Remark: As we saw in the proof of part (c), the two formulas from part (a) can be combined to the more easily stated

$$T^*(f\,dz_I) = T^*(f)\,dT_I.$$

Note also that since $f\,dz_I = f \wedge dz_I$ and $dT_I = T^*(dz_I)$, this is really just saying that the pullback preserves wedge products.

5.9 General Stokes Formula

We will prove the following formula:

$$\int_\varphi d\omega = \int_{\partial\varphi} \omega$$

for $\omega \in \Omega^k(\mathbb{R}^n)$ and $\varphi \in C_{k+1}(\mathbb{R}^n)$, where '$\partial\varphi$' will represent a "boundary" for the k-cell φ, that we will make formal. From this formula, we will derive the formulas you saw in multivariable calculus: Green's theorem, Divergence theorem, Stokes theorem.

Def: A k-chain in $\mathbb{R}^n$ is a formal linear combination of k-cells in $\mathbb{R}^n$:

$$\Phi = \sum_{j=1}^{N} a_j \varphi_j$$

$a_1, \ldots, a_N \in \mathbb{R}$, $\varphi_1, \ldots, \varphi_N \in C_k(\mathbb{R}^n)$. The integral of $\omega \in \Omega^k(\mathbb{R}^n)$ over the k-chain Φ is defined as

$$\int_\Phi \omega := \sum_{j=1}^{N} a_j \int_{\varphi_j} \omega.$$

Remark: we really mean formal sum here; because the explicit sum (thinking of $\varphi_1, \ldots, \varphi_N : [0,1]^k \to \mathbb{R}^n$)

will not give the desired formula for integrating a k-form. (Exercise: check this).
Instead, you should think of a k-chain Φ as parametrizing the chain

$$\bigcup_{j=1}^{N} \varphi_j([0,1]) \subseteq \mathbb{R}^n$$

and weighting each subset $\varphi_j([0,1])$ by a_j.

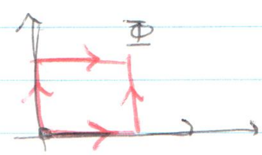
Ex: Define $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in C_1(\mathbb{R}^2)$

$$\varphi_1(t) = (t, 0)$$
$$\varphi_2(t) = (1, t)$$
$$\varphi_3(t) = (t, 1)$$
$$\varphi_4(t) = (0, t)$$

Then

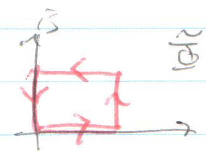
$$\Phi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \text{ is}$$

a 1-chain that we should picture as:



Perhaps a more "natural" 1-chain is

$$\tilde{\Phi} = \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 :$$



Def For $\varphi \in C_{k+1}(\mathbb{R}^n)$, the boundary of φ is the k-chain

$$\partial\varphi = \sum_{j=1}^{k+1} (-1)^{j+1} \left(\varphi \circ \iota^{j,1} - \varphi \circ \iota^{j,0} \right)$$

where $\iota^{j,1}, \iota^{j,0} : [0,1]^k \to [0,1]^{k+1}$ are defined by:

$$\iota^{j,0}(u_1, \dots, u_k) = (u_1, \dots, u_{j-1}, 0, u_j, \dots, u_k)$$

$$\iota^{j,1}(u_1, \dots, u_k) = (u_1, \dots, u_{j-1}, 1, u_j, \dots, u_k)$$

$\iota^{j,0}$ and $\iota^{j,1}$ are called the jth rear face and jth front face, respectively, of the identity map $\iota : [0,1]^{k+1} \to [0,1]^{k+1}$.
The jth dipole of φ is the k-chain

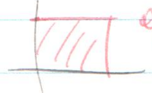
$$\delta^j\varphi := \varphi\circ\iota^{j,1} - \varphi\circ\iota^{j,0}$$

