## Exercises:

§20, 21

1. Let $X$ be a metric space with metric $d$. Prove the **reverse triangle inequality**: for all $x, y, z \in X$

$$|d(x,y) - d(y,z)| \leq d(x,z).$$

2. Recall that the uniform metric on $\mathbb{R}^{\mathbb{N}}$ is defined as

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup_{n \in \mathbb{N}} \overline{d}(x_n, y_n),$$

where $\overline{d}(x_n, y_n) = \min\{|x_n - y_n|, 1\}$ is the standard bounded metric on $\mathbb{R}$.

   (a) Show that $\overline{\rho}$ is a metric.

   (b) Let $C \subset \mathbb{R}^{\mathbb{N}}$ be the subset from Exercise 4 on Homework 6. Determine $\overline{C}$ when $\mathbb{R}^{\mathbb{N}}$ has the topology induced by $\overline{\rho}$.

   (c) Let $h\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the function from Exercise 1 on Homework 7. Find necessary and sufficient conditions on the sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ which guarantee $h$ is continuous when $\mathbb{R}^{\mathbb{N}}$ has the topology induced by $\overline{\rho}$.

   (d) For $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ and $\epsilon > 0$, show that

$$U := (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots$$

   is **not** open with respect to the topology induced by $\overline{\rho}$.

3. Let $X$ be a metric space with metric $d$. For fixed $x_0 \in X$, show that the function $f\colon X \to \mathbb{R}$ defined by $f(x) = d(x, x_0)$ is continuous.

4. Let $X$ be a metric space with metric $d$, and let $(x_i)_{i\in I} \subset X$ be a net.

   (a) Show that $(x_i)_{i\in I}$ converges to $x_0 \in X$ if and only if the net $(d(x_i, x_0))_{i\in I} \subset \mathbb{R}$ converges to 0.

   (b) Show that if $(x_i)_{i\in I}$ converges to $x_0 \in X$, then one can find a sequence $(x_n)_{n\in\mathbb{N}} \subset \{x_i \mid i \in I\}$ converging to $x_0$.

5. For each $n \in \mathbb{N}$, define $f_n\colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{1}{1 + (x-n)^2}.$$

   Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to the zero function pointwise but **not** uniformly.

6*. Let $\ell^2 \subset \mathbb{R}^{\mathbb{N}}$ be the set of sequences $(x_n)_{n\in\mathbb{N}}$ for which the series $\sum_{n=1}^{\infty} x_n^2$ converges. For $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \ell^2$ denote

$$\|\mathbf{x}\|_2 := \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}.$$

   (a) For $\mathbf{x} \in \ell^2$ and $c \in \mathbb{R}$, show that $c\mathbf{x} \in \ell^2$ with $\|c\mathbf{x}\|_2 = |c|\|\mathbf{x}\|_2$.

   (b) For $\mathbf{x}, \mathbf{y} \in \ell^2$, show that the series $\sum_{n=1}^{\infty} |x_n y_n|$ converges and is bounded by $\|\mathbf{x}\|_2 \|\mathbf{y}\|_2$.

   (c) For $\mathbf{x}, \mathbf{y} \in \ell^2$, show that $\mathbf{x} + \mathbf{y} \in \ell^2$ with $\|\mathbf{x} + \mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$.

   (d) Show that $d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2$ defines a metric on $\ell^2$.

   (e) Show that the topology induced by $d_2$ is finer than the uniform topology but coarser than the box topology on $\ell^2$.

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## Solutions:

1. Let $x, y, z \in X$. We must show

$$-d(x,z) \leq d(x,y) - d(y,z) \leq d(x,z).$$

Observe that the first inequality is equivalent to $d(y,z) \leq d(x,y) + d(x,z)$. Using the symmetry of $d$, this is futher equivalent to $d(y,z) \leq d(y,x) + d(x,z)$, which holds by the usual triangle inequality on $d$. The second inequality above is equivalent to $d(x,y) \leq d(x,z) + d(y,z)$. Once more using the symmetry of $d$ this is further equivalent to $d(x,y) \leq d(x,z) + d(z,y)$, which again follows from the usual triangle inequality on $d$. □

