Exercises:

§27, 28, 31

1. Let X be a compact topological space and (Y, d) a metric space. Let $C(X, Y)$ denote the set of all continuous functions $f: X \to Y$.

   (a) For $f, g \in C(X, Y)$, show that $h: X \to \mathbb{R}$ defined by $h(x) := d(f(x), g(x))$ is continuous.

   (b) Show that $D(f, g) := \sup_{x \in X} d(f(x), g(x))$ exists and defines a metric on $C(X, Y)$.

   (c) Let $\varphi: X \to X$ be a continuous function. Show that the map $\Phi: C(X, Y) \to C(X, Y)$ defined by $\Phi(f) := f \circ \varphi$ is uniformly continuous with respect to the metric D.

2. Let (X, d) be a metric space. For $x \in X$ and nonempty $A \subset X$, recall that $d(x, A) := \inf_{a \in A} d(x, a)$.

   (a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

   (b) Suppose $A \subset V$ for A compact and V open. Show that there exists $\epsilon > 0$ so that

   $$\bigcup_{a \in A} B_d(a, \epsilon) \subset V.$$

   [**Hint:** consider the function $f(x) = d(x, X \setminus V)$.]

3. Let (X, d) be a compact metric space and let $f: X \to X$ be a function satisfying $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. (We call such a function an **isometry**.) Show that f is a homeomorphism.

4. Let X be a normal topological space and let $A, B \subset X$ be disjoint closed subsets of X. Show that there are open subsets $U, V \subset X$ satisfying $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

5. Let X be a normal topological space. We say $A \subset X$ is a G_δ set if it is a countable intersection of open sets. Show that $A \subset X$ is a closed G_δ set if and only if there exists a continuous function $f: X \to [0, 1]$ with $f(x) = 0$ for all $x \in A$ and $f(x) > 0$ for all $x \notin A$. [**Hint:** use Urysohn's Lemma.]

6*. For $d \in \mathbb{N}$ and $a = 0, 1, \ldots, d-1$ define

$$U_{d,a} := \{dn + a \mid n \in \mathbb{Z}\} \subset \mathbb{Z}.$$

In this exercise you will use topology to show that there are infinitely many prime numbers.

   (a) Show that the collection $\mathcal{B} := \{U_{d,a} \mid d \in \mathbb{N},\ a = 0, 1, \ldots, d-1\}$ forms a basis for a topology on $\mathbb{Z}$.

   (b) Show that $U_{d,a}$ is clopen in this topology.

   (c) Show that if $U \subset \mathbb{Z}$ is nonempty and open in this topology, then U is infinite.

   (d) Let $P \subset \mathbb{N}$ be the subset of prime numbers. Consider

   $$A := \bigcup_{p \in P} U_{p,0}.$$

   Show that $\mathbb{Z} \setminus A$ is finite.

   (e) Deduce that P is infinite.

Solutions:

1. (a) Observe that h is the composition of the functions $f \times g: X \to Y \times Y$ and $d: Y \times Y \to \mathbb{R}$. The former is continuous by Exercise 1 on Homework 6, and the latter is continuous by Exercise 4 on Homework 7. Hence h is continuous. □

(b) By the previous part, $x \mapsto d(f(x), g(x))$ is continuous. Since X is compact, the extreme value theorem implies the supremum $D(f, g)$ is achieved, and in particular exists. Now, $D(f, g) \geq 0$ since $d(f(x), g(x)) \geq 0$ for all $x \in X$. If $f = g$, then $d(f(x), g(x)) = 0$ for all $x \in X$ and hence $D(f, g) = 0$. Conversely, if $D(f, g) = 0$, then $d(f(x), g(x)) = 0$ for all $x \in X$, which means $f(x) = g(x)$ for all $x \in X$. That is, $f = g$. The symmetry $D(f, g) = D(g, f)$ follows from the corresponding symmetry of d. Finally, for $f, g, h \in C(X, Y)$ and each $x \in X$ we have

$$d(f(x), h(x)) \leq d(f(x), g(x)) + d(g(x), h(x)) \leq D(f, g) + D(g, h).$$

Taking a supremum over $x \in X$ on the left yields $D(f, h) \leq D(f, g) + D(g, h)$. Hence D is a metric.

