$\boxed{4.1}$ Eigenvectors, Eigenvalues, and Spectrum

$\boxed{\text{Ex}}$ Recall our now standard example of $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by reflecting over a line. If $S = \{\vec{e}_1, \vec{e}_2\}$, then usually $[T]_S^S$ is hard to compute, but if $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ is given by $\vec{b}_1$ parallel to the line and $\vec{b}_2$ perpendicular to it then

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We know $[T]_S^S$ and $[T]_{\mathcal{B}}^{\mathcal{B}}$ are similar, we can use the homework to see

$$\mathrm{Tr}([T]_S^S) = \mathrm{Tr}([T]_{\mathcal{B}}^{\mathcal{B}}) = 0$$

and

$$\det([T]_S^S) = \det([T]_{\mathcal{B}}^{\mathcal{B}}) = (1)\cdot(-1) = -1.$$

Also, since

$$([T]_{\mathcal{B}}^{\mathcal{B}})^d = \begin{cases} [T]_{\mathcal{B}}^{\mathcal{B}} & \text{if } d \text{ odd} \\ I_2 & \text{if } d \text{ even} \end{cases}$$

we see

$$\underbrace{T \circ T \circ \cdots \circ T}_{d} = \begin{cases} T & \text{if } d \text{ odd} \\ I & \text{if } d \text{ even.} \end{cases}$$

All of this is to say that find the basis $\mathcal{B}$ on which T acted simply, allows us do computations for T much more easily. $\square$

- In this section, given a lin. trans. $T: V \to V$ we will look for vectors $\vec{v}$ on which T acts as simply as possible; namely, $T(\vec{v}) = \lambda \vec{v}$ for some scalar λ.

$\boxed{\text{Def}}$ Let $T: V \to V$ be a linear transformation. A non-zero vector $\vec{v} \in V$ is called an eigenvector of T if

$$T(\vec{v}) = \lambda \vec{v}$$

for some scalar λ. We call λ an eigenvalue of T. The set of all eigenvalues of T is called the spectrum of T, and is denoted $\sigma(T)$.

- Observe that $\vec{v}$ is an eigenvector of T with eigenvalue λ iff $\vec{v} \neq \vec{0}$ and
$$(T - \lambda I)(\vec{v}) = \vec{0}.$$
That is, iff $\vec{v} \in \mathrm{Ker}(T - \lambda I) \setminus \{\vec{0}\}$.

$\boxed{\text{Def}}$ If λ is an eigenvalue of T, we call $\mathrm{Ker}(T - \lambda I)$ the eigenspace of λ for T.

$\boxed{\text{Ex}}$ Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $y = 2x$. Then

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

So $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are eigenvectors of T with eigenvalues 1 and -1, respectively. It turns out these are all of the eigenvalues of T, so $\sigma(T) = \{-1, 1\}$. Moreover, one can show

$$\operatorname{Ker}(T - 1 \cdot I) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$$

and $$\operatorname{Ker}(T - (-1) \cdot I) = \operatorname{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}.$$ $\square$

Finding Eigenvectors and Eigenvalues

- Suppose you knew λ is an eigenvalue of a lin. trans. T. Then computing $\operatorname{Ker}(T - \lambda I)$ yields all the eigenvectors of T with eigenvalue λ. If you do this for all λ in the spectrum of T, you'll find all the eigenvectors of T. But how do you find the eigenvalues of T in the first place?

- Consider the special case $A : \mathbb{F}^n \to \mathbb{F}^n$ with $A \in M_{n \times n}$.

Prop 4.1 For $A \in M_{n\times n}$, λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.

Proof We have

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } A &\iff \operatorname{Ker}(A - \lambda I_n) \text{ contains a non-zero vector} \\
&\iff (A - \lambda I_n)\vec{x} = \vec{0} \text{ does } \underline{\text{not}} \text{ have a unique solution} \\
\text{count pivots } &\iff A - \lambda I_n \text{ is not invertible} \\
\text{Section 3.3 } &\iff \det(A - \lambda I_n) = 0.
\end{aligned}$$

$\square$

Def Let z be a variable. Then for $A \in M_{n \times n}$

$$\operatorname{char}_A(z) := \det(A - z I_n)$$

is a degree n polynomial in z called the <u>characteristic polynomial</u> of A

- The previous proposition says λ is an eigenvalue of A iff it is a <u>root</u> of its characteristic polynomial.

Ex ① Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\operatorname{char}_A(z) = \det(A - zI) = \left| \begin{pmatrix} 1-z & 0 \\ 0 & -1-z \end{pmatrix} \right| = (1-z)(-1-z) = (z-1)(z+1)$$

The roots of $\operatorname{char}_A(z)$ are $z = \pm 1$. Thus $\sigma(A) = \{-1, 1\}$.

② Let $A = \begin{pmatrix} a_{1,1} & & * \\ & a_{2,2} & \\ 0 & & \ddots \ a_{n,n} \end{pmatrix}$ be upper triangular.

$$\operatorname{char}_A(\lambda) = \det(A - zI) = \det \underbrace{\begin{pmatrix} a_{1,1} - z & & * \\ & a_{2,2} - z & \\ 0 & & \ddots \ a_{n,n} - z \end{pmatrix}}_{\text{upper triangular}} = (a_{1,1} - z)\cdot(a_{2,2} - z) \cdots (a_{n,n} - z)$$

which has roots $z = a_{1,1}, a_{2,2}, \ldots, a_{n,n}$. Thus $\sigma(A) = \{a_{1,1}, a_{2,2}, \ldots, a_{n,n}\}$. $\square$

• Let's record the last example as a proposition. Note a similar argument works for lower triangular.