Hence $\partial\varphi$ is the alternating sum

$$\partial\varphi = \sum_{j=1}^{k+1} (-1)^{j+1}\,\delta^j\varphi.$$

<u>Ex</u>: ① Consider $\varphi \in C_2(\mathbb{R}^2)$

$$\varphi(u_1,u_2) = (u_1,u_2)$$


φ

That is $\varphi = \iota$. Then

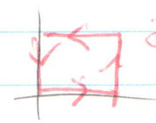
$$\iota^{1,0}(t) = (0,t)$$
$$\iota^{1,1}(t) = (1,t)$$
$$\iota^{2,0}(t) = (t,0)$$
$$\iota^{2,1}(t) = (t,1)$$

So

$$\partial\varphi = (-1)^{1+1}(\varphi\circ\iota^{1,1}(t) - \varphi\circ\iota^{1,0}(t)) + (-1)^{2+1}(\varphi\circ\iota^{2,1}(t) - \varphi\circ\iota^{2,0}(t))$$
$$= \varphi(1,t) - \varphi(0,t) - \varphi(t,1) + \varphi(t,0)$$

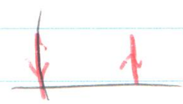
FORMAL SUM $\quad = (1,t) - (0,t) - (t,1) + (t,0)$

From previous example $\quad = \varphi_2(t) - \varphi_4(t) - \varphi_3(t) + \varphi_1(t)$:


$\partial\varphi$

In particular

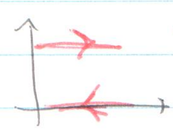
$$\delta^1\varphi(t) = \varphi\circ\iota^{1,1} - \varphi\circ\iota^{1,0}$$
$$= \varphi(1,t) - \varphi(0,t) \; :$$


$\delta^1\varphi$

and

$$\delta^2\varphi(t) = \varphi\circ\iota^{2,1} - \varphi\circ\iota^{2,0}$$
$$= \varphi(t,1) - \varphi(t,0) \; :$$

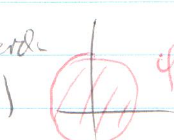
gets reversed in alternating sum.


$\delta^2\varphi$

② ~~Let k+2 ... let k+1 = 3. Then~~

Consider $\varphi \in C_2(\mathbb{R}^2)$ given by polar coords

$$\varphi(u_1,u_2) = (u_1\cos(2\pi u_2),\ u_1\sin(2\pi u_2))$$

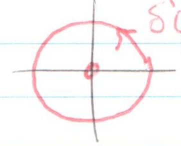

φ

Then

$$\cancel{\delta^1\varphi(t) = \varphi(t,1) - \varphi(t,0)}$$
$$\cancel{= (t\cos(2\pi\cdot 1), t\sin(2\pi\cdot 1)) - (t\cos(2\pi\cdot 0), t\sin(2\pi\cdot 0))}$$
$$\cancel{= (t,0) - (t,0)} \; :$$

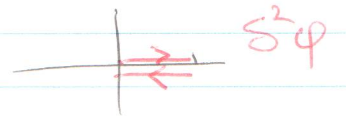

~~$\delta^1\varphi$~~

$$\delta^1\varphi(t) = \varphi(1,t) - \varphi(0,t)$$
$$= (\cos(2\pi t), \sin(2\pi t)) - (0,0):$$

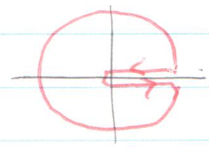


and

$$\delta^2\varphi(t) = \varphi(t,1) - \varphi(t,0)$$
$$= (t\cos(2\pi), t\sin(2\pi)) - (t\cos(0), t\sin(0))$$
$$= (t,0) - (t,0):$$



Thus $\partial\varphi = \delta^1\varphi - \delta^2\varphi$:



(3) For $k=2$, $k+1=3$

$\iota^{1,0}(u_1,u_2) = (0, u_1, u_2)$ $\qquad$ $\iota^{1,1}(u_1,u_2) = (1, u_1, u_2)$

$\iota^{2,0}(u_1,u_2) = (u_1, 0, u_2)$ $\qquad$ $\iota^{2,1}(u_1,u_2) = (u_1, 1, u_2)$

$\iota^{3,0}(u_1,u_2) = (u_1, u_2, 0)$ $\qquad$ $\iota^{3,1}(u_1,u_2) = (u_1, u_2, 1)$

Exercise: (a) For the cube

$$\varphi(u_1,u_2,u_3) = (u_1, u_2, u_3) \in C_3(\mathbb{R}^3)$$

Determine $\partial\varphi$ and $\delta^1\varphi$, $\delta^2\varphi$, $\delta^3\varphi$

(b) For the sphere

$$\varphi(u_1,u_2,u_3) = (u_1\cos(2\pi u_2)\sin(\pi u_3),\ u_1\sin(2\pi u_2)\sin(\pi u_3),\ u_1\cos(\pi u_3))$$

determine $\partial\varphi$ and its dipoles.

