Exercises:

§9, 10, 11, 12

1. Let $f\colon A \to B$ be a function.
   (a) Use the axiom of choice to show that if f is surjective, then there exists $g\colon B \to A$ with $f \circ g(b) = b$ for all $b \in B$.
   (b) Without using the axiom of choice show that if f is injective, then there exists $h\colon B \to A$ with $h \circ f(a) = a$ for all $a \in A$.
2. Show that the well-ordering theorem implies the axiom of choice.
3. Let S_Ω be the minimal uncountable well-ordered set from §10.
   (a) Show that S_Ω has no largest element.
   (b) Show that for every $x \in S_\Omega$, the subset $\{y \in S_\Omega \mid x < y\}$ is uncountable.
   (c) Consider the subset
   $$X := \{x \in S_\Omega \mid (a, x) \neq \emptyset \text{ for all } a < x\}.$$
   Show that X is uncountable. [**Hint:** proceed by contradiction and use the fact that for any $y \in S_\Omega$ there exists $z \in S_\Omega$ with $(y, z) = \emptyset$.]
4. In this exercise you will use Zorn's lemma to prove the following fact from linear algebra: every vector space V has a basis. For a subset $A \subset V$, recall: the **span** of A is the set of all finite linear combinations of vectors in A; A is said to be **independent** if the only way to write the zero vector as a linear combination of elements in A is via the trivial linear combination with all zero scalar coefficients; and A is said to be a **basis** for V if it is independent and its span is all of V.
   (a) Suppose $A \subset V$ is independent. Show that if v is not in the span of A, then $A \cup \{v\}$ is independent.
   (b) Show that the collection of independent subsets of V, ordered by inclusion, has a maximal element.
   (c) Show that V has a basis.
5. Let X be a topological space and let $A \subset X$ be a subset. Suppose that for all $x \in A$, there exists an open set U satisfying $x \in U \subset A$. Show that A is open.

Solutions:

1. (a) Let $\mathcal{A}$ be the collection of subsets of the form $f^{-1}(\{b\})$ for $b \in B$. Since f is surjective, each set in this collection is non-empty. Moreover, if $b \neq b'$, then $f^{-1}(\{b\})$ and $f^{-1}(\{b'\})$ are disjoint since any common element $a \in A$ would satisfy $f(a) = b$ and $f(a) = b'$, contradicting $b \neq b'$. Thus $\mathcal{A}$ is a collection of disjoint non-empty sets. Let C be as in the axiom of choice: $C \cap f^{-1}(\{b\})$ consists of exactly one element, which we will denote by $g(b)$. Thus $g\colon B \to A$ is a function, and for $b \in B$ we have $f \circ g(b) = f(g(b)) = b$ since $g(b) \in f^{-1}(b)$. $\square$

   (b) Fix an arbitrary $a_0 \in A$. Since f is injective, if $b \in f(A)$ then there is a *unique* $a \in A$ with $f(a) = b$. Let us denote this unique element by a_b. Define $h\colon B \to A$ by
   $$h(b) := \begin{cases} a_b & \text{if } b \in f(A) \\ a_0 & \text{otherwise} \end{cases}.$$
   For $a \in A$, note that $a_{f(a)} = a$ since a is the unique element whose image under f is $f(a)$. Thus we have $h(f(a)) = a_{f(a)} = a$. $\square$

2. Let $\mathcal{A}$ be a collection of disjoint non-empty sets. Define

$$X := \bigcup_{A \in \mathcal{A}} A.$$

By the well-ordering theorem, there exists an order relation $<$ on X making it well-ordered. For each $A \in \mathcal{A}$, we have $A \subset X$ and A is non-empty by assumption. Consequently, by the well-ordering there exists a smallest element in A, which we denote x_A. Define $C := \{x_A \mid A \in \mathcal{A}\}$. Then for all $A \in \mathcal{A}$ we have $\{x_A\} \subset C \cap A$. If this reverse containment does not hold, then by definition of C, we must have $x_B \in C \cap A$ for some other $B \in \mathcal{A}$. But then $x_B \in A \cap B$ contradicts the set in $\mathcal{A}$ being disjoint. Thus $C \cap A = \{x_A\}$, and so the axiom of choice holds. □

3. (a) Suppose, towards a contradiction, that $x_0 \in S_\Omega$ is a largest element. This means means $x \leq x_0$ for all $x \in S_\Omega$ and so

$$S_\Omega = \{x \mid x < x_0\} \cup \{x_0\}.$$

However, $\{x \mid x < x_0\} = S_{x_0}$ is the section of S_Ω by x_0 and is countable by definition of S_Ω. So the above equality shows S_Ω is a finite union of countable sets and hence is countable, a contradiction. Thus S_Ω has no largest element. □

(b) Fix $x \in S_\Omega$ and observe

$$S_\Omega = \{y \mid y < x\} \cup \{x\} \cup \{y \mid x < y\} = S_x \cup \{x\} \cup \{y \mid x < y\}.$$

The first two sets in the last union are countable, and so we must have $\{y \mid x < y\}$ lest we contradict S_Ω being uncountable. □