Prop 4.2 If $A = (a_{ij}) \in M_{n\times n}$ is triangular, then $\sigma(A) = \{a_{1,1}, a_{2,2}, \dots, a_{n,n}\}$.

Lemma 4.3 Suppose $A, B \in M_{n\times n}$ are similar, then $\mathrm{char}_A(z) = \mathrm{char}_B(z)$.

Proof Since A and B are similar, there exists invertible Q s.t.

$$A = Q^{-1} B Q.$$

Note that

$$Q^{-1}(B - \lambda I_n) Q = Q^{-1} B Q - \lambda Q^{-1} Q = A - \lambda I_n.$$

Thus $A - \lambda I_n$ and $B - \lambda I_n$ are similar for scalars λ. By Exercise 5 on Homework 8, they have the same determinant. Hence for all scalars λ

$$\mathrm{char}_A(\lambda) = \det(A - \lambda I_n) = \det(B - \lambda I_n) = \mathrm{char}_B(\lambda). \qquad \square$$

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• Let us return to the abstract setting $T: V \to V$, but assume V is finite-dim'l. By taking a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ for V, we can always return to our concrete setting:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\vec{v}]_{\mathcal{B}}}"'] & V \arrow[d, "{[\vec{v}]_{\mathcal{B}}}"] \\ \mathbb{F}^n \arrow[r, "{[T]_{\mathcal{B}}^{\mathcal{B}}}"'] & \mathbb{F}^n \end{tikzcd}$$

Suppose λ is an eigenvalue of $[T]_{\mathcal{B}}^{\mathcal{B}}$ with eigenvector $\vec{x} = (x_1, \dots, x_n)^T \in \mathbb{F}^n$. Consider

$$\vec{v} := x_1 \vec{v}_1 + \dots + x_n \vec{v}_n \in V.$$

Then $\vec{v}$ is an eigenvector of T. Indeed,

$$[T(\vec{v})]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}} [\vec{v}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}} \vec{x} = \lambda \vec{x} = \lambda [\vec{v}]_{\mathcal{B}} = [\lambda \vec{v}]_{\mathcal{B}}.$$

Since $T(\vec{v})$ and $\lambda \vec{v}$ have same coordinate vector, it must be that $T(\vec{v}) = \lambda \vec{v}$. In particular, λ is also an eigenvalue of T.

• But what if we choose another basis $\mathcal{A}$? Will we still get the same eigenvalues and eigenvectors? Yes, because the two matrix rep's are similar:

$$[T]_{\mathcal{A}}^{\mathcal{A}} = [I]_{\mathcal{B}}^{\mathcal{A}} [T]_{\mathcal{B}}^{\mathcal{B}} [I]_{\mathcal{A}}^{\mathcal{B}} = \left([I]_{\mathcal{A}}^{\mathcal{B}}\right)^{-1} [T]_{\mathcal{B}}^{\mathcal{B}} [I]_{\mathcal{A}}^{\mathcal{B}}$$

so by the previous lemma

$$\mathrm{char}_{[T]_{\mathcal{A}}^{\mathcal{A}}}(z) = \mathrm{char}_{[T]_{\mathcal{B}}^{\mathcal{B}}}(z).$$

In light of this, we make the following definition:

Def Let V be a finite-dimensional vector space and let $T: V \to V$ be a linear transformation. Then the characteristic polynomial of T is

$$\mathrm{char}_T(z) = \det([T]_{\mathcal{B}}^{\mathcal{B}} - z I)$$

where $\mathcal{B}$ is any basis for V.

$\boxed{\text{Ex}}$ Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection over the line $y = 2x$. Then for the two bases

$$S = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \quad \text{and} \quad B = \{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \}$$

we have seen that

$$[T]_S^S = \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix} \quad \text{and} \quad [T]_B^B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We've already seen the characteristic polynomial of the latter matrix is $(z-1)(z+1)$, so $\text{char}_T(z) = (z-1)(z+1)$. We should get the same thing for the other matrix:

$$\det([T]_S^S - zI_2) = \det \begin{pmatrix} -3/5 - z & 4/5 \\ 4/5 & 3/5 - z \end{pmatrix} = (-\tfrac{3}{5} - z)(\tfrac{3}{5} - z) - \tfrac{16}{25}$$
$$= -\tfrac{9}{25} + z^2 - \tfrac{16}{25}$$
$$= z^2 - 1 = (z-1)(z+1)$$

$\square$

Real vs Complex Roots

$\boxed{\text{Fact}}$ (Fundamental Theorem of Algebra) Every non-constant polynomial has at least one complex root.

- So for any non-constant $p(z) \in \mathbb{P}_d(\mathbb{C})$, there exists $\lambda \in \mathbb{C}$ s.t. $p(\lambda) = 0$.