Remark: For $\iota : [0,1]^{k+1} \longrightarrow \mathbb{R}^{k+1}$ the identity inclusion, thought of as a $k+1$-cell in $\mathbb{R}^{k+1}$, have

$$\delta^j \iota = \iota^{j,1} - \iota^{j,0}$$

Thus

$$\delta^j\varphi = \varphi\circ\iota^{j,1} - \varphi\circ\iota^{j,0} = \varphi\circ(\delta^j\iota) = \varphi_* \delta^j\iota$$

We first prove the general Stokes formula for $\varphi = \iota \in C_{k+1}(\mathbb{R}^{k+1})$. Then we will generalize to $\varphi \in C_{k+1}(\mathbb{R}^n)$.

<u>Thm</u> (Stokes' formula for a cube)
Assume $k+1=n$. If $\omega \in \Omega^k(\mathbb{R}^n)$ and $\iota : I^n \to \mathbb{R}^n$ is the identity inclusion so that $\iota \in C_n(\mathbb{R}^n)$, then

$$\int_\iota d\omega = \int_{\partial \iota} \omega .$$

<u>Pf</u>: Since $k = n-1$, we can write

$$\omega = \sum_{i=1}^n f_i(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

where the hat '$\wedge$' above dx_i denotes omission. (In particular, this is the unique ascending presentation of ω). We then have

$$d\omega = \sum_{i=1}^n df_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^n \Big(\sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j \Big) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^n (-1)^{i-1} \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum (-1)^{i+1} \text{———} \text{"} \text{———}$$

Thus

$$\int_\iota d\omega = \sum_{i=1}^n (-1)^{i+1} \int_\iota \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^n (-1)^{i+1} \int_{[0,1]^n} \frac{\partial f_i}{\partial x_i} \circ \iota(u) \frac{\partial \iota}{\partial u} \, du$$

$$= \sum_{i=1}^n (-1)^{i+1} \int_{[0,1]^n} \frac{\partial f_i}{\partial x_i}(u) \cdot 1 \, du$$

Fubini $$= \sum_{i=1}^n (-1)^{i+1} \int_0^1 \cdots \int_0^1 \frac{\partial f_i}{\partial x_i}(x)\, dx_i \, dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

FTC $$= \sum_{i=1}^n (-1)^{i+1} \int_0^1 \cdots \int_0^1 f_i(x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n) - f_i(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) \, dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

notation change $$= \sum_{i=1}^n (-1)^{i+1} \int_0^1 \cdots \int_0^1 f_i(u_1, \dots, u_{j-1}, 1, u_j, \dots, u_k) - f_i(u_1, \dots, u_{j-1}, 0, u_j, \dots, u_k) \, du_1 \cdots du_k$$

Fubini $$= \sum_{i=1}^n (-1)^{i+1} \int_{[0,1]^k} f_i \circ \iota^{i,1}(u) - f_i \circ \iota^{i,0}(u) \, du$$

Let us now compute $\int_{S^j_\iota} \omega$. Note that

$$S^j_\iota = \underbrace{\iota \circ \iota^{j,1} - \iota \circ \iota^{j,0} = \iota^{j,1} - \iota^{j,0}}_{\text{formal sums}}$$

Now, $\iota^{j,1}, \iota^{j,0} \in C_k(\mathbb{R}^n)$ and $I = (1, 2, \ldots, \hat{i}, \ldots n)$

we have $\frac{\partial (\iota^{j,1})_I}{\partial u} = \begin{cases} \det(1) & \text{if } i=j \\ \det\left(\begin{bmatrix} 1 & & \\ 0\,0\,0 & \vdots & 0 \\ & & \ddots\ 1 \end{bmatrix}\right) & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$

