \mathbb{R}$, and denote its domain $\sigma$-algebra by $\mathcal{M}_\mu$. Since $\mu_F$ is the restriction of the outer measure induced by $F$, we have

$$\mu(E) = \inf \left\{ \sum_{n=1}^\infty [F(b_n) - F(a_n)] : E \subset \bigcup_{n=1}^\infty (a_n, b_n] \right\}$$

$$= \inf \left\{ \sum_{n=1}^\infty \mu((a_n, b_n]) : E \subset \bigcup_{n=1}^\infty (a_n, b_n] \right\}$$

for all $E \in \mathcal{M}_\mu$. It turns out one can replace the h-intervals above with open intervals:

**Lemma 1.17** For $E \in \mathcal{M}_\mu$

$$\mu(E) = \inf \left\{ \sum_{n=1}^\infty \mu((a_n, b_n)) : E \subset \bigcup_{n=1}^\infty (a_n, b_n) \right\}$$

**Proof** Denote the quantity on the right by $\nu(E)$. Let $\{(a_n, b_n) : n \in \mathbb{N}\}$ be a cover of $E$. Observe that if $(c_k)_{k \in \mathbb{N}} \subset (a_n, b_n)$ is a strictly increasing sequence with $c_1 = a_n$ and $c_k \xrightarrow{k} b_n$, then

$$(a_n, b_n) = \bigcup_{k=1}^\infty (c_k, c_{k+1}].$$

Since the latter union is disjoint

$$\mu((a_n, b_n)) = \sum_{k=1}^\infty \mu((c_k, c_{k+1}]).$$

Thus we can find $\{(c_n, d_n] : n \in \mathbb{N}\}$ covering $E$ with

$$\sum_{n=1}^{\infty} \mu((c_n, d_n]) = \sum_{n=1}^{\infty} \mu((a_n, b_n)).$$

Since $\{(a_n, b_n) : n \in \mathbb{N}\}$ was an arbitrary cover, it follows that $\mu(E) \le \nu(E)$. Conversely, let $\varepsilon > 0$ and let $\{(c_n, d_n] : n \in \mathbb{N}\}$ be a cover of $E$ by h-intervals satisfying

$$\sum_{n=1}^{\infty} \mu((c_n, d_n]) \le \mu(E) + \varepsilon$$

Using the right-continuity of the function defining $\mu$, we can find $\delta_n > 0$ for each $n \in \mathbb{N}$ such that

$$\mu((c_n, d_n + \delta_n)) < \mu((c_n, d_n]) + \frac{\varepsilon}{2^n}$$

Then

$$\nu(E) \le \sum_{n=1}^{\infty} \mu((c_n, d_n + \delta_n)) < \sum_{n=1}^{\infty} \mu((c_n, d_n]) + \frac{\varepsilon}{2^n} \le \mu(E) + 2\varepsilon$$

Letting $\varepsilon \to 0$ yields $\mu(E) = \nu(E)$. □

We will now prove several nice "regularity" properties of the Lebesgue-Stieltjes measure $\mu$.

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$\boxed{\text{Theorem 1.18}}$ For $E \in \mathcal{M}_\mu$

$$\mu(E) = \inf\{\mu(U) : \text{open } U \supset E\}$$
$$= \sup\{\mu(K) : \text{compact } K \subset E\}$$

Proof By Lemma 1.17, for $\varepsilon > 0$ there exists a cover of $E$ $\{(a_n, b_n) : n \in \mathbb{N}\}$ satisfying

$$\sum_{n=1}^{\infty} \mu((a_n, b_n)) \le \mu(E) + \varepsilon.$$

Thus for the open set $U := \bigcup_n (a_n, b_n)$, we have $\mu(U) \le \mu(E) + \varepsilon$. Also, $\mu(V) \ge \mu(E)$ for any open $V \supset E$. The first equality follows.