$\boxed{\text{Warning}}$ This is not true over $\mathbb{R}$. That is, if $p(x) \in \mathbb{P}_d(\mathbb{R})$ (i.e. has real coefficients) it may be that $p(\lambda) \neq 0$ $\forall \lambda \in \mathbb{R}$. For example $p(x) = x^2 + 1$ has no real roots, but

$$p(x) = (x - i)(x + i)$$

so it does have complex ones. We can always treat $p(x)$ as an complex polynomial (that is $\mathbb{P}_d(\mathbb{R}) \subseteq \mathbb{P}_d(\mathbb{C})$), so in order to use the Fundamental Theorem of Algebra, we will usually treat all polynomials as complex ones.

Multiplicities of Eigenvalues

- Suppose $p(z) \in \mathbb{P}_d(\mathbb{C})$ has a root $\lambda \in \mathbb{C}$. Recall that this implies

$$p(z) = (z - \lambda) q(z)$$

for some $q(z) \in \mathbb{P}_d(\mathbb{C})$. We say $z - \lambda$ divides $p(z)$. If λ is also a root of $q(z)$, then

$$p(z) = (z - \lambda)^2 r(z)$$

for some $r(z) \in \mathbb{P}_{d-2}(\mathbb{C})$. Thus $(z-\lambda)^2$ divides $p(z)$.

$\boxed{\text{Def}}$ Let λ be an eigenvalue of a linear transformation T. The algebraic multiplicity of λ is the largest integer k s.t. $(z - \lambda)^k$ divides $\text{char}_T(z)$. We denote $m_\lambda(T) := k$.

The geometric multiplicity of λ is $\dim(\ker(T - \lambda I))$.

- Let $\lambda_1, \ldots, \lambda_d$ be the distinct eigenvalues of T. It follows that

$$\text{char}_T(z) = c\,(z - \lambda_1)^{m_{\lambda_1}(T)} (z - \lambda_2)^{m_{\lambda_2}(T)} \cdots (z - \lambda_d)^{m_{\lambda_d}(T)}$$

for some $c \in \mathbb{C}$. Observe that this shows the degree of $\text{char}_T(z)$ is:
$$m_{\lambda_1}(T) + m_{\lambda_2}(T) + \cdots + m_{\lambda_d}(T).$$

On the other hand
$$\text{char}_T(z) = \det([T]_{\mathcal{B}}^{\mathcal{B}} - zI)$$
where $\mathcal{B} \subset V$ is a basis and so the degree is also $\dim(V)$. Thus we have the following:

$\boxed{\text{Prop 4.3}}$ Let $\lambda_1, \ldots, \lambda_d$ be the distinct eigenvalues of $T: V \to V$. Then
$$\sum_{i=1}^{d} m_{\lambda_i}(T) = \dim(V)$$

Given $A \in M_{n \times n}$, if we say $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its eigenvalues "counting multiplicities", we mean each <u>distinct</u> eigenvalue λ is repeated $m_\lambda(A)$ times in this list.

$\boxed{\text{Prop 4.4}}$ Let λ be an eigenvalue of T. Then $\dim(\ker(T - \lambda I)) \leq m_\lambda(T)$.

<u>Proof</u> Suppose $\dim(\ker(T-\lambda I)) = k$, and let $\vec{v}_1, \ldots, \vec{v}_k \in \ker(T - \lambda I)$ be a basis for this subspace. Since $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent (in V), we can extend this to a basis $\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n$ for V by Prop 2.12. Denote $\mathcal{B} := \{\vec{v}_1, \ldots, \vec{v}_n\}$. Now, for each $j = 1, \ldots, k$, $\vec{v}_j \in \ker(T - \lambda I)$ implies
$$[T(\vec{v}_j)]_{\mathcal{B}} = [\lambda \vec{v}_j]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{th position}$$

Consequently
$$[T]_{\mathcal{B}}^{\mathcal{B}} = \left([T(\vec{v}_1)]_{\mathcal{B}} \cdots [T(\vec{v}_n)]_{\mathcal{B}}\right) = \begin{pmatrix} \begin{matrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{matrix} & A \\ 0 & C \end{pmatrix} = \begin{pmatrix} \lambda I_k & A \\ 0 & C \end{pmatrix}$$
for some $A \in M_{k \times (n-k)}$, $C \in M_{(n-k) \times (n-k)}$. Also, we have
$$[T]_{\mathcal{B}}^{\mathcal{B}} - zI_n = \begin{pmatrix} (\lambda - z) I_k & A \\ 0 & C - zI_{(n-k)} \end{pmatrix}$$

So using Exercise 6.(c) on Homework 7, we have
$$\text{char}_T(z) = \det([T]_{\mathcal{B}}^{\mathcal{B}} - zI_n) = \det((\lambda - z) I_k) \det(C - zI_{(n-k)}) = (\lambda - z)^k \det(C - zI_{(n-k)}).$$

Thus $(\lambda - z)^k$ divides $\text{char}_T(z)$, and so $k \leq m_\lambda(T)$ by definition of the algebraic multiplicity. Since $k = \dim(\ker(T - \lambda I))$, we are done. $\square$

The inequality in the above proposition can be strict, as the following example demonstrates:

$\boxed{\text{Ex}}$ Define $T: \mathbb{P}_2 \to \mathbb{P}_2$ by $T(p(x)) = p'(x)$. Recall that for $S = \{1, x, x^2\}$

$$[T]_S^S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

So

$$\text{char}_T(z) = \det([T]_S^S - zI_3) = \det \underbrace{\begin{pmatrix} -z & 1 & 0 \\ 0 & -z & 2 \\ 0 & 0 & -z \end{pmatrix}}_{} = -z^3 = -(z-0)^3$$

upper triangular

Thus $\sigma(T) = \{0\}$ and $m_T(0) = 3$. On the other hand

$$\ker(T - 0I) = \ker(T) = \{\text{constant polynomials}\} = \text{span}\{1\},$$

so $\dim(\ker(T - 0 \cdot I)) = 1 < 3 = m_T(0)$ $\square$

The trace, the determinant, and the eigenvalues

We conclude this section by relating the trace and determinant of a matrix to its eigenvalues:

$\boxed{\text{Thm 4.5}}$ Let $A \in M_{n \times n}$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ counting multiplicities.