2. (a) Clearly $\overline{\rho}(\mathbf{x},\mathbf{y}) \geq 0$ and if $\mathbf{x} = \mathbf{y}$ then we have equality. Conversely, if $\overline{\rho}(\mathbf{x},\mathbf{y}) = 0$ then we must have $\overline{d}(x_n, y_n) = 0$ for all $n \in \mathbb{N}$, which implies $x_n = y_n$ for all $n \in \mathbb{N}$ or $\mathbf{x} = \mathbf{y}$. The symmetry $\overline{\rho}(\mathbf{x},\mathbf{y}) = \overline{\rho}(\mathbf{y},\mathbf{x})$ follows from the symmetry of $\overline{d}$. Finally, let $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^{\mathbb{N}}$. For each $n \in \mathbb{N}$ we have by the triangle inequality applied to $\overline{d}$

$$\overline{d}(x_n, z_n) \leq \overline{d}(x_n, y_n) + \overline{d}(y_n, z_n) \leq \overline{\rho}(\mathbf{x},\mathbf{y}) + \overline{\rho}(\mathbf{y},\mathbf{z}).$$

Consequently,

$$\overline{\rho}(\mathbf{x},\mathbf{z}) = \sup_{n\in\mathbb{N}} \overline{d}(x_n, z_n) \leq \overline{\rho}(\mathbf{x},\mathbf{y}) + \overline{\rho}(\mathbf{y},\mathbf{z}).$$

So $\overline{\rho}$ satisfies the triangle inequality and consequently is a metric. □

(b) Let $D \subset \mathbb{R}^{\mathbb{N}}$ be the set of sequences $(x_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} x_n = 0$. We claim $\overline{C} = D$. We first show that $D$ is closed with respect to the topology induced by $\overline{\rho}$. Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in D^c$, then the sequence $(x_n)_{n\in\mathbb{N}}$ does **not** converge to zero. This means there exists $\epsilon > 0$ so that for any $N \in \mathbb{N}$ there exists $n \geq N$ with $|x_n - 0| = |x_n| \geq \epsilon$. Note that this still holds after replacing $\epsilon$ with $\min\{\epsilon, 1\}$, so we may assume $\epsilon \leq 1$, in which case we have $\overline{d}(x_n, 0) \geq \epsilon$. We claim that $B_{\overline{\rho}}(\mathbf{x}, \frac{\epsilon}{2}) \subset D^c$. Indeed, for $\mathbf{y}$ in this ball we have for all $n \in \mathbb{N}$ $\overline{d}(x_n, y_n) \leq \overline{\rho}(\mathbf{x},\mathbf{y}) < \frac{\epsilon}{2}$. Applying Exercise 1 to $\overline{d}$ yields

$$|y_n| \geq \overline{d}(0, y_n) \geq |\overline{d}(0, x_n) - \overline{d}(x_n, y_n)| \geq \overline{d}(0, x_n) - \overline{d}(x_n, y_n) > \overline{d}(0, x_n) - \frac{\epsilon}{2}.$$

Thus whenever $n \in \mathbb{N}$ is such that $\overline{d}(x_n, 0) \geq \epsilon$, we have $|y_n| \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$. Since this happens infinitely often for $(x_n)_{n\in\mathbb{N}}$, it follows that $(y_n)_{n\in\mathbb{N}}$ does not converge to zero. Thus $\mathbf{y} \in D^c$ and therefore $B_{\overline{\rho}}(\mathbf{x}, \frac{\epsilon}{2}) \subset D^c$. Since $\mathbf{x} \in D^c$ was arbitrary, this shows $D^c$ is open in the topology induced by $\overline{\rho}$.

Thus $D$ is a closed set and since $C \subset D$ we have $\overline{C} \subset D$. For the reverse inclusion, let $\mathbf{x} \in D$. We'll show that $B_{\overline{\rho}}(\mathbf{x}, \epsilon) \cap C \neq \emptyset$ for all $\epsilon > 0$, which will imply $\mathbf{x} \in \overline{C}$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $|x_n| < \frac{\epsilon}{2}$ for all $n \geq N$. Consider $\mathbf{y} \in C$ defined by $y_n := x_n$ if $n < N$ and $y_n := 0$ if $n \geq N$. Thus for each $n \in \mathbb{N}$

$$\overline{d}(x_n, y_n) =\leq |x_n - y_n| = \begin{cases} 0 & \text{if } n < N \\ |x_n| & \text{if } n \geq N \end{cases} < \frac{\epsilon}{2}.$$