~~There~~ Similarly for $\iota^{j,0}$. Thus

$$\int_{S^j_\iota} \omega = \int_{\iota^{j,1}} \omega - \int_{\iota^{j,0}} \omega = \int_{\iota^{j,1}} \sum_i f_i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n - \int_{\iota^{j,0}} \text{—} \text{''} \text{—}$$

$$= \int_{[0,1]^k} f_j \circ \iota^{j,1} \, du - \int_{[0,1]^k} f_j \circ \iota^{j,0} \, du$$

Consequently

$$\int_{\partial \varphi} \omega = \sum_{j=1}^{n} (-1)^{j+1} \int_{[0,1]^k} f_j \circ \iota^{j,1}(u) - f_j \circ \iota^{j,0}(u) \, du$$

which agrees with our computation for $\int_\varphi d\omega$. $\square$

covered on 3/14/08

Thm (General Stokes' Formula)

Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\varphi \in C_{k+1}(\mathbb{R}^n)$, then

$$\int_\varphi d\omega = \int_{\partial\varphi} \omega.$$

Pf: Consider φ as ~~a~~ smooth function $\varphi : [0,1]^k \to \mathbb{R}^n$ and let $\iota : [0,1]^{k+1} \to \mathbb{R}^{k+1}$ be the identity inclusion (so $\iota \in C_{k+1}(\mathbb{R}^{k+1})$.) Then ~~by~~ $\varphi \circ \iota = \varphi_* \iota$ and so by our theorem on pullbacks we have:

$$\int_\varphi d\omega = \int_{\varphi \circ \iota} d\omega = \int_{\varphi_* \iota} d\omega = \int_\iota \varphi^*(d\omega)$$

$$= \int_\iota d(\varphi^* \omega)$$

(previous thm) $$= \int_{\partial\iota} \varphi^* \omega = \sum_{j=1}^{k+1} (-1)^{j+1} \int_{S^j_\iota} \varphi^* \omega$$

3/16/2018

$$\cancel{\text{Now} \int_{\varphi_* \partial I} \omega = \sum_{j=1}^{k-1} \ldots \int_{\varphi_* \delta}}$$

$$= \sum_{j=1}^{k+1} (-1)^{j+1} \int_{\varphi_* \delta^j I} \omega$$

We previously noted that $\varphi_* \delta^j I = \delta^j \varphi$. Thus, continuing our previous computation, we have:

$$\int_\varphi d\omega = \sum_{j=1}^{k+1} (-1)^{j+1} \int_{\delta^j \varphi} \omega = \int_{\partial \varphi} \omega \qquad \square$$

3/19/2018

Vector Calculus

~~The fundamental theorem~~

Application 1: The fundamental theorem of calculus

Let $f: \mathbb{R} \to \mathbb{R}$ be a smooth function (we actually just need diff'ble) and let $\varphi \in C_1(\mathbb{R})$

$$\varphi(t) = (1-t)a + tb$$

so $\varphi([0,1]) = [a,b]$. Then $\partial\varphi \in \Omega^0(\mathbb{R})$ is given by the single chain

$$\delta^1 \varphi = \underbrace{\varphi(1) - \varphi(0)}_{\text{formal sum}} = b - a$$

So really $\partial\varphi$ is the 0-chain whose image is $\{a\} \cup \{b\}$ and $\{a\}$ is weighted by -1.

Stokes' formula:

$$f(b) - f(a) = \int_{\partial\varphi} f = \int_\varphi df = \int_0^1 f'((1-t)a + tb)\, dt$$

$$= \int_a^b f'(s)\, ds$$

Application 2: Green's theorem

Let $\varphi \in C_2(\mathbb{R}^2)$ and denote $D = \varphi([0,1]^2)$ and $C = \partial\varphi([0,1]^2)$. Then for $\omega = f\,dx + g\,dy$ we have

$$\int_C f\,dx + g\,dy = \int_{\partial\varphi} \omega = \int_\varphi d\omega = \iint_D \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{\text{previously computed.}} dx\,dy$$

Application 3: Gauss Divergence Theorem.