1. $\text{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$
2. $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

Proof Homework 9. $\square$

4.2 Diagonalization

Def Let V be a finite-dimensional vector space. A linear transformation $T:V\to V$ is diagonalizable if there exists a basis $\mathcal{B}$ for V s.t. $[T]_{\mathcal{B}}^{\mathcal{B}}$ is diagonal.

- We've seen that any reflection $T:\mathbb{R}^2\to\mathbb{R}^2$ is diagonal.

4.7

Thm $T:V\to V$ is diagonalizable if and only if there exists a basis for V consisting of eigenvectors.

Proof ($\Rightarrow$) Suppose T is diagonalizable with basis $\mathcal{B}=\{\vec{v}_1,\dots,\vec{v}_n\}$ for V and

$$[T]_{\mathcal{B}}^{\mathcal{B}}=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

Then for each $i=1,\dots,n$

$$[T(\vec{v}_i)]_{\mathcal{B}}=[T]_{\mathcal{B}}^{\mathcal{B}}[\vec{v}_i]_{\mathcal{B}}=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\cdot\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\overset{\text{ith position}}{}=\begin{pmatrix}0\\ \vdots\\ \lambda_i\\ \vdots\\ 0\end{pmatrix}=[\lambda_i\vec{v}_i]_{\mathcal{B}}$$

Since coordinate vectors are unique, we must have $T(\vec{v}_i)=\lambda_i\vec{v}_i$. So $\mathcal{B}$ consists of eigenvectors.

($\Leftarrow$) Let $\mathcal{B}=\{\vec{v}_1,\dots,\vec{v}_n\}$ be a basis for V of eigenvectors for T with eigenvalues $\lambda_1,\dots,\lambda_n$, respectively. Then the jth column of $[T]_{\mathcal{B}}^{\mathcal{B}}$ is:

$$[T(v_j)]_{\mathcal{B}}=[\lambda_j v_j]_{\mathcal{B}}=\begin{pmatrix}0\\ \vdots\\ \lambda_j\\ \vdots\\ 0\end{pmatrix}\leftarrow \text{jth position}$$

Hence $[T]_{\mathcal{B}}^{\mathcal{B}}$ is diagonal and T is diagonalizable. $\square$

Ex $T:P_2\to P_2$ defined by $T(p(x))=p'(x)$ is not diagonalizable. Recall that $\sigma(T)=\{0\}$. So every eigenvector of T belongs to $\ker(T-0\cdot I)\setminus\{\vec{0}\}$. So any basis of eigenvectors would need to contained in the subspace $\ker(T-0\cdot I)$. But we already seen that this subspace is 1-dimensional, so it cannot contain a basis for P_2. $\square$

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4.8

Prop $A\in M_{n\times n}$ is diagonalizable if and only if it is similar to a diagonal matrix.

Proof ($\Rightarrow$) Suppose A is diagonalizable for a basis $\mathcal{B}$ of $\mathbb{F}^n$. Then for the standard basis S we have $A=[A]_S^S$, which is similar to $[A]_{\mathcal{B}}^{\mathcal{B}}$.

($\Leftarrow$) Suppose $A=QDQ^{-1}$ for $D,Q\in M_{n\times n}$ with $D=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ and Q invertible. Since Q is invertible and square, its columns form a basis $\mathcal{B}=\{\vec{v}_1,\dots,\vec{v}_n\}$ for $\mathbb{F}^n$. Observe that

$$A\vec{v}_j=A(Q\hat{e}_j)=(QDQ^{-1})(Q\hat{e}_j)=Q(D\hat{e}_j)=Q(\lambda_j\hat{e}_j)=\lambda_j Q\hat{e}_j=\lambda_j\vec{v}_j.$$

Thus $\vec{v}_j$ is an eigenvector of A (with eigenvalue λ_j). Since this holds for each $j=1,\dots,n$ and $\vec{v}_1,\dots,\vec{v}_n$ forms a basis for $\mathbb{F}^n$, the previous theorem implies A is diagonalizable. $\square$

Rem Note that $Q[\vec{v}_j]_{\mathcal{B}}=Q\vec{e}_j=\vec{v}_j=[\vec{v}_j]_S$. This implies $Q=[I]_{\mathcal{B}}^S$. Also

$$D=Q^{-1}AQ=[I]_S^{\mathcal{B}}[A]_S^S[I]_{\mathcal{B}}^S=[A]_{\mathcal{B}}^{\mathcal{B}}.$$

Def For $A \in M_{n\times n}$ diagonalizable, we call a factorization $A = QDQ^{-1}$ a <u>diagonalization</u> of A when $D \in M_{n\times n}$ is diagonal.

