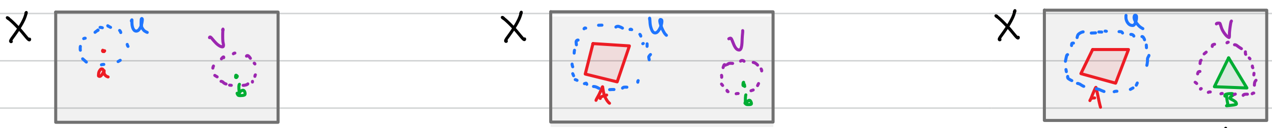
§ 31 Separation Axioms

Def Let X be a topological space such that all singleton sets $\{x\}$ are closed for $x \in \{x\}$. We say X is <u>regular</u> if for any closed set $A \subset X$ and $b \in X \setminus A$, there exists disjoint open sets $U, V \subset X$ with $A \subset U$ and $b \in V$. We say X is <u>normal</u> if for any disjoint closed sets $A, B \subset X$, there exists disjoint open sets $U, V \subset X$ with $A \subset U$ and $B \subset V$. We say A and B are <u>separated</u> by U and V.

- Observe that, since singleton sets are assumed to closed, normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff



Hausdorff   Regular   Normal

Ex ① If X is compact and Hausdorff, then X is normal. $A, B \subset X$ closed are also compact, and so can be separated using Exercise 4 on Homework 11.

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② If (X, d) is a metric space, then X is normal. For $A, B \subset X$ closed and disjoint, define $f, g : X \to \mathbb{R}$ by

$$f(x) := d(x, A) \quad \text{and} \quad g(x) := d(x, B),$$

which we saw before are continuous. Since A is closed, $f(x) = 0$ iff $x \in \bar{A} = A$. Consequently, $f(x) + g(x) > 0$ for all $x \in X$ since $A \cap B = \phi$. Therefore

$$h(x) := \frac{f(x)}{f(x) + g(x)}$$

is continuous. Note that for $a \in A$

$$h(a) = \frac{f(a)}{f(a) + g(a)} = \frac{0}{0 + g(a)} = 0.$$

and for $b \in B$

$$h(b) = \frac{f(b)}{f(b) + g(b)} = \frac{f(b)}{f(b) + 0} = 1.$$

So $A \subset h^{-1}(\{0\})$ and $B \subset h^{-1}(\{1\})$. Consequently

$$U := h^{-1}((-\tfrac{1}{2}, \tfrac{1}{2})) \quad \text{and} \quad V := h^{-1}((\tfrac{1}{2}, \tfrac{3}{2}))$$

give disjoint open sets containing A and B.

③ Recall that $\mathbb{R}_K$ denotes the real numbers with the K-topology: the topology generated by the basis consisting of sets of the form (a, b) and $(a, b) \setminus K$ where $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\mathbb{R}_K$ is Hausdorff but not regular. It is Hausdorff because $\mathbb{R}$ with the standard topology (which is coarser than the K-topology) is Hausdorff. To see that it is not regular, consider $A := K$ and $b := 0$. Suppose, towards a contradiction, that $U, V \subset \mathbb{R}$ are disjoint open sets with $K \subset U$ and $0 \in V$. Since V is open with $V \cap K = \phi$, there exists a basis set of the form $(a, b) \setminus K$ with

$$0 \in (a, b) \setminus K \subset V.$$

So $a < 0 < b$ and therefore we can find $n \in \mathbb{N}$ so that $0 < \frac{1}{n} < b$. Since $\frac{1}{n} \in K \subset \mathcal{U}$, there exists an open interval satisfying

$$\tfrac{1}{n} \in (c, d) \subset \mathcal{V}.$$

Thus we can find $x < \frac{1}{n}$ but with $x > \max\{c, \frac{1}{n+1}\}$, so that $x \in \mathcal{V} \setminus K$. However, we also have $0 \le x < \frac{1}{n} < b$, so $x \in (a,b) \setminus K \subset \mathcal{U}$. This contradicts $\mathcal{U} \cap \mathcal{V} = \emptyset$. $\square$

Def In a topological space X, a neighborhood of a subset $A \subset X$ is any open set $\mathcal{U} \subset X$ containing A: $A \subset \mathcal{U}$.

Prop Let X be a topological space such that all singleton sets are closed.