Let $F: U \to \mathbb{R}^3$ be smooth for $U \subseteq \mathbb{R}^3$ open. Write $F = (f, g, h)$ for smooth, real-valued functions f, g, h. Then F is a smooth vector field and its divergence is

$$\nabla \cdot F = \operatorname{div} F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

If $\varphi \in C_3(\mathbb{R}^3)$ with $D = \varphi([0,1]^3)$ and $S = \partial\varphi([0,1]^3)$ Then

$$\iiint_D \nabla \cdot F = \int_\varphi \nabla \cdot F \overset{dx \wedge dy \wedge dz}{} = \int_\varphi d(f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy)$$

$$= \int_{\partial\varphi} f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy$$

$$= \iint_S (F \cdot \vec{n})\, dS$$

Application 4: Stokes' Curl Theorem

Let F be as before, then

$$\operatorname{curl}(F) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

If $\varphi \in C_2(\mathbb{R}^3)$ with $S = \varphi([0,1]^2)$ $C = \partial\varphi([0,1]^2)$

Then

$$\iint_S \operatorname{curl}(F) \cdot \vec{n}\, dA = \int_\varphi \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$= \int_\varphi d(f\, dx + g\, dy + h\, dz)$$

$$= \int_{\partial\varphi} f\, dx + g\, dy + h\, dz$$

$$= \int_C f\, dx + g\, dy + h\, dz.$$

3/28/2018

Closed Forms and Exact Forms

Def: We say $\omega \in \Omega^k(\mathbb{R}^n)$ is closed if $d\omega = 0$.

We say $\omega \in \Omega^k(\mathbb{R}^n)$ is exact if $\exists\, \alpha \in \Omega^{k-1}(\mathbb{R}^n)$ s.t. $d\alpha = \omega$.

Since $d^2=0$, exact forms are always ~~exact~~ closed.
Question: Are closed forms always exact?
That is, if $d\omega=0$, can we find α s.t. $d\alpha=\omega$?

We will see that for the type of forms we have considered (namely whose coefficient functions are defined on all of $\mathbb{R}^n$), the answer is yes.
~~However, for $\omega\in\Omega^k(U)$~~

Def: Let $U\subseteq\mathbb{R}^n$ be open. The k-forms on U are all sums

$$\sum_I f_I\, dy_I$$

where $I\in\{1,\dots,n\}^k$ and $f_I : U\to\mathbb{R}$ are smooth functions only assumed to be defined on U. The set of k-forms on U is denoted $\Omega^k(U)$.

- The basic k-forms on U are exactly the basic k-forms on $\mathbb{R}^n$, but the simple k-forms on U are not all simple k-forms on $\mathbb{R}^n$.

- Note that if $\varphi\in C_k(\mathbb{R}^n)$ satisfies $\varphi([0,1]^k)\subseteq U$, then ~~for~~ $\omega=\sum f_I\, dy_I\in C_k(U)$ we can define

$$\int_\varphi \omega=\int_{[0,1]^k}\sum_I f_I\circ\varphi\cdot\frac{\partial\varphi_I}{\partial u}\,du$$

just as before.

Def: ~~The~~ The k-cells in U, denoted $C_k(U)$, are k-forms in $\mathbb{R}^n$ φ s.t. $\varphi([0,1]^k)\subseteq U$.

Question: Are closed k-forms on U always exact?
Answer: ~~No~~ No, it depends on the "topology" of U (i.e. does U have any holes?).

Ex ① $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$

Then

$$d\omega = \left(\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)dx + \frac{\partial}{\partial y}(\sim)dy\right)\wedge dx + \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)dx + \frac{\partial}{\partial y}(\sim)dy\right)\wedge dy$$

$$= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)dy\wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)dx\wedge dy$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}dy\wedge dx + \frac{y^2-x^2}{x^2+y^2}dx\wedge dy = 0$$

However, $\omega \neq df$. Can check this directly, or by noting $\varphi(t) = (\cos(2\pi t), \sin(2\pi t))$