- The previous remark says that the diagonalization of A is given by $Q = [I]_B^S$ where B is a basis of eigenvectors of A and $D = [A]_B^B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues.

Ex Let $A = \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Since A is triangular, $\sigma(A) = \{-2, 1, 3\}$ and on Quiz 9 you showed it had eigenvalues:

$$\lambda_1 = -2 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\lambda_2 = 1 \qquad \vec{v}_2 = \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$$
$$\lambda_3 = 3 \qquad \vec{v}_3 = \begin{pmatrix} 2/5 \\ 1 \\ 1 \end{pmatrix}$$

so if we define

$$D := \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} 1 & 1/3 & 2/5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

note Q is invertible: $\det(Q) = \underbrace{1 \cdot 1 \cdot 1}_{\text{diagonal entries.}} = 1 \neq 0$.

then we should have

$$A = QDQ^{-1}.$$

This is equivalent to $AQ = QD$ (which does not require us to compute Q^{-1}), so we'll check that:

$$AQ = \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1/3 & 2/5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1/3 & 6/5 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

$$QD = \begin{pmatrix} 1 & 1/3 & 2/5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1/3 & 6/5 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix} = \checkmark$$

$\square$

Functional Calculus

Let $A \in M_{n\times n}$ have diagonalization $A = QDQ^{-1}$ with $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Observe that

$$A^2 = (QDQ^{-1})(QDQ^{-1}) = QD^2Q^{-1} = Q\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}Q^{-1} = Q\begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix}Q^{-1}$$

Iterating this argument we have

$$A^k = QD^kQ^{-1} = Q\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}Q^{-1}.$$

Now, let $p(t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0$ be a degree d polynomial. Then using the above we have:

$$p(A) = a_d A^d + \cdots + a_1 A + a_0 I_n = a_d QD^dQ^{-1} + \cdots + a_1 QDQ^{-1} + a_0 QIQ^{-1}$$
$$= Q[a_d D^d + \cdots + a_1 D + a_0 I]Q^{-1} = Q\,p(D)\,Q^{-1}$$
$$= Q\left[a_d\begin{pmatrix} \lambda_1^d & & \\ & \ddots & \\ & & \lambda_n^d \end{pmatrix} + \cdots + a_1\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} + a_0\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\right]Q^{-1}$$
$$= Q\begin{pmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{pmatrix}Q^{-1}$$

That is, $p(A) = Q\,p(D)\,Q^{-1}$ and $p(D)$ is just $p(t)$ applied to the diagonal entries (eigenvalues of A).

• Note that $\sigma(p(A)) = p(\sigma(A))$.

• We can extend this from polynomials to power series whose interval of convergence contains $\sigma(A)$: If $f(t) = \sum_{k=0}^{\infty} a_k (t-c)^k$ has radius of convergence R and $\sigma(A) \subseteq (c-R, c+R)$, then $f(A)$ exists and

$$f(A) = Q\begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{pmatrix}Q^{-1}.$$

$\boxed{\text{Ex}}$ Let $A = \begin{pmatrix} -2 & 6 \\ -3 & 7\end{pmatrix}$. Then $A = \begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4\end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}^{-1}$ (Exercise). Recall $f(t) = \sqrt{t}$ can be written as a power series

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(t-c)^k$$

for any $c > 0$, with radius of convergence $R = c$, so that the interval of convergence contains $(0, 2c)$. Choosing $c = 3$ (for example) yields $\sigma(A) = \{1, 4\} \subseteq (0, 6)$. Thus

$$\sqrt{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{4}\end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2\end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2\end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 2\end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2\end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 3\end{pmatrix}.$$

Note that

$$(\sqrt{A})^2 = \begin{pmatrix} 0 & 2 \\ -1 & 3\end{pmatrix}\begin{pmatrix} 0 & 2 \\ -1 & 3\end{pmatrix} = \begin{pmatrix} -2 & 6 \\ -3 & 7\end{pmatrix} = A$$

That is, $\sqrt{A}$ acts just like the square root of a number. $\square$

Distinct eigenvalues

It turns out if all of the eigenvalues of T are distinct, then A is diagonalizable.

4.9

$\boxed{\text{Thm}}$ Let $T: V \to V$ be a linear transformation with $\lambda_1, \ldots, \lambda_r$ distinct eigenvalues and $\vec{v}_1, \ldots, \vec{v}_r \in V$ corresponding eigenvectors. Then $\vec{v}_1, \ldots, \vec{v}_r$ are linearly independent.

Proof We will use proof by induction on r. For the base case $r=1$, since $\vec{v}_1 \neq \vec{0}$ by definition of an eigenvector, the system $\vec{v}_1$ is linearly independent.