1. X is regular if and only if for $x \in X$ and a neighborhood $\mathcal{U}$ of x, there is a neighborhood V of x with $\overline{V} \subset \mathcal{U}$.
2. X is normal if and only if for a closed set $A \subset X$ and a neighborhood $\mathcal{U}$ of A, there exists a neighborhood V of A with $\overline{V} \subset \mathcal{U}$.

Proof We will only prove 2, since the proof of 1 is identical after replacing 'A' with the singleton set $\{x\}$.

($\Longrightarrow$): Assume X is normal and let $A \subset X$ be closed with neighborhood $\mathcal{U}$. Then $A \subset \mathcal{U}$ implies the closed set $B := X \setminus \mathcal{U}$ is disjoint from A. By normality, there exists disjoint open sets $V, W \subset X$ with $A \subset V$ and $B \subset W$. So V is a neighborhood of A and

$$V \subset X \setminus W \subset X \setminus B = \mathcal{U}.$$

Since $X \setminus W$ is a closed set containing V, we have $\overline{V} \subset X \setminus W \subset \mathcal{U}$.

($\Longleftarrow$): Let $A, B \subset X$ be disjoint closed sets. Then $\mathcal{U} := X \setminus B$ is a neighborhood of A, and therefore there exists a neighborhood V of A with $\overline{V} \subset \mathcal{U}$. Consequently $W := X \setminus \overline{V}$ is an open set disjoint from V and

$$W = X \setminus \overline{V} \supset X \setminus \mathcal{U} = B.$$

So A and B are separated by V and W, and thus X is normal. $\square$

§ 33 The Urysohn Lemma

- Let X be a topological space space such that all singleton sets are closed. Suppose for any pair of disjoint closed subsets $A, B \subset X$ there exist a closed interval $[a,b] \subset \mathbb{R}$ and a continuous function $f: X \to [a,b]$ such that $f|_A \equiv a$ and $f|_B \equiv b$. Then X is normal because A and B are separated by the open sets
$$\mathcal{U} := f^{-1}([a, \tfrac{a+b}{2})) \quad \text{and} \quad \mathcal{V} := f^{-1}((\tfrac{a+b}{2}, b])$$

In fact, this was how we proved metric spaces are normal above. The converse of this fact is the content of the next theorem

$\boxed{\text{Thm}}$ (Urysohn Lemma)
Let X be a normal space. Let $A, B \subset X$ be disjoint closed sets and let $[a,b] \subset \mathbb{R}$ be a closed interval. Then there exists a continuous map
$$f: X \to [a,b]$$
so that $f|_A \equiv a$ and $f|_B \equiv b$.

Proof We will construct $f: X \to [0,1]$. Composing with the function $g: [0,1] \to [a,b]$,
$$g(t) := (b-a)t + a$$
will yield the general case. Our construction of f will consist of four steps.

$\boxed{\text{Step 1}}$ We will define a family of open sets $\{ \mathcal{U}_p \mid p \in \mathbb{Q} \cap [0,1] \}$ so that for $p, q \in \mathbb{Q} \cap [0,1]$ if $p < q$ then $\overline{\mathcal{U}_p} \subset \mathcal{U}_q$.

Since the rationals are countable, we may enumerate $\mathbb{Q} \cap [0,1] = \{ p_n \mid n \in \mathbb{N} \}$. Moreover, we can demand $p_1 = 1$ and $p_2 = 0$. We will define the open sets $\mathcal{U}_{p_n}$ by induction on n. Set $\mathcal{U}_1 := X \setminus B$. Since $A \subset \mathcal{U}_1$, the proposition from last section implies there is a neighborhood V of A with $\overline{V} \subset \mathcal{U}_1$. Set $\mathcal{U}_0 := V$.