(c) Let $\epsilon > 0$. Set $\delta = \epsilon$, and suppose $f, g \in C(X, Y)$ satisfy $D(f, g) < \delta = \epsilon$, then $d(f(x), g(x)) \leq D(f, g)$ for all $x \in X$. In particular, this holds for all $x \in \varphi(X)$: $d(f(\varphi(x)), g(\varphi(x))) \leq D(f, g)$ for all $x \in X$. Therefore

$$D(\Phi(f), \Phi(g)) = \sup_{x \in X} d(f \circ \varphi(x), g \circ \varphi(x)) \leq D(f, g) < \delta = \epsilon$$

Hence Φ is uniformly continuous. □

2. (a) We know $x \in \overline{A}$ if and only if $B_d(x, \epsilon) \cap A \neq \emptyset$ for all $\epsilon > 0$. The latter is equivalent to $d(x, A) < \epsilon$ for all $\epsilon > 0$. Since $d(x, A) \geq 0$, this is equivalent to $d(x, A) = 0$. □

(b) Define $f \colon X \to \mathbb{R}$ by $f(x) := d(x, X \setminus V)$, which is continuuous by a lemma from lecture. For each $a \in A \subset V$, there exists $\delta > 0$ so that $B_d(a, \delta) \subset V$ and hence $d(a, y) \geq \epsilon$ for all $y \in X \setminus B_d(x, \epsilon) \supset X \setminus V$. Consequently, $f(a) > 0$ for all $a \in A$. Since A is compact, the extreme value theorem implies f attains a smallest value on A, say at $a_0 \in A$: $f(a_0) \leq f(a)$ for all $a \in A$. Set $\epsilon := f(a_0)$, which is strictly positive by our above argument. We claim

$$\bigcup_{a \in A} B_d(a, \epsilon) \subset V.$$

Indeed, otherwise $B_d(a, \epsilon) \cap (X \setminus V) \neq \emptyset$ for some $a \in A$. Thus $d(a, y) < \epsilon$ for some $y \in X \setminus V$ and therefore

$$f(a) = d(a, X \setminus V) \leq d(a, y) < \epsilon = f(a_0),$$

contradicting $f(a_0)$ being the smallest element. □

3. If $f(x) = f(y)$, then

$$d(x, y) = d(f(x), f(y)) = 0,$$

so that $x = y$. Thus f is injective. We also note that f is continuous according to the ϵ-δ definition of continuity for metric spaces by simply choosing $\delta = \epsilon$.

Suppose, towards a contradiction, that f is not surjective. Then there exists $y \in X \setminus f(X)$. Since X is compact and f is continuous, $f(X)$ is compact and in particular closed (since metric spaces are Hausdorff). Consequently, $X \setminus f(X)$ is open and so there exists $\epsilon > 0$ so that $B_d(y, \epsilon) \subset X \setminus f(X)$. For each $n \in \mathbb{N}$, define $x_n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(y)$. Thus $(x_n)_{n \in \mathbb{N}} \subset X$ is a sequence and for $m < n$ we have

$$d(x_n, x_m) = d(f(x_{n-1}), f(x_{m-1})) = d(x_{n-1}, x_{m-1}) = \cdots = d(x_{n-m+1}, x_1) = d(f(x_{n-m}), f(y)) = d(x_{n-m}, y).$$

Since $x_{n-m} = f(x_{n-m-1}) \in f(X)$, the above distance is at least ϵ. Thus $d(x_n, x_m) > \epsilon$ for all distinct $n, m \in \mathbb{N}$. Now, since X is compact, the sequence $(x_n)_{n \in \mathbb{N}}$ necessarily has a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$, say converging to $x \in X$. Thus there exists $K \in \mathbb{N}$ so that $k \geq K$ implies

$$d(x_{n_k}, x) < \frac{\epsilon}{2}.$$

But then for $k, \ell \geq K$ we have

$$d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, x) + d(x, x_{n_\ell}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

contradicting $d(x_n, x_m) \geq \epsilon$ for all $n, m \in \mathbb{N}$. Thus f must be surjective, and therefore bijective.