For the induction step, suppose we knew the statement is true for $r-1$. Suppose

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r = \vec{0}.$$

Applying $A - \lambda_r I$ to each side yields:

$$\alpha_1(\lambda_1 - \lambda_r)\vec{v}_1 + \cdots + \alpha_{r-1}(\lambda_1 - \lambda_{r-1})\vec{v}_r + \vec{0} = \vec{0}$$

Since $\vec{v}_1, \ldots, \vec{v}_{r-1}$ are lin. indep. by our induction hypothesis, we must have

$$\alpha_1(\lambda_1 - \lambda_r) = \cdots = \alpha_{r-1}(\lambda_1 - \lambda_{r-1}) = 0.$$

Recall that the eigenvalues are distinct, so $\lambda_i - \lambda_r \neq 0$ and so it must be that $\alpha_i = 0$ for $i = 1, \ldots, r-1$. So our original linear combination reduces to

$$\vec{0} + \alpha_r \vec{v}_r = \vec{0}.$$

Since $\vec{v}_r \neq \vec{0}$ (by virtue of being an eigenvector) we must have $\alpha_r = 0$. Thus $\vec{v}_1, \ldots, \vec{v}_r$ are linearly independent. Therefore induction yields the theorem for all $r \in \mathbb{N}$. $\square$

4.10

$\boxed{\text{Cor}}$ If $T: V \to V$ has $\dim(V)$ distinct eigenvalues, then it is diagonalizable.

Proof Let $n = \dim(V)$, and let $\vec{v}_1, \ldots, \vec{v}_n$ be eigenvectors of T with distinct eigenvalues. The theorem implies they are linearly independent. Since there are $n = \dim(V)$ of them, the form a basis, which means T is diagonalizable. $\square$

Criterion for Diagonalizability

Thm 4.11 Let V be a finite-dimensional vector space with $\dim(V) = n$. Let $T: V \to V$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ counting multiplicities. The T is diagonalizable if and only if

$$\dim(\ker(T - \lambda I)) = m_\lambda(T)$$

for all $\lambda \in \sigma(T)$.

Proof ($\Longrightarrow$) Let B be basis s.t. $[T]_B^B$ is diagonal. By reordering B, we can assume

$$[T]_B^B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Thus $\dim(\ker(T - \lambda I))$ is the number of $\lambda_i = \lambda$, which equals $m_\lambda(T)$.

($\Longleftarrow$) Suppose $\dim(\ker(T - \lambda I)) = m_\lambda(T)$ $\forall \lambda \in \sigma(T)$. Let $\lambda_1, \ldots, \lambda_p$ be the distinct eigenvalues of T. For each $k = 1, \ldots, p$, let $V_k = \ker(T - \lambda_k P)$ and let B_k be a basis for V_k. We claim

$$B := \bigcup_{k=1}^{p} B_k$$

is a basis for V, which will imply T is diagonalizable since B consists of eigenvectors. Note that by Proposition 4.3 from Section 4.1

$$\sum_{k=1}^{p} \dim(V_k) = \sum_{k=1}^{p} m_{\lambda_i}(T) = \dim(V).$$

So B contains $\dim(V)$ vectors and therefore it suffices to show they are lin. indep. Let $B = \{\vec{u}_1, \ldots, \vec{u}_n\}$ and set $I_k = \{i : \vec{u}_i \in V_k\}$, $k = 1, \ldots, n$. Suppose

$$\alpha_1 \vec{u}_1 + \cdots + \alpha_n \vec{u}_n = \vec{0}.$$

Write $\vec{v}_k := \sum_{i \in I_k} \alpha_i \vec{u}_i \in V_k = \ker(T - \lambda_k I)$. We claim $\vec{v}_1 = \cdots = \vec{v}_p = \vec{0}$. Indeed, $\vec{v}_1 + \cdots + \vec{v}_p = \vec{0}$ and so if any are non-zero then we have a non-trivial linear comb. summing to zero. At the same time, the non-zero vectors are eigenvectors with distinct eigenvalues and hence are linearly independent, a contradiction. Thus

$$\sum_{i \in I_k} \alpha_i \vec{u}_i = \vec{v}_k = \vec{0}$$

and since B_k is a basis for V_k, we must have $\alpha_i = 0$ $\forall i \in I_k$, and each $k = 1, \ldots, n$. Thus B is a lin. indep. system and therefore a basis. $\square$

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Ex Recall $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ has $\dim(\ker(A - 0 \cdot I)) = 1 < 3 = m_0(A)$. So A is not diagonalizable. Consequently, for any invertible matrix Q, QAQ^{-1} is not diagonal. $\square$

- Note that A in the previous example is at least upper triangular. It turns out this is the best we can hope for in general.

4.12

Thm Let V be finite-dimensional with $\dim(V)=n$. Let $T:V\to V$ have eigenvalues $\lambda_1,\dots,\lambda_n$. Then there exists a basis $\mathcal{B}$ for V s.t. arbitrary

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$$

Proof We proceed by induction on n.
Base Case: $n=1$. Let $\vec{v}\in V$ be s.t. $V=\operatorname{span}\{\vec{v}\}$. Then $T:V\to V$ implies $\vec{v}$ is an eigenvector of T, say with eigenvalue λ. Then for $\mathcal{B}=\{\vec{v}\}$, we have $[T]_{\mathcal{B}}^{\mathcal{B}}=(\lambda)$.
Induction Step: Suppose the theorem holds for dimension $n-1$. Let $\vec{v}_1\in V$ be an eigenvector of T with eigenvalue λ_1. Complete to a basis $\mathcal{B}=\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\}$. Then