Now, suppose that $\mathcal{U}_{p_1}, \ldots, \mathcal{U}_{p_n}$ for $n \geq 2$ have been defined so that $p_i < p_j$ implies $\overline{\mathcal{U}_{p_i}} \subset \mathcal{U}_{p_j}$. Since $n+1 > 2$, $p_{n+1} \neq 0, 1$. This means p_{n+1} is neither the largest or smallest element of $\{p_1, \ldots, p_n, p_{n+1}\}$. Since this is a finite subset of $[0,1]$, p_{n+1} has an immediate predecessor p_i and an immediate successor p_j; that is, $p_i < p_{n+1} < p_j$ and no p_k lies in (p_i, p_{n+1}) or (p_{n+1}, p_j). Since $p_i < p_j$, 12/4 we have
$$\overline{\mathcal{U}_{p_i}} \subset \mathcal{U}_{p_j}.$$
$\overline{\mathcal{U}_{p_i}}$ is closed, the proposition from last section implies there is a neighborhood V of $\overline{\mathcal{U}_{p_i}}$ with $\overline{V} \subset \mathcal{U}_{p_j}$. Set $\mathcal{U}_{p_{n+1}} := V$. Then
$$\overline{\mathcal{U}_{p_i}} \subset \mathcal{U}_{p_{n+1}} \subset \overline{\mathcal{U}_{p_{n+1}}} \subset \mathcal{U}_{p_j}$$
This completes the inductive step, and so induction yields the desired family of open sets.
Note that for all $p \in \mathbb{Q} \cap [0,1]$ we have
$$A \subset \mathcal{U}_0 \subset \overline{\mathcal{U}_0} \subset \mathcal{U}_p \subset \overline{\mathcal{U}_p} \subset \mathcal{U}_1 = X \setminus B$$
Thus $A \subset \mathcal{U}_p$ and $B \subset X \setminus \overline{\mathcal{U}_p}$ for all $p \in \mathbb{Q} \cap [0,1]$.

$\boxed{\text{Step 2}}$ We extend our family $\{U_p \mid p \in \mathbb{Q} \cap [0,1]\}$ to be indexed by all of $\mathbb{Q}$, and still satisfy $p<q \Rightarrow \bar{U}_p \subset U_q$.

For $p \in \mathbb{Q}$ with $p<0$, set $U_p := \phi$. Note that $\bar{U}_p = \phi \subset U_q$ for all $q \in \mathbb{Q} \cap [0,1]$. For $p \in \mathbb{Q}$ with $p>1$, set $U_p := X$. Note that $\bar{U}_q \subset X = U_p$ for all $q \in \mathbb{Q} \cap (-\infty, 0]$. Thus $\{U_p \mid p \in \mathbb{Q}\}$ is the desired family. Observe that $A \subset U_p$ for $p \geq 0$ and $B \subset X \setminus \bar{U}_p$ for $p \leq 1$.

$\boxed{\text{Step 3}}$ For $x \in X$, define

$$\mathbb{Q}(x) := \{p \in \mathbb{Q} \mid x \in U_p\}.$$

Then $\inf \mathbb{Q}(x)$ exists and lies in $[0,1]$.

First observe that $\mathbb{Q}(x) \neq \phi$, since $x \in X = U_p$ for all $p>1$. Also $x \notin \phi = U_p$ for all $p<0$, so $\mathbb{Q}(x)$ is bounded below by 0. Consequently, $\inf \mathbb{Q}(x) \in [0,1]$ for all $x \in X$. Note that $A \subset U_p$ for all $p>0$ implies $\inf \mathbb{Q}(x) = 0$ for $x \in A$. Also $B \subset X \setminus \bar{U}_p$ for all $p \leq 1$ and $B \subset X = U_p$ for all $p>1$ implies $\inf \mathbb{Q}(x) = 1$ for $x \in B$.

$\boxed{\text{Step 4}}$ Define $f: X \to [0,1]$ by $f(x) := \inf \mathbb{Q}(x)$. Then f is the desired function.

By our final observations in the last step, we knew $f|_A \equiv 0$ and $f|_B \equiv 1$. So it remains to show f is continuous. Observe for $x \in X$,

$$x \in \bar{U}_r \Longrightarrow f(x) \leq r \quad (1)$$

$$x \notin U_r \Longrightarrow f(x) \geq r \quad (2)$$

Indeed, if $x \in \bar{U}_r$, then $x \in \bar{U}_r \subset U_s$ for all $s>r$. Thus $s \in \mathbb{Q}(x)$ for all $s>r$ and so

$$f(x) = \inf \mathbb{Q}(x) \leq \inf \{s \mid s>r\} = r$$

This proves (1). If $x \notin U_r$, then $x \notin U_s \subset U_r$ for any $s \leq r$. Hence r is a lower bound for $\mathbb{Q}(x)$, which means $f(x) \geq r$. This proves (2)