It remains to show $f^{-1}\colon X \to X$ is continuous. But if $f(x) = a$ and $f(y) = b$ then

$$d(f^{-1}(a), f^{-1}(a)) = d(x, y) = d(f(x), f(y)) = d(a, b).$$

Thus f^{-1} is an isometry and is therefore continuous by the same argument as above. □

4. Since X is normal, there exists disjoint open sets $U_1, V_1 \subset X$ with $A \subset U_1$ and $B \subset V_1$. By a proposition from the lecture on §31, there exists a neighborhood U of A satisfying $\overline{U} \subset U_1$. Set $V := V_1$, and note that $V \subset X \setminus U_1$, which is a closed set. Hence $\overline{V} \subset X \setminus U_1 \subset X \setminus \overline{U}$. Consequently, $\overline{U} \cap \overline{V} = \emptyset$. □

5. ($\Longrightarrow$): Suppose A is a closed G_δ set. Let $\{U_n \colon n \in \mathbb{N}\}$ be open subsets of X with

$$\bigcap_{n\in\mathbb{N}} U_n = A.$$

Using Urysohn's Lemma, for each $n \in \mathbb{N}$ we can find a continuous function $f_n\colon X \to [0, 1]$ with $f_n\mid_A \equiv 0$ and $f_n\mid_{X\setminus U_n} \equiv 1$. Define $f\colon X \to [0, 1]$ by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

Note that $0 \leq f_n(x) \leq 1$ implies

$$0 \leq f(x) \leq \sum_{n=1}^{\infty} 2^{-n} = 1,$$

so f is well-defined. Also, $f(x) = 0$ for all $x \in A$ since $f_n(x) = 0$. On the other hand, if $x \notin A$ then there must be some $n \in \mathbb{N}$ such that $x \notin U_n$. Consequently, $f_n(x) = 1$ and so $f(x) \geq 2^{-n} > 0$. It remains to show that f is continuous. We will show that the partial sums

$$S_N(x) := \sum_{n=1}^{N} 2^{-n} f_n(x)$$

converge uniformly to f. Since each S_N is continuous (as a finite sum of continuous functions), the Uniform Limit Theorem will imply f is continuous. Let $\epsilon > 0$. Let $N_0 \in \mathbb{N}$ be large enough so that $2^{-N_0} < \epsilon$. Then for all $N \geq N_0$

$$\sum_{n=N+1} 2^{-n} = 2^{-N} \sum_{n=1}^{\infty} 2^{-n} = 2^{-N} < \epsilon.$$

Consequently, for all $x \in X$ and all $N \geq N_0$ we have

$$0 \leq f(x) - S_N(x) = \sum_{n=N+1}^{\infty} 2^{-n} f_n(x) \leq \sum_{n=N+1}^{\infty} 2^{-n} < \epsilon.$$

Thus

$$\sup_{x\in X} |f(x) - S_N(x)| \leq \epsilon$$

for all $N \geq N_0$, and so $(S_N)_{N\in\mathbb{N}}$ converges uniformly to f.