~~Calc~~ Calc: $\int_\varphi \omega = \int_0^1 \frac{-\sin(2\pi t)(-2\pi\sin(2\pi t))}{1} + \frac{\cos(2\pi t)(2\pi\cos(2\pi t))}{1}dt$

$$= \int_0^1 2\pi\, dt = 2\pi$$

while if $\omega = df$ then by Stokes theorem:

$$\int_\varphi \omega = \int_\varphi df = \int_{\partial\varphi} f = f(\varphi(1)) - f(\varphi(0)) = 0.$$

③ $\omega = \frac{x}{(x^2+y^2+z^2)^{3/2}}dy\wedge dz + \frac{y}{(x^2+y^2+z^2)^{3/2}}dz\wedge dx + \frac{z}{(x^2+y^2+z^2)^{3/2}}dx\wedge dy$

$\in \Omega^2(\mathbb{R}^3 \setminus \{(0,0,0)\})$

~~Exerc~~

Exercise: Show ω is closed, but not exact.

Thm (Poincaré Lemma)

If $\omega \in \Omega^k(\mathbb{R}^n)$ is closed, then it is exact.

Pf We will show the existence of "integration operators"

$$L_k : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n) \qquad \forall k \in \mathbb{N}$$

s.t. for $\omega \in \Omega^k(\mathbb{R}^n)$

$$* \qquad (L_{k+1}d + dL_k)(\omega) = \omega$$

Assuming we have such operators, the proof will be complete. Indeed, if ω is closed, then:

$$\omega = (L_{k+1}d + dL_k)(\omega) = 0 + dL_k(\omega) = d(L_k(\omega))$$

So ω is exact.

Let $\beta \in \Omega^k(\mathbb{R}^{n+1})$, which we can write uniquely as:

$$\beta = \sum_I f_I dx_I + \sum_J g_J\, dx_{n+1}\wedge dx_J$$

where the first sum is over ascending k tuples I in $\{1,\dots,n\}$, the second sum is over asc. $(k-1)$ tuples J in $\{1,\dots,n\}$, and $f_I, g_J : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$.
Here we write an element of $\mathbb{R}^{n+1}$ as:

$$(x,t) \in \mathbb{R}^n \times \mathbb{R}$$

That is, we're thinking of 't' as the $(n+1)$st coordinate.

Now

$$d\beta = \sum_{I,\ell} \frac{\partial f}{\partial x_\ell} dx_\ell \wedge dx_I + \sum_I \frac{\partial f}{\partial t} dt \wedge dx_I + \sum_{J,\ell} \frac{\partial g}{\partial x_\ell} dx_\ell \wedge dt \wedge dx_J$$

where $\ell = 1,\dots,n$. Define $N: \Omega^k(\mathbb{R}^{n+1}) \longrightarrow \Omega^{k-1}(\mathbb{R}^n)$ on forms β by:

$$N_k(\beta) := \sum_J \left(\int_0^1 g_J(x,t)\, dt \right) dx_J$$

(thus N ignores the terms f_I where 'dt' does not appear).
We claim

$$** \quad (dN_k + N_{k+1} d)(\beta) = \sum_I \left(f_I(x,1) - f_I(x,0) \right) dx_I$$

Let us compute:

signed comm.

$$N_{k+1}(d\beta) = \sum_I \left(\int_0^1 \frac{\partial f_I}{\partial t} \right) dx_I - \sum_{J,\ell} \left(\int_0^1 \frac{\partial g_J}{\partial x_\ell} dt \right) dx_\ell \wedge dx_J$$

$$(\text{FTC}) \quad = \sum_I \left(f_I(x,1) - f_I(x,0) \right) dx_I - \sum_{J,\ell} \left(\int_0^1 \frac{\partial g_J}{\partial x_\ell} dt \right) dx_\ell \wedge dx_J$$

Next we compute:

$$d(N_k(\beta)) = d\left(\sum_J \left(\int_0^1 g_J(x,t)\, dt \right) \right) dx_J$$

thm from 5.2

$$= \sum_{J,\ell} \int_0^1 \frac{\partial g_J}{\partial x_\ell} dt \; dx_\ell \wedge dx_J$$

Thus

4/2/2015

$$(dN_k + N_{k+1} d)(\beta) = \sum_J \left(f_I(x,1) - f_I(x,0) \right) dx_I$$

as claimed.