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \lambda_1 & X \\ 0 & Y \end{pmatrix}$$

for some $X\in M_{1\times(n-1)}$, $Y\in M_{(n-1)\times(n-1)}$. Let $\mathcal{C}:=\{\vec{v}_2,\dots,\vec{v}_n\}$, $W:=\operatorname{span}\mathcal{C}$, and define $S:W\to W$ by $[S]_{\mathcal{C}}^{\mathcal{C}}=Y$. By the proof of Proposition 4.4

$$\operatorname{char}_T(z) = (\lambda_1 - z)\cdot \operatorname{char}_Y(z) = (\lambda_1 - z)\operatorname{char}_S(z).$$

It follows that $\lambda_2,\dots,\lambda_n$ are eigenvalues of S. Since $\dim(W)=n-1$, our induction hypothesis implies there is a basis $\mathcal{C}'$ for W s.t.

$$[S]_{\mathcal{C}'}^{\mathcal{C}'} = \begin{pmatrix} \lambda_2 & * \\ 0 & \ddots \ \lambda_n \end{pmatrix}.$$

Define $\mathcal{B}':=\{\vec{v}_1\}\cup\mathcal{C}'$. Then

$$[T]_{\mathcal{B}'}^{\mathcal{B}'} = [I]_{\mathcal{B}}^{\mathcal{B}'}[T]_{\mathcal{B}}^{\mathcal{B}}[I]_{\mathcal{B}'}^{\mathcal{B}}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & [I]_{\mathcal{C}}^{\mathcal{C}'} \end{pmatrix}\begin{pmatrix} \lambda_1 & X \\ 0 & [S]_{\mathcal{C}}^{\mathcal{C}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & [I]_{\mathcal{C}'}^{\mathcal{C}} \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1 & * \\ 0 & [I]_{\mathcal{C}}^{\mathcal{C}'}[S]_{\mathcal{C}}^{\mathcal{C}}[I]_{\mathcal{C}'}^{\mathcal{C}} \end{pmatrix} = \begin{pmatrix} \lambda_1 & * \\ 0 & [S]_{\mathcal{C}'}^{\mathcal{C}'} \end{pmatrix} = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$$

Induction then completes the proof. □

4.13

Cor Every $A\in M_{n\times n}(\mathbb{C})$ is similar to an upper triangular matrix.

- Since det and Tr are invariant under similarity, this gives a much simpler proof of the formulas

$$\operatorname{Tr}(A) = \lambda_1 + \cdots + \lambda_n$$
$$\det(A) = \lambda_1\cdots\lambda_n.$$

Fact For every $A\in M_{n\times n}(\mathbb{C})$ there exists a unitary matrix $U\in M_{n\times n}(\mathbb{C})$ $(U^*=U^{-1})$ such that $UAU^{-1}=UAU^*$ is upper triangular. In particular, A is diagonalizable iff there exists a unitary U s.t. UAU^* is diagonal.

Def We say $A\in M_{n\times n}$ is <u>normal</u> if $A^*A = AA^*$

$\boxed{\text{EX}}$ ① If $A=A^*$ is self-adjoint, then $A^*A=A^2=AA^*$, so A is normal.
② A unitary matrix $A^*A=I_n=AA^*$ is normal. □

$\boxed{\text{Thm 4.13}}$ $A\in M_{n\times n}$ is diagonalizable if and only if it is normal. In particular, if A is self-adjoint then $\sigma(A)\subset\mathbb{R}$ and if A is unitary then $\sigma(A)\subset\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$.

~~Proof~~ ($\Rightarrow$) Suppose A is diagonalizable. Using the fact, we have $A=UDU^*$ for D diagonal and U unitary. If $D=\text{diag}(\lambda_1,\dots,\lambda_n)$, then observe that

$$D^*D=\text{diag}(|\lambda_1|^2,\dots,|\lambda_n|^2)=DD^*$$

Thus

$$A^*A=(UDU^*)^*UDU^*=(UD^*U^*)(UDU^*)=UD^*DU^*$$
$$=UDD^*U^*=UDU^*(UD^*U^*)=UDU^*(UDU^*)^*=AA^*.$$

Hence A is normal.

($\Leftarrow$) Suppose A is normal. Using the fact, we have $A=UTU^*$ for U unitary and T triangular. Without loss of generality, T is upper triangular: $[T]_{ij}=0$ for $i>j$. Since $T=U^*AU$, the same computation as in the previous part shows the normality of A implies T is normal. Hence $T^*T=TT^*$. Now, for $i=1,\dots,n$, observe that

$$[T^*T]_{ii}=\sum_{j=1}^n[T^*]_{ij}[T]_{ji}=\sum_{j=1}^n\overline{[T]_{ji}}[T]_{ji}=\sum_{j=1}^n|[T]_{ji}|^2=\sum_{j=1}^i|[T]_{ji}|^2$$

and

$$[TT^*]_{ii}=\sum_{j=1}^n[T]_{ij}[T^*]_{ji}=\sum_{j=1}^n[T]_{ij}\overline{[T]_{ij}}=\sum_{j=1}^n|[T]_{ij}|^2=\sum_{j=i}^n|[T]_{ij}|^2.$$

In particular, we have:

$$|[T]_{11}|^2=[T^*T]_{11}=[TT^*]_{11}=|[T]_{11}|^2+|[T]_{12}|^2+\cdots+|[T]_{1n}|^2.$$

Thus we must have $[T]_{1j}=0$ for $j=2,\dots,n$. Next

$$|\cancelto{0}{[T]_{12}}|^2+|[T]_{22}|^2=[T^*T]_{22}=[TT^*]_{22}=|[T]_{22}|^2+|[T]_{23}|^2+\cdots+|[T]_{2n}|^2$$

So $[T]_{2j}=0$ for $j=3,\dots,n$. Continuing in this way, we see that $[T]_{ij}=0$ for all $i<j$. That is, T is lower triangular. Since it was also upper triangular, we see that T is diagonal and thus $A=UTU^*$ is diagonalizable.