Toward showing the second equality, first suppose $E$ is bounded so that $\bar{E}$ is compact. Using the first equality, for $\varepsilon > 0$ we can find $U \supset \bar{E} \setminus E$ open satisfying

$$\mu(U) \le \mu(\bar{E} \setminus E) + \varepsilon$$

Set $K := \bar{E} \setminus U$, which is compact and

$$K = \bar{E} \cap U^c \subset \bar{E} \cap (\bar{E} \setminus E)^c = \bar{E} \cap (\bar{E}^c \cup E) = E.$$


So $K = E \setminus U$ and hence

$$\mu(K) = \mu(E) - \mu(E \cap U)$$
$$= \mu(E) - [\mu(U) - \mu(U \setminus E)]$$
$$\ge \mu(E) - \mu(U) + \mu(\bar{E} \setminus E) \ge \mu(E) - \varepsilon.$$

Thus $\mu(E) \le \mu(K) + \varepsilon$. Since $\mu(C) \le \mu(E)$ for any compact $C \subset E$, this yields the second equality when $E$ is bounded.

If $E$ is unbounded, let $E_n := E \cap (n, n+1]$. For $\varepsilon > 0$, the above argument gives $K_n \subset E_n$ with $\mu(E_n) \leq \mu(K_n) + \frac{\varepsilon}{2^n}$. Define

$$H_N := \bigcup_{n=-N}^{N} K_n \subset E$$

which is compact with

$$\mu(H_N) = \sum_{n=-N}^{N} \mu(K_n) \geq \sum_{n=-N}^{N} \mu(E_n) - \frac{\varepsilon}{2^n} \geq \mu\left(\bigcup_{k=-N}^{N} E_n\right) - 3\varepsilon$$

Now,

$$\mu(E) = \lim_{N \to \infty} \mu\left(\bigcup_{k=-N}^{N} E_n\right)$$

by continuity from below (or even just countable additivity). So for sufficiently large $N$ we have $\mu(H_N) \geq \mu(E) - 4\varepsilon$, and the second equality follows. □

**Theorem 1.19** For $E \subset \mathbb{R}$, the following are equivalent:

1. $E \in \mathcal{M}_\mu$
2. $E = V \setminus N_1$ for $V$ a $G_\delta$ set and $\mu(N_1) = 0$
3. $E = H \cup N_2$ for $H$ an $F_\sigma$ set and $\mu(N_2) = 0$

*Proof* Since $V, H \in \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_\mu$, 2 and 3 each imply 1. Conversely, suppose $E \in \mathcal{M}_\mu$. First assume $\mu(E) < \infty$. Then Theorem 1.18 allows us to find $K_n \subset E \subset U_n$ compact and open sets satisfying

$$\mu(E_n) - \frac{1}{n} \leq \mu(K_n) \leq \mu(E) \leq \mu(U_n) \leq \mu(E) + \frac{1}{n}$$

Then $V := \bigcap U_n \supset E$ and $H := \bigcup K_n \subset E$ are $G_\delta$ and $F_\sigma$ sets, respectively, satisfying $\mu(V) = \mu(E) = \mu(H)$. Thus $N_1 := V \setminus E$ and $N_2 := E \setminus H$ are $\mu$-null sets. Now, suppose $\mu(E) = \infty$. Since $\mu$ is $\sigma$-finite we have $E = \bigcup_n E_n$ with $\mu(E_n) < \infty$, and we can assume the $E_n$'s are disjoint. Construct an $F_\sigma$ set $H_n \subset E_n$ with $\mu(E_n \setminus H_n) = 0$ as above. Then $H := \bigcup H_n$ is also an $F_\sigma$ set and

$$E \setminus H = \bigcup_{n=1}^{\infty} E_n \setminus H = \bigcup_{n=1}^{\infty} E_n \setminus H_n$$

so $E \setminus H$ is a $\mu$-null set as the countable union of $\mu$-null sets. Next, let $H_1 \subset E^c$ be a $F_\sigma$ set with $0 = \mu(E^c \setminus H_1) = \mu(E^c \cap H_1^c)$. Then $V := H_1^c$ is a $G_\delta$ set containing $E$ and $\mu(V \setminus E) = \mu(H_1^c \cap E^c) = 0$. □

The above theorem tells us sets in $\mathcal{M}_\mu$ (in particular, Borel sets) can be understood in terms of $G_\delta$ or $F_\sigma$ sets, up to $\mu$-null sets. So even though $\mathcal{M}_\mu$ contains a great variety of sets beyond $G_\delta$ and $F_\sigma$ sets, the measure only witnesses complexity up to this level. We can even replace $G_\delta$ and $F_\sigma$ sets with finite unions of open intervals if we are willing to tolerate more "error":