Next we define a cone map $\rho: \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\rho(x,t) = tx.$$

We then define $L_k := N_k \circ \rho^*$. Recall that pullbacks commute with d. Thus:

$$L_{k+1}d + dL_k = N_{k+1}\rho^* d + dN_k\rho^* = (N_{k+1}d + dN_k)\rho^*$$

Now, let $\omega = h\,dx_I \in \Omega^k(\mathbb{R}^n)$ be a simple k-form in $\mathbb{R}^n$. Then if $I = (i_1, \ldots, i_k)$

$$\begin{aligned}\rho^*(h\,dx_I) &= (\rho^* h)\, d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_k} \\ &= h(tx)\, d(tx_{i_1}) \wedge \cdots \wedge d(tx_{i_k}) \\ &= h(tx)\,(t\,dx_{i_1} + x_{i_1}dt) \wedge \cdots \wedge (t\,dx_{i_k} + x_{i_k}dt) \\ &= h(tx)\,(t^k dx_{i_1} \wedge \cdots \wedge dx_{i_k}) + \text{terms involving } dt.\end{aligned}$$

Since $N_{k+1}d + dN_k$ ignores dt, we therefore have: by $(**)$

$$(N_{k+1}d + dN_k)(\rho^*(h\,dx_I)) = (h(1x)1^k - h(0 \cdot x)0^k)dx_I = h\,dx_I$$

That is,

$$(L_{k+1}d + dL_k)(\omega) = \omega.$$

Since everything is linear this holds for all $\omega \in \Omega^k(\mathbb{R}^n)$, and so $(*)$ holds. $\square$

<u>Cor:</u> If U is diffeomorphic to $\mathbb{R}^n$, then all closed forms on U are exact.

<u>Pf:</u> Let $T: U \to \mathbb{R}^n$ be a diffeomorphism. Assume $\omega \in \Omega^k(U)$ is closed. Set $\alpha := (T^{-1})^*\omega \in \Omega^k(\mathbb{R}^n)$. Since pullbacks commute with d, we have:

$$d\alpha = d(T^{-1})^*\omega = (T^{-1})^*(d\omega) = 0$$

so α is closed. By the Poincaré lemma, there exists $\mu \in \Omega^{k-1}(\mathbb{R}^n)$ s.t. $\alpha = d\mu$. But then

$$dT^*\mu = T^*d\mu = T^*\alpha = T^*(T^{-1})^*\omega = (T \circ T^{-1})^*\omega = id^*\omega = \omega.$$

holds by duality equation

Thus ω is exact. $\square$

Def A subset $U \subseteq \mathbb{R}^n$ is starlike if $\exists p \in U$ s.t. $\forall q \in U$, $[p,q] \subseteq U$.

Cor If $U \subseteq \mathbb{R}^n$ is open and starlike, then every closed form on U is exact. In particular, if U is convex, then this holds

Pf By the previous corollary, it suffices to show any open starlike set is diffeomorphic to $\mathbb{R}^n$ — exercise. $\square$

Cohomology

The set of exact k-forms on U is denoted $B^k(U)$ ('B' for "boundary"), while the set of closed k-forms on U is denoted $Z^k(U)$ ('Z' for "Zyklus" or "cycle") we always have

$$B^k(U) \subseteq Z^k(U) \subseteq \Omega^k(U)$$

~~closed~~ Note that these are all vector spaces and so we can consider

$$H^k(U) := Z^k(U) / B^k(U)$$
$$= Z^k(U) / \alpha \sim \beta \text{ if } \alpha - \beta \text{ is exact.}$$

This is called the kth de Rham cohomology group of U.

If $U \subseteq \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$ ~~open and starlike~~, then

$$H^k(U) = \{0\} \qquad \forall k \geq 1$$

But, as we saw in our examples

$$H^1(\mathbb{R}^2 \setminus \{0,0\}) \neq \{0\} \quad \text{and} \quad H^2(\mathbb{R}^3 \setminus \{(0,0,0)\}) \neq \{0\}.$$

In general, $\{H^k(U)\}_{k \in \mathbb{N}}$ are invariants of U (up to diffeomorphism) that depend on the "topology" of U. This is studied more extensively in algebraic topology.