Now we prove f is continuous at each $x \in X$. Fix $x \in X$ and let $(c,d) \subset \mathbb{R}$ be a neighborhood of $f(x)$. Let $p, q \in \mathbb{Q}$ be such that

$$c < p < f(x) < q < d$$

(which we can find by the density of $\mathbb{Q}$). Consider the open set $U := U_q \setminus \bar{U}_p$. Since $f(x) < q$, the contrapositive of (2) implies $x \in U_q$. Also $f(x) > p$ and the contrapositive of (1) implies $x \notin \bar{U}_p$. Thus $x \in U$ and so U is a neighborhood of x. Also, if $y \in U$, then $y \in \bar{U}_q$ so that $f(y) \leq q < d$ by (1), and $y \notin U_p$ so that $f(y) \geq p > c$ by (2). Thus $f(U) \subset (c,d)$, which implies f is continuous at x. Since $x \in X$ was arbitrary, we see that f is continuous. $\square$

§ 35 The Tietze Extension Theorem

- We present an application of the Urysohn Lemma:

Thm (Tietze Extension Theorem)
Let X be a normal space and $A \subset X$ a closed subspace. Then for any continuous function $f: A \to \mathbb{R}$ can be extended to a continuous function $F: X \to \mathbb{R}$ with $F|_A = f$. Moreover, if f is bounded by $R>0$, then F can be chosen to be bounded by R as well.

Proof We will construct F as the uniform limit of continuous functions that approximate f on A. We start by considering the case of f bounded, say $f: \mathbb{R} \to [-R, R]$ for some $R>0$. We need a claim:

Claim If $h: A \to [-r, r]$ is continuous with $r>0$, then there exists a continuous function $g: X \to \mathbb{R}$ satisfying

① $|g(x)| \le \frac{1}{3} r$ for all $x \in X$

② $|h(a) - g(a)| \le \frac{2}{3} r$ for all $a \in A$.

Consider $B := h^{-1}([-r, -\frac{1}{3}r])$ and $C := h^{-1}([\frac{1}{3}r, r])$. These are disjoint and — by the continuity of h — closed in A. Since A is closed, B and C are closed in X. Thus the Urysohn lemma gives a continuous function $g: X \to [-\frac{1}{3}r, \frac{1}{3}r]$ with $g|_B \equiv -\frac{1}{3}r$ and $g|_C \equiv \frac{1}{3}r$. The prescribed range of g implies ①. Now, A is the disjoint union of B, C, and

$$A \setminus (B \cup C) = h^{-1}((-\tfrac{1}{3}r, \tfrac{1}{3}r))$$

Thus for $a \in A$

$$a \in \begin{cases} B \\ C \\ A \setminus (B \cup C) \end{cases} \implies \left.\begin{array}{l} h(a), g(a) \in [-r, -\frac{1}{3}r] \\ h(a), g(a) \in [\frac{1}{3}r, r] \\ h(a), g(a) \in [-\frac{1}{3}r, \frac{1}{3}r] \end{array}\right\} \implies |h(a) - g(a)| \le \tfrac{2}{3} r.$$

So ② holds. □

Now, we will inductively construct a sequence of continuous functions $g_n: X \to \mathbb{R}$, $n \in \mathbb{N}$, satisfying:

① $|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} R$ for all $x \in X$

② $\left|f(a) - \sum_{k=1}^{n} g_k(a)\right| \le \left(\frac{2}{3}\right)^n R$ for all $a \in A$.

For the base case $n=1$, simply apply the claim to $h=f$ and $r=R$ to obtain a continuous function $g_1: X \to \mathbb{R}$ satisfying

$$|g_1(x)| \le \tfrac{1}{3} R = \tfrac{1}{3}\left(\tfrac{2}{3}\right)^0 R \qquad \text{for all } x \in X$$

and

$$|f(a) - g_1(a)| \le \tfrac{2}{3} R = \left(\tfrac{2}{3}\right)^1 R \qquad \text{for all } a \in A.$$

Assume $g_1, \dots, g_n$ have been defined. Note that ② implies $\left(f - \sum_{k=1}^{n} g_k\right): A \to [-(\frac{2}{3})^n R, (\frac{2}{3})^n R]$.