Consequently, $\overline{\rho}(\mathbf{x},\mathbf{y}) \leq \frac{\epsilon}{2} < \epsilon$ and thus $\mathbf{y} \in B_{\overline{\rho}}(\mathbf{x}, \epsilon)$. This shows $B_{\overline{\rho}}(\mathbf{x}, \epsilon) \cap C \neq \emptyset$ and therefore $\mathbf{x} \in \overline{C}$. Therefore $D \subset \overline{C}$ and hence $D = \overline{C}$. □

(c) We claim that $h$ is continuous if and only if $(a_n)_{n\in\mathbb{N}}$ is bounded. First suppose $(a_n)_{n\in\mathbb{N}}$ is bounded and let $A := \sup_{n\in\mathbb{N}} a_n$. Recall that $a_n > 0$ for all $n \in \mathbb{N}$ and so $A > 0$. We will use the $\epsilon$-$\delta$ definition of continuity. Let $0 < \epsilon < 1$ and set $\delta := \frac{\epsilon}{A+1}$ so that $0 < \delta < 1$. If $\mathbf{x},\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ satisfy $\overline{\rho}(\mathbf{x},\mathbf{y}) < \delta$, then $\overline{d}(x_n, y_n) < \delta < 1$ for all $n \in \mathbb{N}$ and therefore $\overline{d}(x_n, y_n) = |x_n - y_n|$. Now, for such $\mathbf{x}, \mathbf{y}$ we have

$$\begin{aligned}
\overline{\rho}(h(\mathbf{x}), h(\mathbf{y})) &= \overline{\rho}\left((a_n x_n + b_n)_{n\in\mathbb{N}}, (a_n y_n + b_n)_{n\in\mathbb{N}}\right) \\
&= \sup_{n\in\mathbb{N}} \overline{d}(a_n x_n + b_n, a_n y_n + b_n) \\
&\leq \sup_{n\in\mathbb{N}} |a_n x_n + b_n - (a_n y_n + b_n)| \\
&= \sup_{n\in\mathbb{N}} a_n |x_n - y_n| \\
&\leq A \sup_{n\in\mathbb{N}} |x_n - y_n| \\
&= A \sup_{n\in\mathbb{N}} \overline{d}(x_n, y_n) \\
&= A\overline{\rho}(\mathbf{x}, \mathbf{y}) < A\delta < \epsilon.
\end{aligned}$$

Thus $h$ is continuous.

We will argue the converse by contrapositive. Suppose $(a_n)_{n\in\mathbb{N}}$ is not bounded. Let $\mathbf{0}$ be the sequence of all zeros and let $\epsilon = 1$. For any $\delta > 0$ let $\mathbf{y}$ be the constant sequence $(\frac{\delta}{2}, \frac{\delta}{2}, \ldots)$. Then

$$\overline{\rho}(\mathbf{0}, \mathbf{y}) = \sup_{n\in\mathbb{N}} \overline{d}(0, \frac{\delta}{2}) \leq \frac{\delta}{2} < \delta,$$

however,

$$\overline{\rho}(h(\mathbf{0}), h(\mathbf{y})) = \sup_{n\in\mathbb{N}} \overline{d}(0 + b_n, a_n\frac{\delta}{2} + b_n) = \sup_{n\in\mathbb{N}} \left(\min\{a_n \frac{\delta}{2}, 1\}\right).$$

Since $(a_n)_{n\in\mathbb{N}}$ is unbounded, there exists $n \in \mathbb{N}$ so that $a_n \frac{\delta}{2} \geq 1$ and consequently $\overline{\rho}(h(\mathbf{0}), h(\mathbf{y})) \geq 1 = \epsilon$. This shows $h$ is not continuous at $\mathbf{0}$. □