Finally, suppose $A=A^*$ is self-adjoint. Then A is normal and so diagonalizable by the above. Using the fact, $A=UDU^*$ for U unitary and D diagonal. So $D=U^*AU$ and

$$D^*=(U^*AU)^*=U^*A^*U=U^*AU=D.$$

Thus if $D=\text{diag}(\lambda_1,\dots,\lambda_n)$ then $\overline{\lambda_j}=\lambda_j$ for each $j=1,\dots,n$. This means $\lambda_j\in\mathbb{R}$.
Hence $\sigma(A)\subset\mathbb{R}$.

If A is unitary, then similarly we have $A=UDU^*$ for U unitary and D diagonal. Since $D^*=U^*AU$, we have

$$D^*D=(U^*AU)^*(U^*AU)=U^*A^*UU^*AU=U^*A^*AU=U^*U=I_n$$

Thus if $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then $\operatorname{diag}(|\lambda_1|^2, \dots, |\lambda_n|^2) = D^* D = I_n = \operatorname{diag}(1, \dots, 1)$. That is, $|\lambda_j|^2 = 1$ for $j = 1, \dots, n$. Hence $\sigma(A) \subset \mathbb{T}$. □

$\boxed{\text{EX}}$ Consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Observe

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},\ A^2\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix},\ A^3\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix},\ A^4\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix}, \dots, A^n\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}F_{n+1}\\F_n\end{pmatrix}$$

where F_n is the nth Fibonacci number. Recall from Homework 8 that $\sigma(A) = \left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$.
Denote

$$\tau := \frac{1-\sqrt{5}}{2}$$
$$\varphi := \frac{1+\sqrt{5}}{2} \quad \leftarrow \text{Golden ratio}$$

You also showed on Homework 8 that

$$\ker(A - \tau I) = \operatorname{span}\left\{\begin{pmatrix}\tau\\1\end{pmatrix}\right\}$$
$$\ker(A - \varphi I) = \operatorname{span}\left\{\begin{pmatrix}\varphi\\1\end{pmatrix}\right\}$$

Thus A has the following diagonalization:

$$A = \begin{pmatrix}\varphi & \tau\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi & 0\\ 0 & \tau\end{pmatrix}\frac{1}{\varphi-\tau}\begin{pmatrix}1 & -\tau\\ -1 & \varphi\end{pmatrix}$$

Therefore

$$\begin{aligned} A^n &= \begin{pmatrix}\varphi & \tau\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^n & 0\\ 0 & \tau^n\end{pmatrix}\frac{1}{\varphi-\tau}\begin{pmatrix}1 & -\tau\\ -1 & \varphi\end{pmatrix} \\ &= \begin{pmatrix}\varphi^{n+1} & \tau^{n+1}\\ \varphi^n & \tau^n\end{pmatrix}\cdot\frac{1}{\varphi-\tau}\begin{pmatrix}1 & -\tau\\ -1 & \varphi\end{pmatrix} \\ &= \frac{1}{\varphi-\tau}\begin{pmatrix}\varphi^{n+1}-\tau^{n+1} & \varphi\tau^{n+1} - \tau\varphi^{n+1}\\ \varphi^n - \tau^n & \varphi\tau^n - \tau\varphi^n\end{pmatrix}\end{aligned}$$

Thus

$$\begin{pmatrix}F_{n+1}\\F_n\end{pmatrix} = A^n\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\varphi-\tau}\begin{pmatrix}\varphi^{n+1}-\tau^{n+1}\\ \varphi^n - \tau^n\end{pmatrix}.$$

This gives us a formula for the Fibonacci numbers:

$$F_n = \frac{\varphi^n - \tau^n}{\varphi-\tau} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \quad \left(\tau = -\frac{1}{\varphi}\right)$$

Consider $x, y > 0$ arbitrary. Then

$$\begin{pmatrix}a\\b\end{pmatrix} := A^n\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{\varphi-\tau}\begin{pmatrix}(\varphi^{n+1}-\tau^{n+1})x + \varphi\tau(\tau^n - \varphi^n)y\\ (\varphi^n - \tau^n)x + \varphi\tau(\tau^{n-1} - \varphi^{n-1})y\end{pmatrix}$$

So using $\varphi \approx 1.618$ and $\tau \approx -0.618$ (so that $\varphi > 1$ and $|\tau| < 1$)

$$\frac{b}{a} = \frac{(\varphi^{n+1} - \tau^{n+1})x + \varphi\tau(\tau^n - \varphi^n)y}{(\varphi^n - \tau^n)x + \varphi\tau(\tau^{n-1} - \varphi^{n-1})y} \approx \frac{\varphi^{n+1}x - \varphi^{n+1}\tau y}{\varphi^n x - \varphi^n \tau y} = \varphi$$

□