So applying the claim to $h = f - \sum_{k=1}^{n} g_k$ and $r = \left(\frac{2}{3}\right)^n R$ yields the desired function $g_{n+1}: X \to \mathbb{R}$ since

$$|g_{n+1}(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n R$$

for all $x \in X$, and

$$\left| f(a) - \sum_{k=1}^{n+1} g_k(a) \right| = \left| \left(f(a) - \sum_{k=1}^{n} g_k(a)\right) - g_{n+1}(a) \right| \leq \frac{2}{3}\left(\frac{2}{3}\right)^n R = \left(\frac{2}{3}\right)^{n+1} R$$

for all $a \in A$. Thus induction yields the desired sequence.

Now, for each $n \in \mathbb{N}$, define $S_n: X \to \mathbb{R}$ by $S_n := \sum_{k=1}^{n} g_n$. Then each S_n is continuous as a finite sum of continuous functions, and

$$|f(a) - S_n(a)| \leq \left(\frac{2}{3}\right)^n R$$

for all $a \in A$ by 2. We claim that $(S_n)_{n \in \mathbb{N}}$ converges uniformly (and the limit will be the extension we want). Indeed, for $n < m$ observe that 1 implies for all $x \in X$

$$\begin{aligned} |S_m(x) - S_n(x)| &= \left| \sum_{k=n+1}^{m} g_k(x) \right| \leq \sum_{k=n+1}^{m} |g_k(x)| \\ &\leq \sum_{k=n+1}^{m} \frac{1}{3}\left(\frac{2}{3}\right)^{k-1} R = \frac{1}{3} R \sum_{k=n+1}^{m} \left(\frac{2}{3}\right)^{k-1} \\ &\leq \frac{1}{3} R \sum_{k=n+1}^{\infty} \left(\frac{2}{3}\right)^{k-1} = R\left(\frac{2}{3}\right)^n. \end{aligned}$$

Thus $(S_n(x))_{n \in \mathbb{N}} \subset \mathbb{R}$ is a Cauchy sequence and therefore convergenges because $\mathbb{R}$ is complete. Denote the limit by $F(x)$, and this defines a function $F: X \to \mathbb{R}$ which is the pointwise limit of $(S_n)_{n \in \mathbb{N}}$. However, the above estimate implies the convergence is uniform: for all $x \in X$

$$|F(x) - S_n(x)| = \lim_{m \to \infty} |S_m(x) - S_n(x)| \leq \lim_{m \to \infty} R\left(\frac{2}{3}\right)^n = R\left(\frac{2}{3}\right)^n.$$

The Uniform Limit Theorem then implies F is continuous. Note that 1 implies for all $x \in X$

$$|F(x)| = \lim_{n \to \infty} |S_n(x)| \leq \lim_{n \to \infty} \sum_{k=1}^{n} |g_k(x)| \leq \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{3}\left(\frac{2}{3}\right)^{k-1} R = R$$

so F is bounded by R. By 2 we have for all $a \in A$

$$|f(a) - F(a)| = \lim_{n \to \infty} |f(a) - S_n(a)| \leq \lim_{n \to \infty} \left(\frac{2}{3}\right)^n R = 0$$

So $f(a) = F(a)$ for all $a \in A$. This completes the proof when f is bounded.

Suppose f is not bounded. Let $\theta: \mathbb{R} \to (-1,1)$ be any homeomorphism (e.g. $\theta(x) = \frac{x}{1+|x|}$). Then $\tilde{f} := \theta \circ f$ is bounded by 1, and so the above argument implies it has a continuous extension $\tilde{F}: X \to [-1,1]$. Consider the closed set $B := \tilde{F}^{-1}(\{-1,1\})$. This is disjoint from A since $\tilde{F}(a) = \tilde{f}(a) = \theta(f(a)) \in (-1,1)$ for $a \in A$. Thus Urysohn's Lemma implies there is a continuous function $\phi: X \to [0,1]$ with $\phi|_B \equiv 0$ and $\phi|_A \equiv 1$. Then $\phi(x)\tilde{F}(x) \in (-1,1)$ for all $x \in X$. Define $F: X \to \mathbb{R}$ by $F(x) := \theta^{-1}(\phi(x)\tilde{F}(x))$. It is continuous as compositions and products of continuous functions. And for $a \in A$ we have $F(a) = \theta^{-1}(\phi(a) \cdot \tilde{F}(a)) = \theta^{-1}(1 \cdot \tilde{f}(a)) = f(a)$. So F extends f. $\square$