(c) Suppose, towards a contradiction, that X is countable. Then by the last theorem in §10 we know X is bounded above by some $y_1 \in S_\Omega$. Let y_2 be the smallest element of the set $\{y \mid y_1 < y\}$. This exists because this set is non-empty and in fact uncountable by the previous part and because S_Ω is well-ordered. Note that $(y_1, y_2) = \emptyset$ since any $y \in S_\Omega$ satisfying $y_1 < y < y_2$ would contradict y_2 being the smallest element. We then inductively define a sequence $(y_n)_{n\in\mathbb{N}} \subset S_\Omega$ by letting y_n be the smallest element of $\{y \mid y_{n-1} < y\}$, which exists by the same reasoning used for y_2. Likewise, we have $(y_{n-1}, y_n) = \emptyset$ for all $n \in \mathbb{N}$. Now, $B := \{y_n \mid n \in \mathbb{N}\}$ is a countable subset of S_Ω and is therefore bounded above by the last theorem in §10. Since S_Ω is well-ordered, the supremum of B exists, which we denote by y_∞.

$$y_1 < y_2 < y_3 < \cdots < y_n < y_{n+1} \cdots \leq y_\infty.$$

Consequently, $y_n < y_\infty$ for all $n \in \mathbb{N}$. We claim that $y_\infty \in X$. Indeed, if $y_\infty \notin X$ then there exists $a < y_\infty$ with $(a, y_\infty) = \emptyset$. Since y_∞ is the smallest upper bound for B, it must be that $a \leq y_n$ for some $n \in \mathbb{N}$. Since $y_n < y_\infty$ and $(a, y_\infty) = \emptyset$, we must have $a = y_n$. But then $a = y_n < y_{n+1} < y_\infty$ contradicts $(a, y_\infty) = \emptyset$. Thus we must have $y_\infty \in X$. However, this contradicts y_1 being an upper bound for X. Thus X cannot be bounded above in S_Ω and is therefore uncountable. □

4. (a) Suppose, towards a contradiction, that $A \cup \{v\}$ is not independent. Then there exist distinct vectors $v_1, \ldots, v_n \in A \cup \{v\}$ and non-zero scalars $c_1, \ldots, c_n$ such that $c_1 v_1 + \cdots + c_n v_n = 0$. Since A is independent, we cannot have $v_1, \ldots, v_n \in A$. Thus one of $v_1, \ldots, v_n$ equals v, say $v_n = v$. But then

$$v = -\frac{c_1}{c_n} v_1 + \cdots + -\frac{c_{n-1}}{c_n} v_{n-1},$$

which contradicts v not being in the span of A. Thus $A \cup \{v\}$ is independent. □

(b) Let $\mathcal{I}$ denote the collection of independent subsets of V. Zorn's lemma will give us the existence of a maximal element if we can show that every chain in $\mathcal{I}$ has an upperbound. Let $\mathcal{C} \subset \mathcal{I}$ be a chain (i.e. a totally ordered subset). Consider the union

$$B := \bigcup_{A \in \mathcal{C}} A.$$

Since $A \subset B$ for all $\mathcal{C}$, if we can show that $B \in \mathcal{I}$ (i.e. that B is independent) then it will be an upper bound for $\mathcal{C}$. Let $v_1, \ldots, v_n \in B$ and suppose $c_1, \ldots, c_n$ are non-zero scalars. We must show $c_1 v_1 + \cdots + c_n v_n \neq 0$. By definition of B, for each $j = 1 \ldots, n$ there is some $A_j \in \mathcal{C}$ with $v_j \in A_j$. Since $\mathcal{C}$ is a chain, we have $A_i \subset A_j$ or $A_j \subset A_i$ for each $1 \leq i, j \leq n$. Consequently, there exists $1 \leq j \leq n$ such that $A_i \subset A_j$ for all $1 \leq i \leq n$. (To find this j, start with $j = 1$, then if $A_1 \subset A_2$ take $j = 2$ otherwise leave $j = 1$, and proceed in this way through all the indices.) So we therefore have $v_1, \ldots, v_n \in A_j$ and since A_j is independent we obtain $c_1 v_1 + \cdots + c_n v_n \neq 0$. Thus B is independent (i.e. $B \in \mathcal{I}$) and is an upper bound for the chain $\mathcal{C}$. Zorn's lemma then completes the proof. □

(c) Using the notation from the previous part, let $B \in \mathcal{I}$ be a maximal element, which exists by the previous part. We claim that B is a basis for V. Indeed, $B \in \mathcal{I}$ implies B is independent. If the span of B is not all V, then there exists $v \in V$ not in the span of B. By part (a), $B \cup \{v\}$ is an independent set and it is strictly larger than B, contradicting the maximality of B. Thus we must have that the span of B is all of V, and therefore B is a basis for V as claimed. □

5. For $x \in A$, let U_x be the open set satisfying $x \in U_x \subset A$. Observe

$$A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} U_x \subset A.$$

Thus the above inclusions must in fact be equalities. In particular, $A = \bigcup_{x \in A} U_x$ is open as a union of open sets. □