(d) Consider $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ defined by $y_n := x_n + \left(1 - \frac{1}{n}\right)\epsilon$. Then $y_n \in (x_n, x_n + \epsilon)$ for all $n \in \mathbb{N}$ and thus $\mathbf{y} \in U$. We will prove that $U$ is not open by showing $B_{\overline{\rho}}(\mathbf{y}, \delta)$ fails to be contained in $U$ for any $\delta > 0$. Let $\delta > 0$. Let $N \in \mathbb{N}$ satisfying $N > \frac{\epsilon}{\delta}$. Note that this implies $\frac{1}{N}\epsilon < \delta$. Let $\mathbf{z} \in \mathbb{R}^{\mathbb{N}}$ be defined by $z_n := y_n + \frac{1}{N}\epsilon$ so that

$$\overline{\rho}(\mathbf{z}, \mathbf{y}) \leq \sup_{n\in\mathbb{N}} |z_n - y_n| = \sup_{n\in\mathbb{N}} \frac{1}{N}\epsilon = \frac{1}{N}\epsilon < \delta.$$

Thus $\mathbf{z} \in B_{\overline{\rho}}(\mathbf{y}, \delta)$. However, $\mathbf{z} \notin U$. Indeed, we have

$$z_N = y_N + \frac{1}{N}\epsilon = x_N + \left(1 - \frac{1}{N}\right)\epsilon + \frac{1}{N}\epsilon = x_N + \epsilon \notin (x_N - \epsilon, x_N + \epsilon).$$

Thus $B_{\overline{\rho}}(\mathbf{y}, \delta)$ is not contained in $U$. Since $\delta > 0$ was arbitrary, this shows $U$ is not open in the topology induced by $\overline{\rho}$. □

3. We will use the $\epsilon$-$\delta$ definition of continuity for metric spaces (where here we let $\mathbb{R}$ have the standard metric). Let $\epsilon > 0$. We claim that for $\delta = \epsilon$, if $x, y \in X$ satisfy $d(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon$. Indeed, using Exercise 1 we have

$$|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| = |d(x, x_0) - d(x_0, y)| \leq d(x, y) < \delta = \epsilon.$$

Thus $f$ is continuous. □

4. (a) Suppose $(x_i)_{i\in I}$ converges to $x_0$. Exercise 3, the function $x \mapsto d(x, x_0)$ is continuous and we know from lecture that continuous functions send convergent nets to convergent nets. Thus $(d(x_i, x_0))_{i\in I}$ converges to $d(x_0, x_0) = 0$.

Conversely, suppose $(d(x_i, x_0))_{i\in I}$ converges to zero. Let $U$ be a neighborhood of $x_0$. Then there exists $\epsilon > 0$ so that $B_d(x_0, \epsilon) \subset U$. Since $(-\epsilon, \epsilon)$ is a neighborhood of zero, there exists $i_0 \in I$ so that $i \geq i_0$ implies $d(x_i, x_0) \in (-\epsilon, \epsilon)$. In particular, $d(x_i, x_0) < \epsilon$ for all $i \geq i_0$ and therefore $x_i \in B_d(x_0, \epsilon) \subset U$. That is, for all $i \geq i_0$ we have $x_i \in U$. □

(b) For each $n \in \mathbb{N}$ the ball $B_d(x_0, \frac{1}{n})$ is a neighborhood of $x_0$. So the convergence of the net implies there exists $i_n \in I$ so that $i \geq i_n$ implies $x_i \in B_d(x_0, \frac{1}{n})$. Define $x_n := x_{i_n}$. Then

$$d(x_n, x_0) < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Thus $(d(x_n, x_0))_{n\in\mathbb{N}}$ converges to zero and so by part (a) the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_0$. □

5. We first show pointwise convergence. Fix $x \in \mathbb{R}$ and let $0 < \epsilon < 1$. Let $N \in \mathbb{N}$ satisfy $N > |x| + \left(\frac{1}{\epsilon} - 1\right)^{1/2}$. Then by the reverse triangle in equality we have for any $n \geq N$

$$|n - x| \geq |n - |x|| \geq n - |x| > \left(\frac{1}{\epsilon} - 1\right)^{1/2}.$$

Squaring both sides yields $(x - n)^2 > \frac{1}{\epsilon} - 1$. Adding one to each side and taking reciprocals then yields $f_n(x) < \epsilon$. Since $f_n(x) \geq 0$, it follows that for all $n \geq N$ we have $|f_n(x)| < \epsilon$. Thus $(f_n(x))_{n\in\mathbb{N}}$ converges to zero. Since $x \in \mathbb{R}$ was arbitrary, we see that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to zero.