($\Longleftarrow$): Suppose there is a continuous function $f\colon X \to [0, 1]$ satisfying $f(x) = 0$ for all $x \in A$ and $f(x) > 0$ for all $x \notin A$. First note that $A = f^{-1}(\{0\})$ is closed since f is continuous. Next, for each $n \in \mathbb{N}$ set $U_n := f^{-1}([0, \frac{1}{n}))$, which is open since f is continuous. Then $A \subset U_n$ for all $n \in \mathbb{N}$, and if $x \in \bigcap U_n$ then $f(x) < \frac{1}{n}$ for all $n \in \mathbb{N}$, which means $f(x) = 0$ and so $x \in A$. Thus

$$A \subset \bigcap_{n\in\mathbb{N}} U_n \subset A.$$

Therefore A is the intersection of the U_n's and therefore G_δ. □

6*. (a) First note that $U_{1,0} = \mathbb{Z}$, and so every element of $\mathbb{Z}$ is contained in a set in the collection $\mathcal{B}$. Next, suppose $x \in U_{d,a} \cap U_{e,b}$. Then $x = dn + a = em + b$ for some $n, m \in \mathbb{Z}$. Let f be the least common multiple of d and e and let c be the remainder one gets after dividing x by f: $x = fm + c$ for some multiple $m \in \mathbb{Z}$. Then $x \in U_{f,c}$, and in fact $U_{f,c} = \{x + fn : n \in \mathbb{Z}\}$. Observe that fn is a multiple of both d and e since f is the least common multiple, and therefore $x + fn \in U_{d,a} \cap U_{e,b}$ for all $n \in \mathbb{Z}$. Hence $U_{f,c} \subset U_{d,a} \cap U_{e,b}$, and therefore $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$. □

(b) $U_{d,a}$ is certainly open. To see that it is closed, observe that

$$\mathbb{Z} \setminus U_{d,a} = \bigcup_{b \neq a} U_{d,b}.$$

Indeed, $x \in U_{d,b}$ if and only if $x = dn + b$ for some $n \in \mathbb{Z}$. So if $x \in U_{d,a} \cap U_{d,b}$, then $dn + b = dm + a$ for some $n, m \in \mathbb{Z}$, and therefore $a - b = d(n - m)$. This implies $a - b$ is divisible by d, but $a - b \in \{-(d-1), \ldots, -1, 0, 1, \ldots, d-1\}$ and so this is only possible if $a = b$. This shows $U_{d,a}$ is disjoint from $U_{d,b}$ for all $b \neq a$ and so

$$\bigcup_{b \neq a} U_{d,b} \subset \mathbb{Z} \setminus U_{d,a}.$$

Conversely, if $x \in \mathbb{Z} \setminus U_{d,a}$, let $b \in \{0, \ldots, d-1\}$ be the remainder one gets after dividing x by d: $x = dn + b$ for some $n \in \mathbb{N}$. So $x \in U_{d,b}$, and since $x \notin U_{d,a}$, we must have $b \neq a$. This yields the other inclusion and establishes the desired equality. □

(c) Suppose $U \subset \mathbb{Z}$ is nonempty and open. Let $x \in U$. Since $\mathcal{B}$ is a basis, there exists $d \in \mathbb{N}$ and $a \in \{0, 1, \ldots, d-1\}$ such that $x \in U_{d,a} \subset U$. Define $f : \mathbb{N} \to U$ by $f(n) := dn + a$. This is injective, and hence U is infinite. □

(d) If $x \in \mathbb{Z} \setminus A$, this means $x \notin U_{p,0}$ for any prime number p. Consequently, x is not divisible by p for any prime p. There are only two numbers not divisible by any prime number: ± 1. Thus $\mathbb{Z} \setminus A = \{-1, 1\}$, which is of course finite. □

(e) Suppose, towards a contradiction that P is finite. Then A from the previous part is a finite union of closed sets (recall that $U_{p,0}$ is closed by part (b)), and thus is closed. Therefore its complement $\mathbb{Z} \setminus A$ is open. However, in the previous part we showed $\mathbb{Z} \setminus A$ is finite and nonempty, which contradicts part (c). Thus P must be infinite. □