**Proposition 1.20** If $E \in \mathcal{M}_\mu$ with $\mu(E) < \infty$, then for all $\varepsilon > 0$ there exists a finite union of open intervals $A$ such that $\mu(E \Delta A) < \varepsilon$

**Proof** Exercise 1 on Homework 4. $\square$

**Def** The *Lebesgue measure* is the Lebesgue-Stieltjes measure associated to the function $F(x) = x$, and is denoted $m$. The $\sigma$-algebra on which $m$ is defined (i.e. the completion of $\mathcal{B}_\mathbb{R}$ with respect to $m$) is denoted $\mathcal{L}$, and we say $E \in \mathcal{L}$ is *Lebesgue measurable*. $\square$

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In the following theorem, for $E \subset \mathbb{R}$ and $s, r \in \mathbb{R}$ we denote

$$E + s = \{x + s : x \in E\} \qquad \text{and} \qquad Er := \{xr : x \in E\},$$

and call these sets translations and dilations of $E$, respectively.

**Theorem 1.21** If $E \in \mathcal{L}$ then $E+s, Er \in \mathcal{L}$ for all $s, r \in \mathbb{R}$ with

$$m(E+s) = m(E) \qquad \text{and} \qquad m(Er) = |r|\, m(E)$$

**Proof** By Exercise 1 on Homework 2, $\mathcal{B}_\mathbb{R}$ is closed under translations and dilations and so $m_s(E) := m(E+s)$ and $m^r(E) := m(Er)$ define measures on $\mathcal{B}_\mathbb{R}$ (Exercise check this). Observe

$$m_s((a,b]) = m((a+s, b+s]) = (b+s) - (a+s) = b - a = m((a,b])$$

Thus $m_s$ and $m$ agree on $\mathcal{B}_\mathbb{R}$ by Theorem 1.14. Also

$$m^r((a,b]) = \begin{cases} m((ar, br]) & \text{if } r > 0 \\ m(\{0\}) & \text{if } r = 0 \\ m([br, ar)) & \text{if } r < 0 \end{cases} = \begin{cases} br - ar & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ ar - br & \text{if } r < 0 \end{cases} = |r|(b-a) = |r|\, m((a,b])$$

So $m^r$ and $|r| m$ agree on $\mathcal{B}_\mathbb{R}$.

Now, let $E \in \mathcal{L}$. Since $\mathcal{L}$ is the completion of $\mathcal{B}_\mathbb{R}$ with respect to $m$, we have $E = B \cup N$ where $B \in \mathcal{B}_\mathbb{R}$ and $N \subset C \in \mathcal{B}_\mathbb{R}$ with $m(C) = 0$. Then $N + s \subset C + s$ and $Nr \subset Cr$ and these are still $m$-null sets. Thus $E + s = (B+s) \cup (N+s) \in \mathcal{L}$ and $Er = (Br) \cup (Nr) \in \mathcal{L}$. Using Theorem 1.18 we then have

$$m(E+s) = \inf\{m(U) : U \supset E+s \text{ open}\} = \inf\{m(U-s) : U - s \supset E \text{ open}\} = m(E)$$

and similarly $m(Er) = |r|\, m(E)$. $\square$

Recall that

$$m(E) = \inf\left\{ \sum_{n=1}^\infty (b_n - a_n) : E \subset \bigcup_{n=1}^\infty (a_n, b_n] \right\}$$

Hence an $m$-null set is a null set precisely in the sense of the Riemann-Lebesgue theorem. Thus any countable set $E \subset \mathbb{R}$ is an $m$-null set by Exercise 2 on Homework 1. Consequently, $m(\mathbb{Q}) = 0$ and

$$m([0,1] \setminus \mathbb{Q}) = m([0,1]) - m([0,1] \cap \mathbb{Q}) = 1 - 0 = 1.$$

So the irrationals in $[0,1]$ have full measure. However, some uncountable sets are $m$-null.