Next we show the sequence of functions does not converge uniformly to zero. Consider $\epsilon = 1$. Then for any $n \in \mathbb{N}$ we have for $x = n$ that

$$|f_n(n) - 0| = |1 - 0| = 1 \geq \epsilon.$$

Hence $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to zero. □

6. (a) Observe that for $N \in \mathbb{N}$ we have $\sum_{n=1}^{N} (cx_n)^2 = c^2 \sum_{n=1}^{N} x_n^2$. Thus

$$\|c\mathbf{x}\|_2 = \left(c^2 \sum_{n=1}^{\infty} x_n^2\right)^{1/2} = |c| \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2} = |c|\|\mathbf{x}\|_2.$$

□

(b) For each $N \in \mathbb{N}$, we have by Exercise 6.(c) on Homework 7 (with $p = q = 2$t)

$$\sum_{n=1}^{N} |x_n y_n| \leq \left(\sum_{n=1}^{N} x_n^2\right)^{1/2} \left(\sum_{n=1}^{N} y_n^2\right)^{1/2} \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2.$$

Since this holds for all $N \in \mathbb{N}$, it follows that the series $\sum_{n=1}^{\infty} |x_n y_n|$ converges with the claimed bound. □

(c) Using the previous part, for each $N \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{n=1}^{N} (x_n + y_n)^2 &= \sum_{n=1}^{N} x_n^2 + 2x_n y_n + y_n^2 \\
&\leq \|\mathbf{x}\|_2^2 + 2\sum_{n=1}^{N} |x_n y_n| + \|\mathbf{y}\|_2^2 \\
&\leq \|\mathbf{x}\|_2^2 + 2\|\mathbf{x}\|_2\|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2 = \left(\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2\right)^2.
\end{aligned}$$

Since this holds for all $N \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} (x_n + y_n)^2$ converges. Moreover, the above computation also implies $\|\mathbf{x} + \mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$. □

(d) Note that $d_2$ is defined for all pairs $\mathbf{x}, \mathbf{y} \in \ell^2$ because $-\mathbf{y} \in \ell^2$ by part (a) and $\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) \in \ell^2$ by part (c). Now, The first two conditions in the definition of a metric are clear, and the triangle inequality follows from applying the previous part to $\mathbf{x} - \mathbf{z} = (\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})$. □

(e) Observe that for $\mathbf{x}, \mathbf{y} \in \ell^2$ and each $n \in \mathbb{N}$ we have

$$\overline{d}(x_n, y_n) \leq |x_n - y_n| = ((x_n - y_n)^2)^{1/2} \leq \left( \sum_{n=1}^{\infty} (x_n - y_n)^2 \right)^{1/2} = d_2(\mathbf{x}, \mathbf{y}).$$

Taking a supremum over $n \in \mathbb{N}$ on the left yields $\overline{\rho}(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y})$. It follows that for all $\epsilon > 0$ one has $B_{d_2}(\mathbf{x}, \epsilon) \subset B_{\overline{\rho}}(\mathbf{x}, \epsilon)$. Thus a lemma from lecture implies the topology induced by $d_2$ is finer than the topology induced by $\overline{\rho}$ (i.e. the uniform topology).

Next, let $\mathbf{x} \in \ell^2$ and $\epsilon > 0$. Recall that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Denote

$$\delta := \frac{\epsilon}{(\sum_{n=1}^{\infty} \frac{1}{n^2})^{1/2}}$$

Consider $U := \prod_{n \in \mathbb{N}} (x_n - \frac{\delta}{n}, x_n + \frac{\delta}{n})$, which is a neighborhood of $\mathbf{x}$ in the box topology. We claim that $U \cap \ell^2 \subset B_{d_2}(\mathbf{x}, \epsilon)$. Indeed, for $\mathbf{y} \in U \cap \ell^2$ we have

$$d_2(\mathbf{x}, \mathbf{y}) = \left( \sum_{n=1}^{\infty} (x_n - y_n)^2 \right)^{1/2} < \left( \sum_{n=1}^{\infty} \frac{\delta^2}{n^2} \right)^{1/2} = \delta \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} = \epsilon.$$

Thus $\mathbf{y} \in B_{d_2}(\mathbf{x}, \epsilon)$. Since sets of the form $U \cap \ell^2$ are part of a basis for the box topology on $\ell^2$, this shows that the topology induced by $d_2$ is coarser than the box topology. □