**Ex** (Cantor Set) Recursively define a sequence of sets $[0,1] =: C_0 \supset C_1 \supset C_2 \supset \cdots$ by

$$C_n := C_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \left( \frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

So

$$C_1 = [0,1] \setminus \left(\tfrac{1}{3},\tfrac{2}{3}\right) = \left[0,\tfrac{1}{3}\right] \cup \left[\tfrac{2}{3},1\right]$$
$$C_2 = C_1 \setminus \left(\left(\tfrac{1}{9},\tfrac{2}{9}\right) \cup \left(\tfrac{4}{9},\tfrac{5}{9}\right) \cup \left(\tfrac{7}{9},\tfrac{8}{9}\right)\right) = \left[0,\tfrac{1}{9}\right] \cup \left[\tfrac{2}{9},\tfrac{1}{3}\right] \cup \left[\tfrac{2}{3},\tfrac{7}{9}\right] \cup \left[\tfrac{8}{9},1\right]$$

The Cantor set is then
$$C := \bigcap_{n=1}^{\infty} C_n$$
Note that $C$ is closed as the intersection of closed sets, and hence is compact since it is also bounded. Any $x \in [0,1]$, has a base-3 expansion
$$x = a_1 3^{-1} + a_2 3^{-2} + \cdots \qquad a_1, a_2, \dots \in \{0,1,2\}$$
This unique except if $\exists k$ with $a_k \neq 2$ and $a_\ell = 2 \ \forall \ell > k$, in which case
$$0 \cdot 3^{-k} + 2 3^{-(k+1)} + 2 \cdot 3^{-(k+2)} + \cdots = 1 \cdot 3^{-k} + 0 \cdot 3^{-(k+1)} + 0 \cdot 3^{-(k+2)} + \cdots$$
$$1 \cdot 3^{-k} + 2 3^{-(k+1)} + 2 \cdot 3^{-(k+2)} + \cdots = 2 \cdot 3^{-k} + 0 \cdot 3^{-(k+1)} + 0 \cdot 3^{-(k+2)} + \cdots$$
We adopt the convention of using the expansion with no 1's: the left side when $a_k = 0$ and the right side when $a_k = 1$. Write $x = \sum a_k 3^{-k}$ for this unique expansion. Then
$$C = \left\{ \sum a_k 3^{-k} : a_k \in \{0,2\} \right\}$$

Indeed,
$$\bigcup_{k=0}^{3^{n-1}-1} \left( \frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right) = \left\{ \sum a_k 3^{-k} : a_n = 1 \right\}$$
so that
$$C_n = \left\{ \sum a_k 3^{-k} : a_1, \dots, a_n \in \{0,2\} \right\}.$$
This presentation of $C$ implies it is uncountable either by a diagonalization argument or by observing that $\sum a_k 3^{-k} \mapsto \sum b_k 2^{-k}$, where $b_k = 0$ if $a_k = 0$ and $b_k = 1$ otherwise, defines a surjection from $C$ onto $[0,1]$. However, $m(C) = 0$. Indeed, by continuity from above (note $m(C_0) < \infty$) we have $m(C) = \lim_{n\to\infty} m(C_n)$. Using induction, one can show $C_n$ is the disjoint union of $2^n$ intervals of length $\frac{1}{3^n}$. Thus $m(C_n) = \left(\frac{2}{3}\right)^n \to 0 = m(C)$. $\square$

Note that the function $\sum a_k 3^{-k} \mapsto \sum b_k 2^{-k}$ defined in the previous example is strictly increasing on $C$ except on the $x$'s with a finite expansion, which by our convention means the last non-zero coefficient is 2:
$$x = a_1 3^{-1} + \cdots + 2 3^{-N} \mapsto b_1 2^{-1} + \cdots + 1 \cdot 2^{-N}$$

Since $2^{-N} = 2^{-(N+1)} + 2^{-(N+2)} + \cdots$, $x$ has the same image as
$$y = a_1 3^{-1} + \cdots + 0 \cdot 3^{-N} + 2 \cdot 3^{-(N+1)} + 2 \cdot 3^{-(N+2)} + \cdots = x - 3^{-N}$$
Note that $x$ and $y$ are endpoints of one of the intervals removed from $[0,1]$ to form $C$. Define $f : [0,1] \to [0,1]$ to be this map on $C$ and constant on the missing intervals: $f(z) := f(x) = f(y)$ for $z \in (y,x)$. Then $f$ is increasing and onto, hence continuous (Exercise: prove this). This function is called the Cantor function, and will be a rich source of examples moving forward.